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An Overview of Conditionals and Biconditionals in Probability

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Abstract: Conditional and biconditional statements are a standard part of symbolic logic but they have only recently begun to be explored in probability for applications in artificial intelligence. Here we give a brief overview of the major theorems involved and illustrate them using two standard model problems from conditional probability.

Key–Words: probability of material implication; probability of a conditional; probability of a biconditional

1 Introduction

Elementary treatments of symbolic logic include a discussion of the logical operations of negation, conjunction, disjunction, the conditional and the biconditional. Elementary treatments of probability, however, will discuss probabilities of a negation, conjunction, and disjunction but will leave out probabilities of conditionals and biconditionals: statements of the form "if Q then P" and "P if and only if Q." Our treatment here follows the work of Nguyen, Mukaidono, and Kreinovich [6] where the topic is developed for applications in artificial intelligence. The authors there show that an analog of Bayes' theorem involving probabilities of conditionals called Logical Bayes' theorem, yields comparable results to the standard version of Bayes' theorem.

In this presentation we outline a treatment of this topic suitable for undergraduates. We show how these probabilities are computed and apply them to two standard model problem. Our contention is that once an instructor has developed the laws of probability of involving the logical operations of "and," "or" and "not" and once an instructor has covered conditional probability, the student is fully prepared for a discussion on the probability of a conditional, biconditional, and related issues such as the corresponding logical analog of Bayes' theorem.

We begin with an experiment which produces a sample space S, the set of all possible outcomes of the experiment. Any subset of the sample space is known as an *event*. Let P and Q be any two events. Then we recall the definition of conditional probability for the discrete case as

$$\mathbf{P}(P|Q) = \frac{\#\text{ outcomes in }(P\cap Q)}{\#\text{ outcomes in }Q},$$

from which it readily follows that

$$\mathbf{P}(P|Q) = \frac{\mathbf{P}(P\cap Q)}{\mathbf{P}(Q)}.$$

This may be interpreted as given certain knowledge that Q occurred, our sample space can be restricted from S to Q, and as a result the event P is restricted to $P\cap Q$. We call this the probability of P given Q, or P given knowledge that Q occurred.

Next we recall the logical definitions of the conditional (P implies Q) and the biconditional (P implies Q and Q implies P). We do so using truth tables.

P	Q	$P\to Q$	$P\leftrightarrow Q$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

The operation $P\to Q$ is known as *material implication* or the *conditional* and it is seen to be logically equivalent to $\sim P\cup Q$. It is also logically equivalent to $\sim(P\cap\sim Q)$. It is often read "P implies Q" or "if P then Q." It is pointed out, for example in Copi [2] p.17, that the symbol for material implication should not be thought of as representing the unique meaning of if-then, rather it should be thought of as the partial meaning which all uses of the term if-then have in common. One interesting usage of $P\to Q$ is a statement like "If the team wins, then I'm a monkey's uncle." No real connection is being made between the two statements. Since the statement as a whole $P\to Q$ is asserted by the speaker to be true, and the precedent Q is meant to be absurd, it can only be that the antecedent P is false. This is precisely the speaker's intent in asserting this implication: to assert that P is false.

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2 The Probability of a Conditional

We now wish to consider the probability of a conditional, otherwise known as or material implication, and see how this probability relates to the more well known concept of conditional probability.

2.1 Probability of a Conditional

Definition 1 *Define the event*

$$\begin{aligned} Q &\rightarrow P = \sim Q \cup P \\ &= \sim (Q \cap \sim P) \end{aligned}$$

The following theorem describes the probability of a conditional in terms of conditional probabilities.

