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Product-free Lambek Calculus is NP-complete

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Abstract

In this paper we prove that the derivability problems for product-free Lambek calculus and product-free Lambek calculus allowing empty premises are NP-complete. Also we introduce a new derivability characterization for these calculi.

Introduction

Lambek calculus L was first introduced in [3]. Lambek calculus uses syntactic types that are built from primitive types using three binary connectives: multiplication, left division, and right division. Natural fragments of Lambek calculus are the product-free Lambek calculus $L(\backslash, /)$, which does not use multiplication, and the unidirectional Lambek calculi, which have only one connective left: a division (left or right).

For the non-associative variant of Lambek calculus the derivability can be checked in polynomial time as shown in [2] (for the product-free fragment of the non-associative Lambek calculus this was proved already in [1]).

In [4] NP-completeness was proved for the derivability problem for full associative Lambek calculus. In [5] there was presented a polynomial algorithm for its unidirectional fragments.

We show that the classical satisfiability problem SAT is polynomial time reducible to the $L(\backslash, /)$ -derivability problem and thus $L(\backslash, /)$ is NP-complete.

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1 Product-free Lambek Calculus

Product-free Lambek calculus $L(\backslash, /)$ can be constructed as follows. Let $\mathbf{P} = \{p_0, p_1, \dots\}$ be a countable set of what we call *primitive types*. Let Tp be the set of *types* constructed from primitive types with two binary connectives $/, \backslash$. We will denote primitive types by small letters (p, q, r, \dots) and types by capital letters (A, B, C, \dots). By capital greek letters ($\Pi, \Gamma, \Delta, \dots$) we will denote finite (possibly empty) sequences of types. Expressions like $\Pi \rightarrow A$, where Π is not empty, are called *sequents*.

Axioms and rules of $L(\backslash, /)$:

$$\begin{array}{l}
 A \rightarrow A, \\
 \frac{\Pi A \rightarrow B}{\Pi \rightarrow (B/A)} (\rightarrow /), \\
 \frac{A \Pi \rightarrow B}{\Pi \rightarrow (A \backslash B)} (\rightarrow \backslash), \\
 \frac{\Phi \rightarrow B \quad \Gamma B \Delta \rightarrow A}{\Gamma \Phi \Delta \rightarrow A} (\text{CUT}), \\
 \frac{\Phi \rightarrow A \quad \Gamma B \Delta \rightarrow C}{\Gamma (B/A) \Phi \Delta \rightarrow C} (/ \rightarrow), \\
 \frac{\Phi \rightarrow A \quad \Gamma B \Delta \rightarrow C}{\Gamma \Phi (A \backslash B) \Delta \rightarrow C} (\backslash \rightarrow),
 \end{array}$$

(Here Γ and Δ can be empty.)

In this paper we will consider two calculi — $L(\backslash, /)$ and $L^*(\backslash, /)$, called product-free Lambek calculus allowing empty premises. In $L^*(\backslash, /)$ we allow the antecedent of a sequent to be empty.

It can be shown that in these calculi every derivable sequent has a cut-free derivation where all instances of the axiom are of the form $p \rightarrow p$ where $p \in \mathbf{P}$.

2 Reduction from SAT

Let $c_1 \wedge \dots \wedge c_m$ be a Boolean formula in conjunctive normal form with clauses $c_1 \dots c_m$ and variables $x_1 \dots x_n$. The reduction maps the formula to a sequent, which is derivable in $L(\backslash, /)$ (and in $L^*(\backslash, /)$) if and only if the formula $c_1 \wedge \dots \wedge c_m$ is satisfiable.

For any Boolean variable x_i let $\neg_0 x_i$ stand for the literal $\neg x_i$ and $\neg_1 x_i$ stand for the literal x_i .

Note that $\langle t_1, \dots, t_n \rangle \in \{0, 1\}^n$ is a satisfying assignment for the Boolean formula $c_1 \wedge \dots \wedge c_m$ if and only if for every $j \leq m$ there exists $i \leq n$ such that the literal $\neg_{t_i} x_i$ appears in the clause c_j (as usual, 1 stands for “true” and 0 stands for “false”).

Let $p_i^j, q_i^j, a_i^j, b_i^j, c_i^j, d_i^j; 0 \leq i \leq n, 0 \leq j \leq m$ be distinct primitive types from \mathbf{P} .

We define the following families of types:

$$\begin{aligned}
G^0 &\equiv (p_0^0 \setminus p_n^0) \\
G^j &\equiv (q_n^j / ((q_0^j \setminus p_0^j) \setminus G^{j-1})) \setminus p_n^j \\
G &\equiv G^m \\
A_i^0 &\equiv (a_i^0 \setminus p_i^0) \\
A_i^j &\equiv (q_i^j / ((b_i^j \setminus a_i^j) \setminus A_i^{j-1})) \setminus p_i^j \\
A_i &\equiv A_i^m \\
C_i^0 &\equiv (c_i^0 \setminus p_i^0) \\
C_i^j &\equiv (q_i^j / ((d_i^j \setminus c_i^j) \setminus C_i^{j-1})) \setminus p_i^j \\
C_i &\equiv C_i^m \\
E_i^0(t) &\equiv p_{i-1}^0 \\
E_i^j(t) &\equiv \begin{cases} q_i^j / (((q_{i-1}^j / E_i^{j-1}(t)) \setminus p_{i-1}^j) \setminus p_i^{j-1}), & \text{if } \neg_t x_i \text{ appears in } c_j \\ q_i^j / (q_{i-1}^j / (E_i^{j-1}(t) \setminus p_i^{j-1})) \setminus p_{i-1}^j, & \text{if } \neg_t x_i \text{ does not appear in } c_j \end{cases} \\
F_i^j(t) &\equiv (E_i^j(t) \setminus p_i^j) \\
F_i(t) &\equiv F_i^m(t) \\
H_i^0 &\equiv p_{i-1}^0 \setminus p_i^0 \\
H_i^j &\equiv ((q_{i-1}^j / (q_i^j / H_i^{j-1})) \setminus p_{i-1}^j) \setminus p_i^j \\
H_i &\equiv H_i^m \\
B_i^0 &\equiv a_i^0 \\
B_i^j &\equiv q_{i-1}^j / (((b_i^j / B_i^{j-1}) \setminus a_i^j) \setminus p_{i-1}^{j-1}) \\
B_i &\equiv B_i^m \setminus p_{i-1}^m \\
D_i^0 &\equiv c_i^0 \\
D_i^j &\equiv q_{i-1}^j / (((d_i^j / D_i^{j-1}) \setminus c_i^j) \setminus p_{i-1}^{j-1}) \\
D_i &\equiv D_i^m \setminus p_{i-1}^m.
\end{aligned}$$

Let Π_i denote the following sequences of types:

$$(F_i(0) / (B_i \setminus A_i)) H_i ((D_i \setminus C_i) \setminus F_i(1)).$$

Theorem 2.1. *The following statements are equivalent:*

1. $c_1 \wedge \dots \wedge c_m$ is satisfiable.
2. $L(\setminus, /) \vdash \Pi_1 \dots \Pi_n \rightarrow G$
3. $L^*(\setminus, /) \vdash \Pi_1 \dots \Pi_n \rightarrow G$.

This theorem will be proven in section 4.

3 Derivability Characterization

Let At be the set of *atoms* or *primitive types with superscripts*, $\{p^{(i)} \mid p \in \mathbf{P}, i \in \mathbb{Z}\}$. Let FS be the free monoid (the set of all finite strings) generated by elements of At . We will denote elements of FS by $\mathbb{A}, \mathbb{B}, \mathbb{C}$ and so on, by ε we will denote the empty string.

Consider two mappings:

$$t : \text{FS} \rightarrow \mathbf{P}, t(\mathbb{A}p^{(i)}) = p; \quad d : \text{FS} \rightarrow \mathbb{Z}, d(\mathbb{A}p^{(i)}) = i.$$

Let $\mathbb{A} \sqsubset \mathbb{B}$ denote that \mathbb{A} is a strict prefix of \mathbb{B} (i.e. there is $\mathbb{C} \neq \varepsilon \in \text{FS}$ such that $\mathbb{B} = \mathbb{A}\mathbb{C}$). We will denote such \mathbb{C} as $\mathbb{A} \setminus \mathbb{B}$. By $\mathbb{A} \sqsubseteq \mathbb{B}$ we will denote that either $\mathbb{A} \sqsubset \mathbb{B}$ or $\mathbb{A} = \mathbb{B}$. We can define in the usual way the following notions: $\min_{\sqsubseteq}, \max_{\sqsubseteq}, \inf_{\sqsubseteq}, \sup_{\sqsubseteq}, [\mathbb{A}, \mathbb{B}]_{\sqsubseteq}$, and $(\mathbb{A}, \mathbb{B}]_{\sqsubseteq}$.