Theorem 2

$$\mathbf{P}(Q \rightarrow P) = \mathbf{P}(\sim Q) + \mathbf{P}(P \cap Q)$$

Proof.

$$\begin{aligned} \mathbf{P}(Q \rightarrow P) &= \mathbf{P}(\sim (Q \cap \sim P)) \\ &= 1 - \mathbf{P}(Q \cap \sim P) \\ &= 1 - \mathbf{P}(\sim P|Q)\,\mathbf{P}(Q) \\ &= 1 - (1 - \mathbf{P}(P|Q))\,\mathbf{P}(Q) \\ &= 1 - \mathbf{P}(Q) + \mathbf{P}(P|Q)\,\mathbf{P}(Q) \\ &= \mathbf{P}(\sim Q) + \mathbf{P}(P \cap Q) \end{aligned}$$

■

The next theorem and its corollary can be thought of as describing, among other things, a paradox of small sets: *from a very small set one can draw any conclusion with a high degree of probability.* Or, one may describe this as a paradox of improbable events: *from a very unlikely event, one can draw any conclusion with a high degree of probability.* Not being aware of this paradox in one's reasoning can lead to misleading conclusions. An example of this paradox might be: if a plane crashed (a very unlikely event) then the fleet is unsafe. Technically this is a highly probable inference. An underlying cause might be that whatever caused the plane to crash could be shared by most of the planes. But we can also make the highly probable inference: if a plane crashed then the fleet is safe. The underlying reason being that whatever the hidden causes were which affected the one plane do not pertain to the rest of the fleet.

Corollary 3

$$\begin{aligned} \mathbf{P}(\varnothing \rightarrow P) &= 1 \\ \mathbf{P}(S \rightarrow P) &= \mathbf{P}(P) \\ \mathbf{P}(Q \rightarrow \varnothing) &= \mathbf{P}(\sim Q) \\ \mathbf{P}(Q \rightarrow S) &= 1 \\ \mathbf{P}(Q \rightarrow \varnothing) &= \mathbf{P}(\sim Q) \\ \mathbf{P}(Q \rightarrow Q) &= 1 \\ \mathbf{P}(Q \rightarrow \sim Q) &= \mathbf{P}(\sim Q) \\ \mathbf{P}(\sim Q \rightarrow Q) &= \mathbf{P}(Q) \end{aligned}$$

Now we express standard conditional probability in terms of the probability of a conditional.

Theorem 4

$$\mathbf{P}(P|Q) = \frac{\mathbf{P}(Q \rightarrow P) - \mathbf{P}(\sim Q)}{\mathbf{P}(Q)}$$

From here we may now formulate a version of Bayes' theorem. Recall the simplest version of Bayes' theorem which states

$$\mathbf{P}(P|Q) = \frac{\mathbf{P}(Q|P)\,\mathbf{P}(P)}{\mathbf{P}(Q)}.$$

From this and the previous theorem one can derive the logical analog of Bayes' theorem.

Theorem 5 *Logical Analog of Bayes' Theorem (1).*

$$\mathbf{P}(Q \rightarrow P) = \mathbf{P}(P \rightarrow Q) + \mathbf{P}(P) - \mathbf{P}(Q)$$

Corollary 6 *When P and Q are equiprobable, i.e. $\mathbf{P}(P) = \mathbf{P}(Q)$, then $\mathbf{P}(Q \rightarrow P) = \mathbf{P}(P \rightarrow Q)$.*

Now we recall the law of total probability and the corresponding more general version of Bayes' theorem. The law of total probability states that given a partition of the sample space S into a disjoint union of events Q_i and any event P, we may decompose $\mathbf{P}(P)$ as:

$$\mathbf{P}(P) = \sum_{i=1}^{n} \mathbf{P}(P|Q_i)\,\mathbf{P}(Q_i)$$

Substituting for $\mathbf{P}(P|Q_i)$ using the previous theorem yields the following reformulations of the law of total probability.

Theorem 7 *Reformulation: Law of Total Probability.*

$$\mathbf{P}(P) = \sum_{i=1}^{n} (\mathbf{P}(Q_i \rightarrow P) - \mathbf{P}(\sim Q_i))$$

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Finally we recall Bayes' Theorem in its usual general formulation, namely:

$$\mathbf{P}(Q_i|P) = \frac{\mathbf{P}(P|Q_i)\,\mathbf{P}(Q_i)}{\mathbf{P}(P)}$$

and reformulate it in terms of the probability of an implication.