For $\mathbb{A} \in \text{FS}, \mathbb{A} \neq \varepsilon$ let $\mathcal{P}_{\mathbb{A}} = \{\mathbb{B} \mid \mathbb{B} \sqsubseteq \mathbb{A}, \mathbb{B} \neq \varepsilon\}$. The relation \sqsubseteq is a total order on $\mathcal{P}_{\mathbb{A}}$.

Let α be a partial function on $\mathcal{P}_{\mathbb{A}}$. For each such function we can define the following:

$$\begin{aligned} \mathbb{B} <_{\alpha} \mathbb{C} &\Leftrightarrow \exists n \geq 1, \alpha^n(\mathbb{B}) = \mathbb{C}, \\ \mathbb{B} \leq_{\alpha} \mathbb{C} &\Leftrightarrow \mathbb{B} <_{\alpha} \mathbb{C} \vee \mathbb{B} = \mathbb{C}, \\ \mu_{\alpha}^{-}(\mathbb{B}) &= \min_{\sqsubseteq}(\mathbb{B}, \alpha(\mathbb{B})), \\ \mu_{\alpha}^{+}(\mathbb{B}) &= \max_{\sqsubseteq}(\mathbb{B}, \alpha(\mathbb{B})), \\ \mathcal{F}_{\alpha}(\mathbb{B}) &= \{\mathbb{C} \mid \mathbb{C} \leq_{\alpha} \mathbb{B}\}, \\ \nu_{\alpha}^{-}(\mathbb{B}) &= \inf_{\sqsubseteq}(\mathcal{F}_{\alpha}(\mathbb{B})), \\ \nu_{\alpha}^{+}(\mathbb{B}) &= \sup_{\sqsubseteq}(\mathcal{F}_{\alpha}(\mathbb{B})). \end{aligned}$$

Consider two antiendomorphisms $(\cdot)^{\leftarrow}$ and $(\cdot)^{\rightarrow}$ on FS defined by

$$\begin{aligned} (p^{(0)})^{\leftarrow} &= p^{(-1)}, \quad (p^{(0)})^{\rightarrow} = p^{(1)}, \\ (p^{(i)})^{\leftarrow} &= (p^{(i)})^{\rightarrow} = p^{(-i - \text{sgn}(i))}, \text{ for } i \neq 0. \end{aligned}$$

(A function $f: X \rightarrow X$ is an antiendomorphism if $\forall a, b \in X, f(ab) = f(b)f(a)$. In a free monoid it can be defined by its actions on the generators).

Consider $[[\cdot]]: \text{Tp} \rightarrow \text{FS}$, a mapping from Lambek types to elements of the free monoid defined by

$$[[p]] = p^{(0)}, \quad [[(A/B)]] = [[B]]^{\rightarrow}[[A]], \quad [[(A \setminus B)]] = [[B]][[A]]^{\leftarrow}.$$

Let us define φ — the partial function on $\mathcal{P}_{[A]}$ that reflects the structure of the Lambek type A :

$$\varphi(\mathbb{A}) = \begin{cases} \inf_{\sqsubset} \{\mathbb{B} \mid \mathbb{A} \sqsubset \mathbb{B}, |d(\mathbb{B})| = |d(\mathbb{A})| - 1\}, & \text{if } d(\mathbb{A}) > 0; \\ \sup_{\sqsubset} \{\mathbb{B} \mid \mathbb{B} \sqsubset \mathbb{A}, |d(\mathbb{B})| = |d(\mathbb{A})| - 1\}, & \text{if } d(\mathbb{A}) < 0. \end{cases}$$

It can be easily shown that the following facts hold:

1. There is a unique $\mathbb{A}_0 \in \mathcal{P}_{[A]}$ such that $d(\mathbb{A}_0) = 0$.
2. $\varphi(\mathbb{A})$ is defined for every $\mathbb{A} \neq \mathbb{A}_0$.
3. \leq_{φ} is a partial order on $\mathcal{P}_{[A]}$.
4. For every $i \in \mathbb{N}$ such that $i < |d(\mathbb{A})|$ there exists \mathbb{B} such that $|d(\mathbb{B})| = i$ and $\mathbb{A} <_{\varphi} \mathbb{B}$, for instance $\mathbb{A} \leq_{\varphi} \mathbb{A}_0$.
5. If $\mathbb{A} \in [\mu_{\varphi}^{-}(\mathbb{B}), \mu_{\varphi}^{+}(\mathbb{B})]_{\sqsubset}$, then $\mathbb{A} \leq_{\varphi} \mathbb{B}$.

Suppose $\mathbb{A}, \mathbb{B} \in \mathcal{P}_{[A]}$. There exists $\mathbb{C} \in \mathcal{P}_{[A]}$ such that $\mathbb{A} \leq_{\varphi} \mathbb{C}$, $\mathbb{B} \leq_{\varphi} \mathbb{C}$, and for all $\mathbb{C}' \in \mathcal{P}_{[A]}$ such that $\mathbb{A} <_{\varphi} \mathbb{C}'$ and $\mathbb{B} \leq_{\varphi} \mathbb{C}'$, $\mathbb{C} \leq_{\varphi} \mathbb{C}'$. Such \mathbb{C} is called the φ -join of \mathbb{A} and \mathbb{B} .

A set $\mathcal{G} \subset \mathcal{P}_{[A]}$ is called φ -closed if there is no $\mathbb{A} \notin \mathcal{G}$ such that $\varphi(\mathbb{A}) \in \mathcal{G}$. Let $\mathcal{N}_{\mathbb{A}} = \{\mathbb{B} \in \mathcal{P}_{\mathbb{A}} \mid d(\mathbb{B}) = 2i + 1, i \in \mathbb{Z}\}$.

Suppose we have a Lambek sequent $A_1 \dots A_n \rightarrow B$. Let

$$\mathbb{W} = [(\dots (B/A_n) / \dots) / A_1] = [A_1]^{-} \dots [A_n]^{-} [B].$$

Let π be a function on $\mathcal{P}_{\mathbb{W}}$, and ψ be a partial function defined by

$$\psi(\mathbb{A}) = \begin{cases} \pi(\mathbb{A}), & \text{if } \mathbb{A} \in \mathcal{N}_{\mathbb{W}}; \\ \varphi(\mathbb{A}), & \text{if } \mathbb{A} \notin \mathcal{N}_{\mathbb{W}} \text{ and } d(\mathbb{A}) \neq 0. \end{cases}$$

To characterize derivability of the sequent $A_1 \dots A_n \rightarrow B$ we shall use the following conditions, which we call proof conditions.

1. If $\mathbb{A} \in \mathcal{N}_{\mathbb{W}}$, then $\pi(\mathbb{A}) \notin \mathcal{N}_{\mathbb{W}}$ and $\pi^2(\mathbb{A}) = \mathbb{A}$ for all $\mathbb{A} \in \mathcal{P}_{\mathbb{W}}$.
2. $t(\pi(\mathbb{A})) = t(\mathbb{A})$.
3. $\mu_{\pi}^{-}(\mathbb{A}) \sqsubset \mu_{\pi}^{-}(\mathbb{B}) \Rightarrow \mu_{\pi}^{+}(\mathbb{A}) \sqsubset \mu_{\pi}^{-}(\mathbb{B}) \vee \mu_{\pi}^{+}(\mathbb{B}) \sqsubset \mu_{\pi}^{+}(\mathbb{A})$.
4. $\mathbb{A} \in \mathcal{N}_{\mathbb{W}} \Rightarrow \mathbb{A} <_{\psi} \varphi(\mathbb{A})$ or equivalently $\forall \mathbb{A} \in \mathcal{P}_{\mathbb{W}}, \mathcal{F}_{\varphi}(\mathbb{A}) \subset \mathcal{F}_{\psi}(\mathbb{A})$.
5. $\mathbb{A} \notin \mathcal{N}_{\mathbb{W}} \wedge \mathbb{A} \neq \mathbb{A}_0 \Rightarrow \exists \mathbb{B} (\mathbb{B} <_{\psi} \mathbb{A} \wedge \mathbb{B} \not<_{\varphi} \mathbb{A})$.