Theorem 8 *Logical Analog of Bayes' Theorem (2).*

$$\mathbf{P}(P \to Q_i) = \mathbf{P}(Q_i \to P) + \mathbf{P}(Q_i) - \mathbf{P}(P)$$

where $\mathbf{P}(P) = \sum_{i=1}^{n} (\mathbf{P}(Q_i \to P) - \mathbf{P}(\sim Q_i))$ *or any other equivalent expression to the law of total probability.*

We now conclude this section with an important theorem. Conditional probabilities and probabilities of conditionals share the same order relations.

Theorem 9 *Let P, Q and R be events. Then*

$$\mathbf{P}(P|Q) > \mathbf{P}(R|Q)$$

if and only if

$$\mathbf{P}(Q \to P) > \mathbf{P}(Q \to R).$$

Proof. We have

$$\begin{aligned} \mathbf{P}(Q \to P) &> \mathbf{P}(Q \to R) \\ \mathbf{P}(\sim Q) + \mathbf{P}(P|Q)\,\mathbf{P}(Q) &> \mathbf{P}(\sim Q) \\ &\quad + \mathbf{P}(R|Q)\,\mathbf{P}(Q) \\ \mathbf{P}(P|Q) &> \mathbf{P}(R|Q) \end{aligned}$$

and vice-versa. ■

A practical application of this theorem is the fact that probabilities of conditionals and the logical version of Bayes' theorem can be used instead of the standard version of Bayes' theorem, allowing one to draw similar conclusions. We illustrate this in our model problem.

3 Probability of a Biconditional

The probability of the biconditional P if and only if Q, $\mathbf{P}(Q \leftrightarrow P)$, may be thought of intuitively as the probability that P and Q are *either* simultaneously true *or* simultaneously false. The probability of the negated biconditional $P(Q \nleftrightarrow P) = \mathbf{P}(Q \leftrightarrow \sim P) = \mathbf{P}(\sim Q \leftrightarrow P)$, is the probability that P and Q are *neither* simultaneously true *nor* simultaneously false.

Definition 10 *Define the event*

$$\begin{aligned} Q &\leftrightarrow P = (P \cap Q) \cup (\sim P \cap \sim Q) \\ &= (\sim Q \cup P) \cap (\sim P \cup Q). \end{aligned}$$

and the corresponding probability

$$\mathbf{P}(Q \leftrightarrow P) = \mathbf{P}((P \cap Q) \cup (\sim P \cap \sim Q)).$$

The following theorem describes the probability of a biconditional in terms of unions and intersections.

Theorem 11

$$\begin{aligned} \mathbf{P}(Q \leftrightarrow P) &= \mathbf{P}(P \cap Q) + \mathbf{P}(\sim P \cap \sim Q) \\ \mathbf{P}(Q \leftrightarrow P) &= 1 - \mathbf{P}(P \cup Q) + \mathbf{P}(P \cap Q) \\ \mathbf{P}(Q \leftrightarrow P) &= 1 - \mathbf{P}(P) - \mathbf{P}(Q) + 2\mathbf{P}(P \cap Q) \end{aligned}$$

Corollary 12

$$\begin{aligned} \mathbf{P}(P \leftrightarrow P) &= 1 \\ \mathbf{P}(P \leftrightarrow \sim P) &= 0 \\ \mathbf{P}(P \leftrightarrow S) &= \mathbf{P}(P) \\ \mathbf{P}(P \leftrightarrow \varnothing) &= \mathbf{P}(\sim P) \end{aligned}$$

The first two statements can be interpreted as saying that P correlates with itself perfectly and P correlates with its negation not at all. The second two statements say in words that the probability of an event P measures the logical correlation of the event with certainty. The probability of P not occurring measures the logical correlation of the event with impossibility.