We will call $\mathcal{G} \subset \mathcal{P}_{\mathbb{W}}$ π -closed if for all $\mathbb{A} \in \mathcal{G}, \pi(\mathbb{A}) \in \mathcal{G}$. It is readily seen that if π satisfies proof conditions (1) and (3), then for every $\mathbb{A} \in \mathcal{N}_{\mathbb{W}}, [\mu_{\pi}^{-}(\mathbb{A}), \mu_{\pi}^{+}(\mathbb{A})]_{\sqsubset}$ and $\mathcal{P}_{\mathbb{W}} \setminus [\mu_{\pi}^{-}(\mathbb{A}), \mu_{\pi}^{+}(\mathbb{A})]_{\sqsubset}$ are π -closed. If π satisfies proof conditions (1) and (2), then \mathcal{G} cannot be π -closed if for given $p \in \mathbf{P}$ there are odd number of $\mathbb{A} \in \mathcal{G}$ such that $t(\mathbb{A}) = p$.

Lemma 3.1. *If π satisfies proof condition (4), then \leq_{ψ} is a partial order on $\mathcal{P}_{\mathbb{W}}$.*

Proof. Reflexivity and transitivity directly follow from the definition of \leq_{ψ} .

Now lets prove antisymmetry. Suppose that there are $\mathbb{B}, \mathbb{C} \in \mathcal{P}_{\mathbb{W}}$ such that $\mathbb{B} \leq_{\psi} \mathbb{C}$ and $\mathbb{C} \leq_{\psi} \mathbb{B}$. If $\mathbb{B} \neq \mathbb{C}$ then there is $i > 0$ such that $\psi^i(\mathbb{B}) = \mathbb{B}$ and thus for all $j > 0, \psi^j(\mathbb{B})$ is defined.

If π satisfy proof condition (4) then if $\mathbb{A} \leq_{\varphi} \mathbb{B}$ then $\mathbb{A} \leq_{\psi} \mathbb{B}$. There is $\mathbb{A}_0 \in \mathcal{P}_{\mathbb{W}}$ such that $d(\mathbb{A}_0) = 0$, and for all $\mathbb{A} \in \mathcal{P}_{\mathbb{W}}, \mathbb{A} \leq_{\varphi} \mathbb{A}_0$. This means that $\mathbb{B} \leq_{\varphi} \mathbb{A}_0$ and thus $\mathbb{B} \leq_{\psi} \mathbb{A}_0$. The function ψ is not defined on \mathbb{A}_0 . Contradiction. \square

Lemma 3.2. $L^*(\setminus, /) \vdash A_1 \dots A_n \rightarrow B$ if and only if there exists π satisfying proof conditions (1)-(4).

$L(\setminus, /) \vdash A_1 \dots A_n \rightarrow B$ if and only if $n > 0$ and there exists π satisfying proof conditions (1)-(5).

Proof. Suppose that $L^{(*)}(\setminus, /) \vdash A_1 \dots A_n \rightarrow B$. Induction on the length of the derivation.

If the sequent is of the form $p \rightarrow p$, then $\mathbb{W} = p^1 p^0, \mathcal{P}_{\mathbb{W}} = \{p^1, p^1 p^0\}, \mathcal{N}_{\mathbb{W}} = \{p^1\}$ and π such that $\pi(p^1) = p^1 p^0$ and $\pi(p^1 p^0) = p^1$ satisfies all necessary proof conditions.

Suppose that the last step in the derivation of $A_1 \dots A_n \rightarrow B$ was an application of the rule $(\rightarrow /)$. Then $B = (C/D), L^{(*)}(\setminus, /) \vdash A_1 \dots A_n D \rightarrow C$ and for $\mathcal{P}_{\mathbb{W}'}$, where $\mathbb{W}' = \llbracket A_1 \rrbracket^{\rightarrow} \dots \llbracket A_n \rrbracket^{\rightarrow} \llbracket D \rrbracket^{\rightarrow} \llbracket C \rrbracket^{\rightarrow}$ there exists π' satisfying all necessary proof conditions. But in this case $\mathbb{W} = \mathbb{W}'$, and therefore this π' works for the sequent $A_1 \dots A_n \rightarrow B$ too.

Suppose that the last step in the derivation of $A_1 \dots A_n \rightarrow B$ was an application of the rule $(\rightarrow \setminus)$. Then $B = (C \setminus D), \mathbb{W} = \llbracket A_1 \rrbracket^{\rightarrow} \dots \llbracket A_n \rrbracket^{\rightarrow} \llbracket D \rrbracket \llbracket C \rrbracket^{\leftarrow}, L^{(*)}(\setminus, /) \vdash C A_1 \dots A_n \rightarrow D$, and by induction hypothesis for $\mathcal{P}_{\mathbb{W}'}$, where

$$\mathbb{W}' = \llbracket C \rrbracket^{\rightarrow} \llbracket A_1 \rrbracket^{\rightarrow} \dots \llbracket A_n \rrbracket^{\rightarrow} \llbracket D \rrbracket$$

there exists π' satisfying all necessary proof conditions. Consider

$$\beta : \mathcal{P}_{\mathbb{W}'} \rightarrow \mathcal{P}_{\mathbb{W}}, \beta(\mathbb{A}) = \begin{cases} \llbracket A_1 \rrbracket^{\rightarrow} \dots \llbracket A_n \rrbracket^{\rightarrow} \llbracket D \rrbracket (\mathbb{A}^{\rightarrow^{-1}})^{\leftarrow}, & \text{if } \mathbb{A} \sqsubseteq \llbracket C \rrbracket^{\rightarrow}; \\ \llbracket C \rrbracket^{\rightarrow} \setminus \mathbb{A}, & \text{if } \llbracket C \rrbracket^{\rightarrow} \sqsubset \mathbb{A}. \end{cases}$$

Let $\pi(\mathbb{A}) = \beta(\pi'(\beta^{-1}(\mathbb{A})))$. Such π satisfies all necessary proof conditions.

Suppose that the last step in the derivation of $A_1 \dots A_n \rightarrow B$ was an application of the rule ($/ \rightarrow$). Then $A_1 \dots A_n \rightarrow B$ is of the form

$$C_1 \dots (C_i/D) D_1 \dots D_k C_{i+1} \dots C_l \rightarrow C$$

so that $L^{(*)}(\backslash, /) \vdash C_1 \dots C_l \rightarrow C$ and $L^{(*)}(\backslash, /) \vdash D_1 \dots D_k \rightarrow D$.

Consider $\mathbb{W}' = \llbracket C_1 \rrbracket^{\rightarrow} \dots \llbracket C_l \rrbracket^{\rightarrow} \llbracket C \rrbracket$ and $\mathbb{W}'' = \llbracket D_1 \rrbracket^{\rightarrow} \dots \llbracket D_k \rrbracket^{\rightarrow} \llbracket D \rrbracket$. By induction hypothesis there are π' and π'' — functions on $\mathcal{P}_{\mathbb{W}'}$ and $\mathcal{P}_{\mathbb{W}''}$ respectively, satisfying all necessary proof conditions.

Let $\mathbb{C} = \llbracket C_1 \rrbracket^{\rightarrow} \dots \llbracket C_i \rrbracket^{\rightarrow}$ and $\mathbb{D} = \llbracket D_1 \rrbracket^{\rightarrow} \dots \llbracket D_k \rrbracket^{\rightarrow}$. Consider

$$\beta': \mathcal{P}_{\mathbb{W}'} \rightarrow \mathcal{P}_{\mathbb{W}}, \beta'(\mathbb{A}) = \begin{cases} \mathbb{A} & , \text{ if } \mathbb{A} \sqsubseteq \mathbb{C}; \\ \mathbb{C}(\llbracket D \rrbracket^{\rightarrow})^{\rightarrow} \mathbb{D}(\mathbb{C} \setminus \mathbb{A}) & , \text{ if } \mathbb{C} \sqsubseteq \mathbb{A}; \end{cases}$$

$$\text{and } \beta'': \mathcal{P}_{\mathbb{W}''} \rightarrow \mathcal{P}_{\mathbb{W}}, \beta''(\mathbb{A}) = \begin{cases} \mathbb{C}(\llbracket D \rrbracket^{\rightarrow})^{\rightarrow} \mathbb{A} & , \text{ if } \mathbb{A} \sqsubseteq \mathbb{D}; \\ \mathbb{C}((\mathbb{D} \setminus \mathbb{A})^{\rightarrow})^{\rightarrow} & , \text{ if } \mathbb{D} \sqsubseteq \mathbb{A}; \end{cases}$$

$$\text{Let } \pi(\mathbb{A}) = \begin{cases} \beta'(\pi'(\beta'^{-1}(\mathbb{A}))) & , \text{ if } \mathbb{A} \sqsubseteq \mathbb{C} \text{ or } \mathbb{C}(\llbracket D \rrbracket^{\rightarrow})^{\rightarrow} \mathbb{D} \sqsubseteq \mathbb{A}; \\ \beta''(\pi''(\beta''^{-1}(\mathbb{A}))) & , \text{ if } \mathbb{C} \sqsubseteq \mathbb{A} \sqsubseteq \mathbb{C}(\llbracket D \rrbracket^{\rightarrow})^{\rightarrow} \mathbb{D}; \end{cases}$$

Such π satisfies all necessary proof conditions.