The negated biconditional, denoted by $P \nleftrightarrow Q$, is described in set theory as symmetric difference where it is often denoted by $P \vartriangle Q$. It is also described in logic as exclusive or (xor), where it is denoted by P xor Q or by $P \sqcup Q$. Different notations may be employed depending on the context. Since the context here views the biconditional as primary, we will employ the negated biconditional symbol here. We are interested in the probability of the negated biconditional. The following equivalences can be verified:

$$\begin{aligned} P &\nleftrightarrow Q = \sim (P \leftrightarrow Q) \\ &= (P \nrightarrow Q) \cup (Q \nrightarrow P) \\ &= (P \cap \sim Q) \cup (Q \cap \sim P) \\ &= Q \leftrightarrow \sim P \\ &= \sim Q \leftrightarrow P \end{aligned}$$

Theorem 13 *The Negated Biconditional or Exclusive Or.*

$$\mathbf{P}(Q \nleftrightarrow P) = \mathbf{P}(P \cup Q) - \mathbf{P}(P \cap Q)$$

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$$\begin{aligned} P &\nleftrightarrow Q = \sim (P \leftrightarrow Q) \\ &= (P \nrightarrow Q) \cup (Q \nrightarrow P) \\ &= (P \cap \sim Q) \cup (Q \cap \sim P) \\ &= Q \leftrightarrow \sim P \\ &= \sim Q \leftrightarrow P \end{aligned}$$

Finally we describe the probability of a biconditional in terms of the probability of a conditional.

Theorem 14

$$\begin{aligned} \mathbf{P}(Q \leftrightarrow P) &= \mathbf{P}(Q \rightarrow P) + \mathbf{P}(P \rightarrow Q) - 1 \\ \mathbf{P}(Q \leftrightarrow P) &= 2\mathbf{P}(P \rightarrow Q) + \mathbf{P}(P) - \mathbf{P}(Q) - 1 \\ \mathbf{P}(Q \leftrightarrow P) &= 2\mathbf{P}(Q \rightarrow P) + \mathbf{P}(Q) - \mathbf{P}(P) - 1 \end{aligned}$$

We leave the proof of these statements to the reader.

4 A Logical Correlation Coefficient

When $\mathbf{P}(P \leftrightarrow Q)$ is near 1 we can say that P and Q are nearly simultaneously true or simultaneously false. We may also say that the events are nearly the same. This corresponds to P and Q exhibiting a strong direct (positive) logical correlation. When $\mathbf{P}(P \leftrightarrow Q)$ is near 0 we can say that P and $\sim Q$ are nearly simultaneously true or simultaneously false, which corresponds to P and Q exhibiting a strong inverse (negative) logical correlation. We may also say that the events are nearly complementary. It follows that $\mathbf{P}(P \leftrightarrow Q) \approx \frac{1}{2}$ corresponds to P and Q exhibiting little logical correlation. Based on this we may define an effective logical correlation coefficient for two events.

Definition 15 *The logical correlation coefficient for the events P and Q is given by*

$$\boldsymbol{\rho}(P,Q) = 2\mathbf{P}(P \leftrightarrow Q) - 1.$$

It follows that

$$\mathbf{P}(P \leftrightarrow Q) = \frac{\boldsymbol{\rho}(P,Q) + 1}{2}$$

and that

$$-1 \leq \boldsymbol{\rho}(P,Q) \leq 1.$$

The next theorem is quite fundamental in that it gives conceptual justification for the preceding definition. It says that the *logical correlation coefficient of two events is the probability that those events are the same minus the probability that those events are different.*

Theorem 16

$$\boldsymbol{\rho}(P,Q) = \mathbf{P}(P \leftrightarrow Q) - \mathbf{P}(P \nleftrightarrow Q)$$

Proof. By definition

$$\begin{aligned} \boldsymbol{\rho}(P,Q) &= 2\mathbf{P}(P \leftrightarrow Q) - 1 \\ &= \mathbf{P}(P \leftrightarrow Q) + \mathbf{P}(P \leftrightarrow Q) - 1 \\ &= \mathbf{P}(P \leftrightarrow Q) - (1 - \mathbf{P}(P \leftrightarrow Q)) \\ &= \mathbf{P}(P \leftrightarrow Q) - \mathbf{P}(P \nleftrightarrow Q) \end{aligned}$$

■

Corollary 17

$$\boldsymbol{\rho}(P,Q) = 1 - 2\mathbf{P}(P) - 2\mathbf{P}(Q) + 4\mathbf{P}(P \cap Q)$$

The following are some algebraic properties of logical correlation which the reader can verify.