Suppose that the last step in the derivation of $A_1 \dots A_n \rightarrow B$ was an application of the rule ($\backslash \rightarrow$). Then $A_1 \dots A_n \rightarrow B$ is of the form

$$C_1 \dots C_{i-1} D_1 \dots D_k (D \setminus C_i) \dots C_l \rightarrow C$$

so that $L^{(*)}(\backslash, /) \vdash C_1 \dots C_l \rightarrow C$ and $L^{(*)}(\backslash, /) \vdash D_1 \dots D_k \rightarrow D$.

Consider $\mathbb{W}' = \llbracket C_1 \rrbracket^{\rightarrow} \dots \llbracket C_l \rrbracket^{\rightarrow} \llbracket C \rrbracket$ and $\mathbb{W}'' = \llbracket D_1 \rrbracket^{\rightarrow} \dots \llbracket D_k \rrbracket^{\rightarrow} \llbracket D \rrbracket$. By induction hypothesis there are π' and π'' — functions on $\mathcal{P}_{\mathbb{W}'}$ and $\mathcal{P}_{\mathbb{W}''}$ respectively, satisfying all necessary proof conditions.

Let $\mathbb{C} = \llbracket C_1 \rrbracket^{\rightarrow} \dots \llbracket C_{i-1} \rrbracket^{\rightarrow}$ and $\mathbb{D} = \llbracket D_1 \rrbracket^{\rightarrow} \dots \llbracket D_k \rrbracket^{\rightarrow}$. Consider

$$\beta': \mathcal{P}_{\mathbb{W}'} \rightarrow \mathcal{P}_{\mathbb{W}}, \beta'(\mathbb{A}) = \begin{cases} \mathbb{A} & , \text{ if } \mathbb{A} \sqsubseteq \mathbb{C}; \\ \mathbb{C}\mathbb{D}(\llbracket D \rrbracket^{\leftarrow})^{\rightarrow}(\mathbb{C} \setminus \mathbb{A}) & , \text{ if } \mathbb{C} \sqsubseteq \mathbb{A}; \end{cases}$$

$$\text{and } \beta'': \mathcal{P}_{\mathbb{W}''} \rightarrow \mathcal{P}_{\mathbb{W}}, \beta''(\mathbb{A}) = \begin{cases} \mathbb{C}\mathbb{A} & , \text{ if } \mathbb{A} \sqsubseteq \mathbb{D}; \\ \mathbb{C}\mathbb{D}((\mathbb{D} \setminus \mathbb{A})^{\leftarrow})^{\rightarrow} & , \text{ if } \mathbb{D} \sqsubseteq \mathbb{A}; \end{cases}$$

$$\text{Let } \pi(\mathbb{A}) = \begin{cases} \beta'(\pi'(\beta'^{-1}(\mathbb{A}))) & , \text{ if } \mathbb{A} \sqsubseteq \mathbb{C} \text{ or } \mathbb{C}\mathbb{D}(\llbracket D \rrbracket^{\leftarrow})^{\rightarrow} \sqsubseteq \mathbb{A}; \\ \beta''(\pi''(\beta''^{-1}(\mathbb{A}))) & , \text{ if } \mathbb{C} \sqsubseteq \mathbb{A} \sqsubseteq \mathbb{C}\mathbb{D}(\llbracket D \rrbracket^{\leftarrow})^{\rightarrow}; \end{cases}$$

Such π satisfies all necessary proof conditions.

Thus we proved one side of the lemma.

Now suppose that for the given sequent $A_1 \dots A_n \rightarrow B$, for $\mathcal{P}_{\mathbb{W}}$ there exists π satisfying proof conditions (1)-(4).

Induction on total number of connectives in the sequent.

If there are no connectives, the sequent is of the form $p_1 \dots p_n \rightarrow q$ and $\mathbb{W} = p_1^{(1)} \dots p_n^{(1)} q^{(0)}$. The function π satisfies proof condition (1), thus $|\mathcal{N}_{\mathbb{W}}| = |\mathcal{P}_{\mathbb{W}} \setminus \mathcal{N}_{\mathbb{W}}|$. This means that $n = 1$. So $\mathcal{P}_{\mathbb{W}} = \{p_1^{(1)}, p_1^{(1)} q^{(0)}\}$ and $\mathcal{N}_{\mathbb{W}} = \{p_1^{(1)}\}$. The function π satisfies proof condition (2), therefore $p_1 = q$, and the sequent is an axiom.

If $B = (C/D)$, then the sequent $A_1 \dots A_n D \rightarrow C$ has less connectives than the original sequent, but $\llbracket A_1 \rrbracket^{\rightarrow} \dots \llbracket A_n \rrbracket^{\rightarrow} \llbracket D \rrbracket^{\rightarrow} \llbracket C \rrbracket^{\rightarrow} = \mathbb{W}$, and therefore π satisfies all necessary proof conditions for the new sequent. By inductual hypothesis this means that $L^*(\setminus, /) \vdash A_1 \dots A_n D \rightarrow C$ and by applying the rule $(\rightarrow /)$ we get $L^*(\setminus, /) \vdash A_1 \dots A_n \rightarrow B$.

If $B = (C \setminus D)$, then the sequent $CA_1 \dots A_n \rightarrow D$ has less connectives than the original sequent.

Let $\mathbb{W}' = \llbracket C \rrbracket^{\rightarrow} \llbracket A_1 \rrbracket^{\rightarrow} \dots \llbracket A_n \rrbracket^{\rightarrow} \llbracket D \rrbracket^{\rightarrow}$. Consider

$$\beta : \mathcal{P}_{\mathbb{W}'} \rightarrow \mathcal{P}_{\mathbb{W}}, \beta(\mathbb{A}) = \begin{cases} \llbracket A_1 \rrbracket^{\rightarrow} \dots \llbracket A_n \rrbracket^{\rightarrow} \llbracket D \rrbracket^{\rightarrow} (\mathbb{A}^{\rightarrow^{-1}})^{\leftarrow} & , \text{ if } \mathbb{A} \sqsubseteq \llbracket C \rrbracket^{\rightarrow}; \\ \llbracket C \rrbracket^{\rightarrow} \setminus \mathbb{A} & , \text{ if } \llbracket C \rrbracket^{\rightarrow} \sqsubset \mathbb{A}; \end{cases}$$

Let $\pi'(\mathbb{A}) = \beta^{-1}(\pi(\beta(\mathbb{A})))$. Such π' satisfies all necessary proof conditions. By induction hypothesis this means that $L^*(\setminus, /) \vdash CA_1 \dots A_n \rightarrow D$, and by applying the rule $(\rightarrow \setminus)$ we get $L^*(\setminus, /) \vdash A_1 \dots A_n \rightarrow B$.

Now we can only consider sequents of the form $A_1 \dots A_n \rightarrow p$. This means that $\mathbb{W} = \llbracket A_1 \rrbracket^{\rightarrow} \dots \llbracket A_n \rrbracket^{\rightarrow} p^{(0)}$. Let $\mathbb{A}_1 = \pi(\mathbb{W})$. Since π satisfies proof condition (4) and ψ is not defined on \mathbb{W} , $\varphi(\mathbb{A}_1) = \mathbb{W}$. Therefore $d(\mathbb{A}_1) = 1$. Let $\mathbb{A}_1 = \llbracket A_1 \rrbracket^{\rightarrow} \dots \llbracket A_{i-1} \rrbracket^{\rightarrow} \mathbb{A}''$.

Suppose that $A_i = (C/D)$. This means that $\llbracket A_i \rrbracket^{\rightarrow} = \llbracket C \rrbracket^{\rightarrow} (\llbracket D \rrbracket^{\rightarrow})^{\rightarrow}$. There exists a unique $\mathbb{D}_0 \in \mathcal{P}_{\llbracket D \rrbracket^{\rightarrow}}$ such that $d(\mathbb{D}_0) = 0$. Consider

$$\mathbb{A}_2 = \llbracket A_1 \rrbracket^{\rightarrow} \dots \llbracket A_{i-1} \rrbracket^{\rightarrow} \llbracket C \rrbracket^{\rightarrow} (\mathbb{D}_0^{\rightarrow})^{\rightarrow} \in \mathcal{P}_{\mathbb{W}}.$$

$d(\mathbb{A}_2) = -2$, $\varphi(\mathbb{A}_2) = \mathbb{A}_1$, $\psi^2(\mathbb{A}_2) = \mathbb{W}$, and there is no $\mathbb{B} \in \mathcal{P}_{\mathbb{W}}$ such that $\mathbb{A}_2 \sqsubset \mathbb{B}$ and $\varphi(\mathbb{B}) = \mathbb{A}_1$.

Also $\mathcal{F}_{\psi}(\mathbb{A}_2) = [\nu_{\psi}^{-}(\mathbb{A}_2), \nu_{\psi}^{+}(\mathbb{A}_2)]_{\sqsubset}$. Let us prove this statement. Consider $\mathbb{B} \in [\nu_{\psi}^{-}(\mathbb{A}), \nu_{\psi}^{+}(\mathbb{A})]_{\sqsubset}$, $\mathbb{B} \neq \mathbb{A}_2$. There exists $\mathbb{C} \in \mathcal{F}_{\psi}(\mathbb{A}_2)$ such that $\mathbb{B} \sqsubseteq \mathbb{C}$ and $\psi(\mathbb{C}) \sqsubseteq \mathbb{B}$. If $\mathbb{C} \in \mathcal{P}_{\mathbb{W}} \setminus \mathcal{N}_{\mathbb{W}}$, then $\mathbb{B} \in [\mu_{\varphi}^{-}(\mathbb{C}), \mu_{\varphi}^{+}(\mathbb{C})]_{\sqsubset}$, and thus $\mathbb{B} \leq_{\psi} \varphi(\mathbb{C}) \leq_{\psi} \mathbb{A}_2$. If $\mathbb{C} \in \mathcal{P}_{\mathbb{W}}$, then $\pi(\mathbb{B}) \in [\mu_{\varphi}^{-}(\mathbb{C}), \mu_{\varphi}^{+}(\mathbb{C})]_{\sqsubset} \subset [\nu_{\psi}^{-}(\mathbb{A}), \nu_{\psi}^{+}(\mathbb{A})]_{\sqsubset}$. Since $\mathbb{B} <_{\psi} \mathbb{A}_1$, this means that $\mathbb{B} <_{\psi} \mathbb{A}_2$.

Let $\nu_\psi^+(\mathbb{A}_2) = \llbracket A_1 \rrbracket^\rightarrow \dots \llbracket A_l \rrbracket^\rightarrow$. Consider

$$\mathbb{W}' = \llbracket A_1 \rrbracket^\rightarrow \dots \llbracket A_{i-1} \rrbracket^\rightarrow \llbracket C \rrbracket^\rightarrow \llbracket A_{l+1} \rrbracket^\rightarrow \dots \llbracket A_n \rrbracket^\rightarrow p^{(0)}$$

and $\mathbb{W}'' = \llbracket A_{i+1} \rrbracket^\rightarrow \dots \llbracket A_l \rrbracket^\rightarrow \llbracket D \rrbracket^\rightarrow$. Let $\mathbb{C} = \llbracket A_1 \rrbracket^\rightarrow \dots \llbracket A_{i-1} \rrbracket^\rightarrow \llbracket C \rrbracket^\rightarrow$ and $\mathbb{D} = \llbracket A_{i+1} \rrbracket^\rightarrow \dots \llbracket A_l \rrbracket^\rightarrow$. Consider

$$\beta': \mathcal{P}_{\mathbb{W}'} \rightarrow \mathcal{P}_{\mathbb{W}}, \beta'(\mathbb{A}) = \begin{cases} \mathbb{A} & , \text{ if } \mathbb{A} \sqsubseteq \mathbb{C}; \\ \mathbb{C}(\llbracket D \rrbracket^\rightarrow)^\rightarrow \mathbb{D}(\mathbb{C} \setminus \mathbb{A}) & , \text{ if } \mathbb{C} \sqsubset \mathbb{A}; \end{cases}$$

and $\beta'': \mathcal{P}_{\mathbb{W}''} \rightarrow \mathcal{P}_{\mathbb{W}}, \beta''(\mathbb{A}) = \begin{cases} \mathbb{C}(\llbracket D \rrbracket^\rightarrow)^\rightarrow \mathbb{A} & , \text{ if } \mathbb{A} \sqsubseteq \mathbb{D}; \\ \mathbb{C}((\mathbb{D} \setminus \mathbb{A})^\rightarrow)^\rightarrow & , \text{ if } \mathbb{D} \sqsubset \mathbb{A}; \end{cases}$

Functions $\pi' = \beta'^{-1}\pi\beta'$ and $\pi'' = \beta''^{-1}\pi\beta''$ satisfy all necessary proof conditions. By induction hypothesis this means that

$$L^*(\setminus, /) \vdash A_1 \dots A_{i-1} C A_{l+1} \dots A_n \rightarrow p$$

and $L^*(\setminus, /) \vdash A_{i+1} \dots A_l \rightarrow D$. By applying the rule $(/ \rightarrow)$ we get

$$L^*(\setminus, /) \vdash A_1 \dots A_n \rightarrow p.$$

Suppose that $A_i = (D \setminus C)$. This means that $\llbracket A_i \rrbracket^\rightarrow = (\llbracket D \rrbracket^\leftarrow)^\rightarrow \llbracket C \rrbracket^\rightarrow$. There exists a unique $\mathbb{D}_0 \in \mathcal{P}_{\llbracket D \rrbracket^\leftarrow}$ such that $d(\mathbb{D}_0) = 0$. Consider

$$\mathbb{A}_2 = \llbracket A_1 \rrbracket^\rightarrow \dots \llbracket A_{i-1} \rrbracket^\rightarrow (\mathbb{D}_0^\leftarrow)^\rightarrow \in \mathcal{P}_{\mathbb{W}}.$$

$d(\mathbb{A}_2) = 2$, $\varphi(\mathbb{A}_2) = \mathbb{A}_1$, $\psi^2(\mathbb{A}_2) = \mathbb{W}$, and there is no $\mathbb{B} \in \mathcal{P}_{\mathbb{W}}$ such that $\mathbb{B} \sqsubset \mathbb{A}_2$ and $\varphi(\mathbb{B}) = \mathbb{A}_1$. Like in previous case we can say that $\mathcal{F}_\psi(\mathbb{A}_2) = [\nu_\psi^-(\mathbb{A}_2), \nu_\psi^+(\mathbb{A}_2)]_\sqsubset$.

Let $\nu_\psi^-(\mathbb{A}_2) = \llbracket A_1 \rrbracket^\rightarrow \dots \llbracket A_l \rrbracket^\rightarrow q^{(j)}$ for some $q^{(j)} \in \text{At}$. Consider

$$\mathbb{W}' = \llbracket A_1 \rrbracket^\rightarrow \dots \llbracket A_l \rrbracket^\rightarrow \llbracket C \rrbracket^\rightarrow \llbracket A_{i+1} \rrbracket^\rightarrow \dots \llbracket A_n \rrbracket^\rightarrow p^{(0)}$$

and $\mathbb{W}'' = \llbracket A_{l+1} \rrbracket^\rightarrow \dots \llbracket A_{i-1} \rrbracket^\rightarrow \llbracket D \rrbracket^\rightarrow$. Let $\mathbb{C} = \llbracket A_1 \rrbracket^\rightarrow \dots \llbracket A_l \rrbracket^\rightarrow$ and $\mathbb{D} = \llbracket A_{l+1} \rrbracket^\rightarrow \dots \llbracket A_{i-1} \rrbracket^\rightarrow$. Consider

$$\beta': \mathcal{P}_{\mathbb{W}'} \rightarrow \mathcal{P}_{\mathbb{W}}, \beta'(\mathbb{A}) = \begin{cases} \mathbb{A} & , \text{ if } \mathbb{A} \sqsubseteq \mathbb{C}; \\ \mathbb{C}\mathbb{D}(\llbracket D \rrbracket^\leftarrow)^\rightarrow (\mathbb{C} \setminus \mathbb{A}) & , \text{ if } \mathbb{C} \sqsubset \mathbb{A}; \end{cases}$$

and $\beta'': \mathcal{P}_{\mathbb{W}''} \rightarrow \mathcal{P}_{\mathbb{W}}, \beta''(\mathbb{A}) = \begin{cases} \mathbb{C}\mathbb{A} & , \text{ if } \mathbb{A} \sqsubseteq \mathbb{D}; \\ \mathbb{C}\mathbb{D}((\mathbb{D} \setminus \mathbb{A})^\leftarrow)^\rightarrow & , \text{ if } \mathbb{D} \sqsubset \mathbb{A}; \end{cases}$