Theorem 18

$$\begin{aligned} \boldsymbol{\rho}(P,Q) &= \boldsymbol{\rho}(Q,P) \\ \boldsymbol{\rho}(\sim P,Q) &= -\boldsymbol{\rho}(P,Q) \\ \boldsymbol{\rho}(P,P) &= 1 \\ \boldsymbol{\rho}(\sim P,P) &= -1 \\ \boldsymbol{\rho}(P,S) &= 2\mathbf{P}(P) - 1 \\ \boldsymbol{\rho}(P,\varnothing) &= 1 - 2\mathbf{P}(P) \\ \boldsymbol{\rho}(P,Q) &= 0 \textit{ if and only if } \mathbf{P}(P \leftrightarrow Q) = \frac{1}{2} \end{aligned}$$

5 Logical and Independence

We now describe a notion of *logical independence* which is analogous but distinct from the standard notion of *stochastic or statistical dependence*. Recall that two events P and Q are stochastically independent if $\mathbf{P}(P|Q) = \mathbf{P}(P)$ or alternatively if $\mathbf{P}(P \cap Q) = \mathbf{P}(P)\mathbf{P}(Q)$.

Definition 19 *Logical independence. The events P and Q are called logically independent if*

$$\mathbf{P}(Q \rightarrow P) = \mathbf{P}(P).$$

It follows from this definition and the logical version of Bayes' theorem that $\mathbf{P}(Q \rightarrow P) = \mathbf{P}(P)$ whenever $\mathbf{P}(P \rightarrow Q) = \mathbf{P}(Q)$.

Theorem 20 *The events P and Q are logically independent if and only if*

$$\mathbf{P}(P \leftrightarrow Q) = \mathbf{P}(P \cap Q)$$

or

$$\mathbf{P}(P \cup Q) = 1$$

or

$$\mathbf{P}(\sim P \cap \sim Q) = 0.$$

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We see that logically independent events cover the sample space. Equivalently, we see that logically independent events cannot be simultaneously false. Recall that P and Q are statistically independent if $\mathbf{P}(P|Q) = \mathbf{P}(P)$ or alternatively if $\mathbf{P}(P \cap Q) = \mathbf{P}(P)\,\mathbf{P}(Q)$. We may ask: in which circumstance do the two notions of logical independence and statistical independence coincide? The next theorem answers this question.

Corollary 21 *The events P and Q are both logically independent and statistically independent whenever:*

$$\mathbf{P}(P) = 1 \text{ or } \mathbf{P}(Q) = 1.$$

That is, the only way for two events to be both logically and statistically independent is if one of them is certain to occur.

Proof. We begin with the formula

$$\mathbf{P}(P \cup Q) = \mathbf{P}(P) + \mathbf{P}(Q) - \mathbf{P}(P \cap Q)$$

which reduces in our case to

$$\begin{aligned} 1 &= \mathbf{P}(P) + \mathbf{P}(Q) - \mathbf{P}(P)\,\mathbf{P}(Q) \\ 0 &= (1 - \mathbf{P}(P))\,(1 - \mathbf{P}(Q)) \end{aligned}$$

from which the result follows. ■

Corollary 22 *If the events P and Q are logically independent and statistically independent then*

$$\mathbf{P}(P \leftrightarrow Q) = \mathbf{P}(P)\,\mathbf{P}(Q).$$

Corollary 23 *If the events P and Q are logically independent and mutually exclusive then*

$$Q = \sim P.$$

We recall that

$$\mathbf{P}(P \leftrightarrow Q) = \mathbf{P}(P \cap Q) + \mathbf{P}(\sim P \cap \sim Q).$$

Dividing both sides by $\mathbf{P}(P \leftrightarrow Q)$ yields

$$1 = \frac{\mathbf{P}(P \cap Q)}{\mathbf{P}(P \leftrightarrow Q)} + \frac{\mathbf{P}(\sim P \cap \sim Q)}{\mathbf{P}(P \leftrightarrow Q)}.$$

When P and Q are logically independent the first term is 1 and the second term is 0. Therefore the first term can be thought of as measuring the degree to which P and Q are logically independent and the second term can be thought of as measuring the degree to which P and Q are logically dependent.