Functions $\pi' = \beta'^{-1}\pi\beta'$ and $\pi'' = \beta''^{-1}\pi\beta''$ satisfy all necessary proof conditions. By induction hypothesis this means that

$$L^*(\setminus, /) \vdash A_1 \dots A_l C A_{i+1} \dots A_n \rightarrow p$$

and $L^*(\setminus, /) \vdash A_{l+1} \dots A_{i-1} \rightarrow D$. By applying the rule $(\setminus \rightarrow)$ we get

$$L^*(\setminus, /) \vdash A_1 \dots A_n \rightarrow p.$$

Thus we fully proved the lemma for $L^*(\setminus, /)$.

Suppose we have π that satisfies proof conditions (1)-(5). We already proved that $L^*(\setminus, /) \vdash A_1 \dots A_n \rightarrow B$. The construction given provides us with possible last step of the derivation. Hence we can construct a derivation. If π satisfies proof condition (5) this means that there will be no \mathbb{A}_2 such that $\mathcal{F}_\psi(\mathbb{A}_2) = \mathcal{F}_\varphi(\mathbb{A}_2)$, and thus there will be no steps in derivation that require sequents of the form $\rightarrow A$. Thus $L(\setminus, /) \vdash A_1 \dots A_n \rightarrow B$.

The lemma is fully proven. \square

Lemma 3.3. *Suppose we have two sequents $A_1 \dots A_n \rightarrow B$ and $C_1 \dots C_m \rightarrow D$. $L^*(\setminus, /) \vdash A_1 \dots A_n \rightarrow B$. Let $\mathbb{W} = \llbracket A_1 \rrbracket^{\rightarrow} \dots \llbracket A_n \rrbracket^{\rightarrow} \llbracket B \rrbracket$ and $\mathbb{W}' = \llbracket C_1 \rrbracket^{\rightarrow} \dots \llbracket C_m \rrbracket^{\rightarrow} \llbracket D \rrbracket$. Suppose that there is a mapping $\beta: \mathcal{P}_{\mathbb{W}'} \rightarrow \mathcal{P}_{\mathbb{W}}$ such that the following holds:*

1. β is injective,
2. For all $\mathbb{A} \in \mathcal{P}_{\mathbb{W}'}$, $t(\beta(\mathbb{A})) = t(\mathbb{A})$, $d(\beta(\mathbb{A})) = d(\mathbb{A})$,
3. For all $\mathbb{A}, \mathbb{B} \in \mathcal{P}_{\mathbb{W}'}$, $\mathbb{A} \sqsubset \mathbb{B}$ if and only if $\beta(\mathbb{A}) \sqsubset \beta(\mathbb{B})$.

Let $\mathcal{G} = \{\mathbb{A} \in \mathcal{P}_{\mathbb{W}} \mid \neg \exists \mathbb{B} \in \mathcal{P}_{\mathbb{W}'}, \beta(\mathbb{B}) = \mathbb{A}\}$. If \mathcal{G} is π -closed and φ -closed, then $L^*(\setminus, /) \vdash C_1 \dots C_m \rightarrow D$.

Proof. Let φ' be φ for $\mathcal{P}_{\mathbb{W}'}$. Since \mathcal{G} is φ -closed, for all $\mathbb{A} \in \mathcal{P}_{\mathbb{W}'}$, $\varphi'(\mathbb{A}) = \beta^{-1}(\varphi(\beta(\mathbb{A})))$. Since \mathcal{G} is π -closed, π' defined as $\beta^{-1}\pi\beta$ is defined on all $\mathcal{P}_{\mathbb{W}'}$ and satisfies proof conditions (1)-(4). Therefore by lemma 3.2

$$L^*(\setminus, /) \vdash C_1 \dots C_m \rightarrow D.$$

\square

Lemma 3.4. *Suppose $\mathbb{A} \in \mathcal{N}_{\mathbb{W}}$ and \mathbb{B} is the φ -join of \mathbb{A} and $\pi(\mathbb{A})$. If π satisfies proof conditions (1)-(4), then $\mathbb{B} \notin \mathcal{N}_{\mathbb{W}}$.*

Proof. Suppose that $\mathbb{B}_i \in \mathcal{N}_{\mathbb{W}}$. There is \mathbb{B}_1 such that $\mathbb{A} <_\psi \mathbb{B}_1$ and $\varphi(\mathbb{B}_1) = \mathbb{B}$. There is \mathbb{B}_2 such that $\pi(\mathbb{A}) <_\psi \mathbb{B}_2$ and $\varphi(\mathbb{B}_2) = \mathbb{B}$. This means that $\mathbb{A} \leq_\psi \mathbb{B}_1$ and $\pi(\mathbb{A}) \leq_\psi \mathbb{B}_2$ and since $\psi(\mathbb{A}) = \pi(\mathbb{A})$, either $\mathbb{B}_1 <_\psi \mathbb{B}_2$ or $\mathbb{B}_2 <_\psi \mathbb{B}_1$. But since $\psi(\mathbb{B}_1) = \psi(\mathbb{B}_2) = \mathbb{B}$, we get $\mathbb{B} <_\psi \mathbb{B}$. Contradiction. \square

4 Proof of the Main Theorem

Consider $\mathbb{W} = \llbracket F_1(t_1) \rrbracket^{\rightarrow} \dots \llbracket F_n(t_n) \rrbracket^{\rightarrow} \llbracket G \rrbracket$.

If a primitive type occurs in the sequent $F_1(t_1) \dots F_n(t_n) \rightarrow G$ it occurs exactly twice. Let \mathbb{P}_i^{j+} be the element of $\mathcal{N}_{\mathbb{W}}$ such that $t(\mathbb{P}_i^{j+}) = p_i^j$ and \mathbb{P}_i^{j-} be the element of $\mathcal{P}_{\mathbb{W}} \setminus \mathcal{N}_{\mathbb{W}}$ such that $t(\mathbb{P}_i^{j-}) = p_i^j$. In the same way we define \mathbb{Q}_i^{j+} and \mathbb{Q}_i^{j-} .

The function π can only satisfy proof conditions (1) and (2) if for every i and j , $\pi(\mathbb{P}_i^{j+}) = \mathbb{P}_i^{j-}$ and $\pi(\mathbb{Q}_i^{j+}) = \mathbb{Q}_i^{j-}$. If it is so then π satisfies proof conditions (3) and (5).

The following facts hold:

1. $d(P_i^{m-}) = 0$.
2. If $\neg_{t_i} x_i$ does not appear in clause c_j , then $\varphi^3(\mathbb{P}_i^{j-1+}) = \varphi^2(\mathbb{Q}_i^{j-}) = \varphi(\mathbb{Q}_{i-1}^{j+}) = \mathbb{P}_{i-1}^{j-}$.
3. If $\neg_{t_i} x_i$ appears in clause c_j , then $\varphi^3(\mathbb{Q}_{i-1}^{j+}) = \varphi^2(\mathbb{P}_{i-1}^{j-}) = \varphi(\mathbb{P}_i^{j-1+}) = \mathbb{Q}_i^{j-}$.
4. $\varphi^4(\mathbb{Q}_0^{j-}) = \varphi^3(\mathbb{P}_0^{j+}) = \varphi^2(\mathbb{P}_n^{j-1-}) = \varphi(\mathbb{Q}_n^{j+}) = \mathbb{P}_n^{j-}$.