Definition 24 *The logical independence of P and Q is measured by*

$$\begin{aligned} \mathbf{I}_L(P,Q) &= \frac{\mathbf{P}(P \cap Q)}{\mathbf{P}(P \leftrightarrow Q)} \\ &= \frac{\mathbf{P}(P \cap Q)}{\mathbf{P}(P \cap Q) + \mathbf{P}(\sim P \cap \sim Q)} \end{aligned}$$

and the logical dependence of P and Q is measured by

$$\begin{aligned} \mathbf{D}_L(P,Q) &= \frac{\mathbf{P}(\sim P \cap \sim Q)}{\mathbf{P}(P \leftrightarrow Q)} \\ &= \frac{\mathbf{P}(\sim P \cap \sim Q)}{\mathbf{P}(P \cap Q) + \mathbf{P}(\sim P \cap \sim Q)}. \end{aligned}$$

In terms of our notation we have $\mathbf{I}_L(P,Q) + \mathbf{D}_L(P,Q) = 1$.

6 Applications

We apply here probabilities of conditionals, biconditionals and the logical analog of Bayes' theorem to two standard model problems which illustrates many of the concepts we have described.

We stress again that strong conditional and biconditional relationships (logical correlations) do not necessarily imply that there is cause-and-effect. Probabilities of conditionals and biconditionals are simply ways of measuring *association* between two events.

6.1 Application 1: Medical Testing

Example 1 *A patient takes a diagnostic test to check for a disease. Let $D =$ a patient has the disease, $P =$ a patient tests positive for the disease. Suppose the following information is known:*

$\mathbf{P}(D) = 10\%$	$\mathbf{P}(\sim D) = 90\%$

$\mathbf{P}(P\|D) = 95\%$	$\mathbf{P}(\sim P\|D) = 5\%$
$\mathbf{P}(P\| \sim D) = 7\%$	$\mathbf{P}(\sim P\| \sim D) = 93\%$

A standard exercise in total probability and Bayes' theorem would be to use this information to calculate the probability that a patient tests positive and the inverse conditional probabilities. We do so in the next tables. Using the Law of Total Probability

$\mathbf{P}(P) = 16\%$	$\mathbf{P}(\sim P) = 84\%$

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and we can compute using Bayes' theorem

$\mathbf{P}(D\|P) = 60\%$	$\mathbf{P}(D\| \sim P) = 1\%$
$\mathbf{P}(\sim D\| \sim P) = 99\%$	$\mathbf{P}(\sim D\|P) = 40\%$

Now we are interested in comparing and contrasting this standard exercise with the analogous probabilities of material implications and biconditionals. We show the results of the computations in the following tables.

$\mathbf{P}(D \rightarrow P) = 100\%$	$\mathbf{P}(D \rightarrow \sim P) = 91\%$
$\mathbf{P}(\sim D \rightarrow P) = 7\%$	$\mathbf{P}(\sim D \rightarrow \sim P) = 94\%$

$\mathbf{P}(P \rightarrow D) = 94\%$	$\mathbf{P}(\sim P \rightarrow D) = 7\%$
$\mathbf{P}(\sim P \rightarrow \sim D) = 100\%$	$\mathbf{P}(P \rightarrow \sim D) = 91\%$

.

We begin by observing that the probability a patient has the disease, given that the test was positive $\mathbf{P}(D|P)$, is only 60%. This is often surprising to those who first encounter this, and it is explained by the fact that the number of those who have the disease and test positive is only moderately larger than the number of those who don't have the disease and test positive. The surprise is sometimes explained as due to people confusing $\mathbf{P}(D|P)$ with $\mathbf{P}(P|D)$.

We contrast $\mathbf{P}(D|P)$ with $\mathbf{P}(P \rightarrow D) = 94\%$ which would seem to provide the result which people initially expect, except for the fact that $\mathbf{P}(P \rightarrow \sim D) = 91\%$. Apparently, testing positive for the disease implies with about 94% probability that a person has the disease, and it also implies with about 91% probability that a person does not have the disease! One may explain this apparent paradox by recalling that $\mathbf{P}(\varnothing \rightarrow Q) = 1$, for any Q. From here we see that since P is a small set, one is able to infer both a proposition D and its negation $\sim D$ with high probability. We note however that $\mathbf{P}(P \rightarrow D) > \mathbf{P}(P \rightarrow \sim D)$ just as $\mathbf{P}(D|P) > \mathbf{P}(D| \sim P)$.

Arguably, it is the probability of the biconditional, or probable logical correlation, which coincides best with the intuition that people mistakenly apply to $\mathbf{P}(D|P)$ initially. The biconditional probabilities are as follows:

$\mathbf{P}(P \leftrightarrow D) = 93\%$	$\mathbf{P}(P \leftrightarrow \sim D) = 7\%$

.

We may compute their logical correlation coefficients as:

$\boldsymbol{\rho}(P, D) = 0.86$	$\boldsymbol{\rho}(P, \sim D) = -0.86$

We see that testing positive for the disease and actually having the disease have a strong direct logical correlation, as one would expect. Therefore, testing positive and not having the disease, or testing negative and having the disease, have a logical correlation with the same magnitude but the opposite sign.

6.2 Application 2: Manufacturing and Quality Control

Example 2 *A company manufactures items using three machines. 40% of the manufactured items come from Machine 1, 40% from Machine 2 and 20% from Machine 3. Of those items coming from machine 1, 95% work properly, of those items coming from Machine 2, 90% work properly, and of those items coming from Machine 3, 93% work properly.*

Let $M1$, $M2$, and $M3$ denote the events that a manufactured item comes from Machines 1, 2, and 3 respectively. Let W denote the event that a manufactured item works properly. Then we have

$\mathbf{P}(M1) = 40\%$	$\mathbf{P}(\sim M1) = 60\%$
$\mathbf{P}(M2) = 40\%$	$\mathbf{P}(\sim M2) = 60\%$
$\mathbf{P}(M3) = 20\%$	$\mathbf{P}(\sim M3) = 80\%$

and

$\mathbf{P}(W\|M1) = 90\%$	$\mathbf{P}(\sim W\|M1) = 10\%$
$\mathbf{P}(W\|M2) = 95\%$	$\mathbf{P}(\sim W\|M2) = 5\%$
$\mathbf{P}(W\|M3) = 93\%$	$\mathbf{P}(\sim W\|M3) = 7\%$

.

Furthermore we can compute using the Law of Total Probability:

$\mathbf{P}(W) = 93\%$	$\mathbf{P}(\sim W) = 7\%$

and we can compute using Bayes' theorem

Credit	Blame
$\mathbf{P}(M1\|W) = 39\%$	$\mathbf{P}(M1\| \sim W) = 57\%$
$\mathbf{P}(M2\|W) = 41\%$	$\mathbf{P}(M2\| \sim W) = 29\%$
$\mathbf{P}(M3\|W) = 20\%$	$\mathbf{P}(M3\| \sim W) = 19\%$
Total $= 100\%$	Total $= 100\%$

.

At this point a very important but standard textbook exercise is completed. Given the knowledge that the item is working properly we can apportion the credit according to the first column and given that the item is defective we can apportion the blame according to the second column.

While this analysis is standard and correct, we are not entirely satisfied. Which machine correlates most with a working item, (direct logical correlation) and which machine correlates most with a defective item (inverse logical correlation)? To answer this let us begin with the analogs of the above computations for the probability of a conditional.