Lemma 4.1. *For every $0 < i \leq n$ and $j > 0$, $\mathbb{P}_i^{j-1+} <_{\psi} \mathbb{Q}_i^{j-}$.*

Proof. For $i = n$ this is true, because

$$\psi^3(\mathbb{P}_n^{j-1+}) = \pi\varphi\pi(\mathbb{P}_n^{j-1+}) = \pi\varphi(\mathbb{P}_n^{j-1-}) = \pi(\mathbb{Q}_n^{j+}) = \mathbb{Q}_n^{j-}.$$

Now suppose that for all $i' > i$ this was already proven. There are four possibilities:

1. If $\neg_{t_{i+1}} x_{i+1}$ does not appear in clauses c_{j-1} and c_j , then $\psi^2(\mathbb{P}_i^{j-1+}) = \mathbb{P}_{i+1}^{j-1+}$, $\psi^2(\mathbb{Q}_{i+1}^{j-}) = \mathbb{Q}_i^{j-}$, and $\mathbb{P}_{i+1}^{j-1+} <_{\psi} \mathbb{Q}_{i+1}^{j-}$. Thus $\mathbb{P}_i^{j-1+} <_{\psi} \mathbb{Q}_i^{j-}$.
2. If $\neg_{t_{i+1}} x_{i+1}$ does not appear in the clause c_{j-1} , but appears in c_j , then $\psi^3(\mathbb{P}_i^{j-1+}) = \pi\varphi\pi(\mathbb{P}_i^{j-1+}) = \pi\varphi(\mathbb{P}_i^{j-1-}) = \pi(\mathbb{Q}_i^{j+}) = \mathbb{Q}_i^{j-}$.
3. If $\neg_{t_{i+1}} x_{i+1}$ appears in the clause c_{j-1} , but does not appear in c_j , then $\psi^2(\mathbb{P}_i^{j-1+}) = \mathbb{P}_{i+1}^{j-2+}$, $\psi^2(\mathbb{Q}_{i+1}^{j-}) = \mathbb{Q}_i^{j-}$, $\varphi(\mathbb{Q}_{i+1}^{j-1+}) = \mathbb{P}_{i+1}^{j-1+}$, $\mathbb{P}_{i+1}^{j-2+} <_{\psi} \mathbb{Q}_{i+1}^{j-1-}$, and $\mathbb{P}_{i+1}^{j-1+} <_{\psi} \mathbb{Q}_{i+1}^{j-}$. Thus $\mathbb{P}_i^{j-1+} <_{\psi} \mathbb{Q}_i^{j-}$.
4. If $\neg_{t_{i+1}} x_{i+1}$ appears in both clauses c_{j-1} and c_j , then $\psi^2(\mathbb{P}_i^{j-1+}) = \mathbb{P}_{i+1}^{j-2+}$, $\psi^2(\mathbb{Q}_{i+1}^{j-1+}) = \mathbb{Q}_i^{j-}$, and $\mathbb{P}_{i+1}^{j-2+} <_{\psi} \mathbb{Q}_{i+1}^{j-1-}$. Thus $\mathbb{P}_i^{j-1+} <_{\psi} \mathbb{Q}_i^{j-}$.

□

Lemma 4.2. For every $0 \leq i < n$ and $j > 0$, $\mathbb{Q}_i^{j+} <_\psi \mathbb{P}_i^{j-}$.

Proof. For $i = 0$ this is true, because

$$\psi^3(\mathbb{Q}_0^{j+}) = \pi\varphi\pi(\mathbb{Q}_0^{j+}) = \pi\varphi(\mathbb{Q}_0^{j-}) = \pi(\mathbb{P}_0^{j+}) = \mathbb{P}_0^{j-}.$$

Now suppose that for all $i' < i$ this was already proven. There are four possibilities:

1. If $\neg_{t_i} x_i$ does not appear in clauses c_{j+1} and c_j , then $\psi^2(\mathbb{Q}_i^{j+}) = \mathbb{Q}_{i-1}^{j+}$, $\psi^2(\mathbb{P}_{i-1}^{j-}) = \mathbb{P}_{i-1}^{j-}$, and $\mathbb{Q}_{i-1}^{j+} <_\psi \mathbb{P}_{i-1}^{j-}$. Thus $\mathbb{Q}_i^{j+} <_\psi \mathbb{P}_i^{j-}$.
2. If $\neg_{t_i} x_i$ does not appear in the clause c_{j+1} , but appears in c_j , then $\psi^3(\mathbb{Q}_i^{j+}) = \pi\varphi\pi(\mathbb{Q}_i^{j+}) = \pi\varphi(\mathbb{Q}_i^{j-}) = \pi(\mathbb{P}_i^{j+}) = \mathbb{P}_i^{j-}$.
3. If $\neg_{t_i} x_i$ appears in the clause c_{j+1} , but does not appear in c_j , then $\psi^2(\mathbb{Q}_i^{j+}) = \mathbb{Q}_{i-1}^{j+}$, $\psi^2(\mathbb{P}_{i-1}^{j+1-}) = \mathbb{P}_{i-1}^{j-}$, $\varphi(\mathbb{P}_{i-1}^{j+}) = \mathbb{Q}_{i-1}^{j+1+}$, $\mathbb{Q}_{i-1}^{j+} <_\psi \mathbb{P}_{i-1}^{j-}$, and $\mathbb{Q}_{i-1}^{j+1+} <_\psi \mathbb{P}_{i-1}^{j+1-}$. Thus $\mathbb{Q}_i^{j+} <_\psi \mathbb{P}_i^{j-}$.
4. If $\neg_{t_i} x_i$ appears in both clauses c_{j+1} and c_j , then $\psi^2(\mathbb{Q}_i^{j+}) = \mathbb{Q}_{i-1}^{j+1+}$, $\psi^2(\mathbb{P}_{i-1}^{j+1-}) = \mathbb{P}_{i-1}^{j-}$, and $\mathbb{Q}_{i-1}^{j+1+} <_\psi \mathbb{P}_{i-1}^{j+1-}$. Thus $\mathbb{Q}_i^{j+} <_\psi \mathbb{P}_i^{j-}$.

□

From lemmas 4.1 and 4.2 we can conclude that if $i > 0$ and $j \leq j'$ then $\mathbb{P}_i^{j+} <_\psi \mathbb{P}_i^{j'+}$.

Lemma 4.3. If $i < i'$, then $\mathbb{P}_i^{j+} <_\psi \mathbb{P}_{i'}^{j+}$.

Proof. If $\neg_{t_{i+1}} x_{i+1}$ appears in clause c_j , then $\psi^2(\mathbb{P}_i^{j+}) = \mathbb{P}_{i+1}^{j-1+}$ and $\mathbb{P}_{i+1}^{j-1+} <_\psi \mathbb{P}_{i+1}^{j+}$. If $\neg_{t_{i+1}} x_{i+1}$ appears in clause c_{j+1} , then $\psi(\mathbb{P}_i^{j+1-}) = \mathbb{P}_{i+1}^{j+}$ and $\mathbb{P}_i^{j-} <_\psi \mathbb{P}_i^{j+1+}$. If neither of this is the case, then $\psi^2(\mathbb{P}_i^{j+}) = \mathbb{P}_{i+1}^{j+}$. This means that $\mathbb{P}_i^{j+} <_\psi \mathbb{P}_{i+1}^{j+}$ and thus $\mathbb{P}_i^{j+} <_\psi \mathbb{P}_{i'}^{j+}$. □

Lemma 4.4. $\langle t_1, \dots, t_n \rangle$ is a satisfying assignment for $c_1 \wedge \dots \wedge c_m$ if and only if $L^*(\setminus, /) \vdash F_1(t_1) \dots F_n(t_n) \rightarrow G$ and if and only if

$$L(\setminus, /) \vdash F_1(t_1) \dots F_n(t_n) \rightarrow G.$$

Proof. Suppose that $\langle t_1, \dots, t_n \rangle$ is a satisfying assignment for $c_1 \wedge \dots \wedge c_m$. In view of lemmas 4.1 and 4.2 the only members of $\mathcal{N}_{\mathbb{W}}$ for which we have not proved that π satisfies proof condition (4) are \mathbb{P}_0^{j+} .

We now prove that for every j , $\mathbb{P}_0^{j+} <_\psi \varphi(\mathbb{P}_0^{j+}) = \mathbb{P}_n^{j-1-}$. There exist i such that $\neg_{t_i} x_i$ appears in clause c_j . This means that $\psi(\mathbb{P}_{i-1}^{j-}) = \mathbb{P}_i^{j-1+}$ and by

lemma 4.3 $\mathbb{P}_0^{j+} <_\psi \mathbb{P}_i^{j+}$ and $\mathbb{P}_i^{j-1+} <_\psi \mathbb{P}_n^{j-1+}$. Thus $\mathbb{P}_0^{j+} <_\psi \varphi(\mathbb{P}_0^{j+}) = \mathbb{P}_n^{j-1-}$ and by lemma 3.2 we can now say that $L^*(\setminus, /) \vdash F_1(t_1) \dots F_n(t_n) \rightarrow G$.