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$\mathbf{P}(M1 \to W) = 96\%$	$\mathbf{P}(M1 \to \sim W) = 64\%$
$\mathbf{P}(M2 \to W) = 98\%$	$\mathbf{P}(M2 \to \sim W) = 62\%$
$\mathbf{P}(M3 \to W) = 99\%$	$\mathbf{P}(M3 \to \sim W) = 81\%$

We may compute, either directly or using the logical analog of Bayes' theorem the reverse probabilities of conditionals.

Credit	Blame
$\mathbf{P}(W \to M1) = 43\%$	$\mathbf{P}(\sim W \to M1) = 97\%$
$\mathbf{P}(W \to M2) = 45\%$	$\mathbf{P}(\sim W \to M2) = 95\%$
$\mathbf{P}(W \to M3) = 26\%$	$\mathbf{P}(\sim W \to M3) = 94\%$

Although these probabilities are less intuitive, their relative ranking remain the same. That is, we would come to the same decision as to how to rank the machines in terms of credit and blame whether we used conditional probabilities or probabilities of conditionals.

However, it is the *logical correlations*, or the probabilities of biconditionals, that we are most interested in. We compute them here.

Biconditionals	
$\mathbf{P}(W \leftrightarrow M1) = 39\%$	$\mathbf{P}(\sim W \leftrightarrow M1) = 61\%$
$\mathbf{P}(W \leftrightarrow M2) = 43\%$	$\mathbf{P}(\sim W \leftrightarrow M2) = 57\%$
$\mathbf{P}(W \leftrightarrow M3) = 25\%$	$\mathbf{P}(\sim W \leftrightarrow M3) = 75\%$

We can express these results equivalently in terms of their effective correlation coefficients:

Logical Correlations	
$\boldsymbol{\rho}(W, M1) = -0.22$	$\boldsymbol{\rho}(\sim W, M1) = 0.22$
$\boldsymbol{\rho}(W, M2) = -0.14$	$\boldsymbol{\rho}(\sim W, M2) = 0.14$
$\boldsymbol{\rho}(W, M3) = -0.5$	$\boldsymbol{\rho}(\sim W, M3) = 0.5$

The fact that all the logical correlations are negative indicates that for a given machine, it is more likely than not that an item was either defective and produced by it, or working and produced by another machine. This does not speak well for the manufacturer. The logical correlations also lead us to rank the machines in the following order: $M3 < M1 < M2$.

7 Conclusion

Probabilities of conditionals can be used in addition to, or even instead of conditional probabilities in any problem involving Bayes' theorem.

The probability of a biconditional provides a highly intuitive measure of relatedness, or logical correlation, between two events. It measures the degree to which two events are simultaneously true or simultaneously false. One may also define a logical correlation coefficient for two events via the formula $\boldsymbol{\rho}(P, Q) = 2\mathbf{P}(P \leftrightarrow Q) - 1$.

It is our opinion that selections from the topics we have discussed here can be incorporated into undergraduate treatments of probability once the standard material has been developed. Probabilities of conditionals, and especially probabilities of biconditionals, can then provide additional insights into many standard problems.

References

[1] Adams E., "The Logic of Conditionals." Inquiry, 8, (1965), pp. 166-197.

[2] Copi, I. M., *Symbolic Logic*, Macmillan Publishing Co., Inc., New York, 1979

[3] Fulda, Joseph S., Material Implication Revisited, The American Mathematical Monthly, Vol. 96, No.3 (Mar., 1989), pp. 247-250.

[4] Lewis, D., Probabilities of Conditionals and Conditional Probabilities, The Philosophical Review, Vol. 85, No. 3 (Jul., 1976), pp. 297-315.

[5] Milne, P., Bruno de Finetti and the Logic of Conditional Events, Brit. J. Phil. Sci., 48 (1997), 195-232.

[6] Nguyen, Mukaidono, and Kreinovich. Probability of Implication, Logical Version of Bayes' Theorem, and Fuzzy Logic Operations. Proceedings IEEE-FUZZ (2002), Volume 1, p. 530-535.