Suppose that $\langle t_1, \dots, t_n \rangle$ is not a satisfying assignment for $c_1 \wedge \dots \wedge c_m$. There exists j such that no $\neg_{t_i} x_i$ appear in clause c_j . This means that for $i \leq n$, $\psi^{2i}(\mathbb{Q}_n^{j+}) = \mathbb{Q}_{n-i}^{j+}$, $\psi(\mathbb{P}_n^{j-1-}) = \mathbb{Q}_n^{j+}$, and $\psi(\mathbb{Q}_0^{j-}) = \mathbb{P}_0^{j+}$. Thus $\mathbb{P}_n^{j-1-} <_\psi \mathbb{P}_0^{j+}$. This means that π cannot satisfy proof condition (4). Thus by lemma 2.1 $L^*(\setminus, /) \not\vdash F_1(t_1) \dots F_n(t_n) \rightarrow G$.

Since π satisfies proof condition (5),

$$L(\setminus, /) \vdash F_1(t_1) \dots F_n(t_n) \rightarrow G \Leftrightarrow L^*(\setminus, /) \vdash F_1(t_1) \dots F_n(t_n) \rightarrow G$$

and thus the lemma is fully proven. \square

Lemma 4.5. $L(\setminus, /) \vdash \Pi_i \rightarrow F_i(t_i)$, where $t_i \in \{0, 1\}$.

Lemma 4.6. *If the formula $c_1 \wedge \dots \wedge c_m$ is satisfiable, then $L(\setminus, /) \vdash \Pi_1 \dots \Pi_n \rightarrow G$.*

Proof. Suppose $\langle t_1, \dots, t_n \rangle$ is a satisfying assignment for $c_1 \wedge \dots \wedge c_m$. According to Lemma 4.4 $L(\setminus, /) \vdash F_1(t_1) \dots F_n(t_n) \rightarrow G$. Now we apply Lemma 4.5 and the cut rule n times. \square

Suppose $L^*(\setminus, /) \vdash \Pi_1 \dots \Pi_n \rightarrow G$. Consider

$$\begin{aligned} \mathbb{W} = & \llbracket F_1(0)/(B_1 \setminus A_1) \rrbracket^{\rightarrow} \llbracket H_1 \rrbracket^{\rightarrow} \llbracket (D_1 \setminus C_1) \setminus F_1(1) \rrbracket^{\rightarrow} \dots \\ & \dots \llbracket F_n(0)/(B_n \setminus A_n) \rrbracket^{\rightarrow} \llbracket H_n \rrbracket^{\rightarrow} \llbracket (D_n \setminus C_n) \setminus F_n(1) \rrbracket^{\rightarrow} \llbracket G \rrbracket \end{aligned}$$

By Lemma 3.2 for $\mathcal{P}_{\mathbb{W}}$ there exists π satisfying proof conditions (1)-(4). Consider the following abbreviations:

$$\begin{aligned} \mathbb{F}_i^0 &= \llbracket F_1(0)/(B_1 \setminus A_1) \rrbracket^{\rightarrow} \llbracket H_1 \rrbracket^{\rightarrow} \llbracket (D_1 \setminus C_1) \setminus F_1(1) \rrbracket^{\rightarrow} \dots \llbracket F_i(0) \rrbracket^{\rightarrow} \\ \mathbb{A}_i &= \mathbb{F}_i^0(\llbracket A_i \rrbracket^{\leftarrow})^{\leftarrow} \\ \mathbb{B}_i &= \mathbb{A}_i(\llbracket B_i \rrbracket^{\leftarrow})^{\leftarrow} \\ \mathbb{H}_i &= \mathbb{B}_i \llbracket H_i \rrbracket^{\rightarrow} \\ \mathbb{C}_i &= \mathbb{H}_i(\llbracket C_i \rrbracket^{\leftarrow})^{\leftarrow} \\ \mathbb{D}_i &= \mathbb{C}_i(\llbracket D_i \rrbracket^{\leftarrow})^{\leftarrow} \\ \mathbb{F}_i^1 &= \mathbb{D}_i \llbracket F_i(1) \rrbracket^{\rightarrow} \end{aligned}$$

Lemma 4.7. *If $L^*(\setminus, /) \vdash \Pi_1 \dots \Pi_i F_{i+1}(t_{i+1}) \dots F_n(t_n) \rightarrow G$, then there is $t_i \in \{0, 1\}$ such that $L^*(\setminus, /) \vdash \Pi_1 \dots \Pi_{i-1} F_i(t_i) \dots F_n(t_n) \rightarrow G$*

Proof. Consider $\mathbb{W}' = \mathbb{F}_i^1 \mathbb{W}''$, where $\mathbb{W}'' = \llbracket F_{i+1}(t_{i+1}) \rrbracket \rightarrow \dots \llbracket F_n(t_n) \rrbracket \rightarrow \llbracket G \rrbracket$. By Lemma 3.2 for $\mathcal{P}_{\mathbb{W}'}$ there exists π satisfying proof conditions (1)-(5).

Let $\mathbb{W}'_0 = \mathbb{F}_{i-1}^1 \llbracket F_i(0) \rrbracket \rightarrow \mathbb{W}''$ and $\mathbb{W}'_1 = \mathbb{F}_{i-1}^1 \llbracket F_i(1) \rrbracket \rightarrow \mathbb{W}''$.

For each j there are only two elements of $\mathcal{P}_{\mathbb{W}'}$ such that $t(\mathbb{A}) = a_i^j$, two elements such that $t(\mathbb{A}) = b_i^j$, two elements such that $t(\mathbb{A}) = c_i^j$, and two elements such that $t(\mathbb{A}) = d_i^j$. This means that these pairs of elements are π -closed.

For each j there are six elements of $\mathcal{P}_{\mathbb{W}'}$ such that $t(\mathbb{A}) = p_i^0$. Let us denote them by $\mathbb{P}_1, \dots, \mathbb{P}_6$ so that $\mathbb{P}_1 \sqsubset \dots \sqsubset \mathbb{P}_6$. The following holds:

$$\mathbb{F}_{i-1}^1 \sqsubset \mathbb{P}_1 \sqsubseteq \mathbb{F}_i^0 \sqsubset \mathbb{P}_2 \sqsubseteq \mathbb{A}_i \sqsubset \mathbb{B}_i \sqsubset \mathbb{P}_3 \sqsubseteq \mathbb{H}_i \sqsubset \mathbb{P}_4 \sqsubseteq \mathbb{C}_i \sqsubset \mathbb{D}_i \sqsubset \mathbb{P}_5 \sqsubseteq \mathbb{F}_i^1 \sqsubset \mathbb{P}_6.$$

$\{\mathbb{P}_1, \dots, \mathbb{P}_6\}$ is π -closed. $\mathbb{P}_1, \mathbb{P}_3, \mathbb{P}_5 \in \mathcal{N}_{\mathbb{W}'}$. $[\mathbb{P}_1, \mathbb{P}_2]_{\sqsubset}, [\mathbb{P}_3, \mathbb{P}_6]_{\sqsubset}$, and $[\mathbb{P}_4, \mathbb{P}_5]_{\sqsubset}$ cannot be π -closed, therefore there are only two possibilities: either $\pi(\mathbb{P}_1) = \mathbb{P}_4$, $\pi(\mathbb{P}_3) = \mathbb{P}_2$, and $\pi(\mathbb{P}_5) = \mathbb{P}_6$, or $\pi(\mathbb{P}_1) = \mathbb{P}_6$, $\pi(\mathbb{P}_3) = \mathbb{P}_4$, and $\pi(\mathbb{P}_5) = \mathbb{P}_2$.

Suppose that $\pi(\mathbb{P}_1) = \mathbb{P}_4$, $\pi(\mathbb{P}_3) = \mathbb{P}_2$, and $\pi(\mathbb{P}_5) = \mathbb{P}_6$. Since $[\mathbb{P}_1, \mathbb{P}_4]_{\sqsubset}$ is π -closed, in $(\mathbb{F}_{i-1}^1, \mathbb{D}_i]_{\sqsubset}$ the only elements for which we had not determined $\pi(\mathbb{A})$ are elements in $(\mathbb{F}_{i-1}^1, \mathbb{F}_i^0]_{\sqsubset}$ and in $(\mathbb{C}_i, \mathbb{D}_i]_{\sqsubset}$ with $t(\mathbb{A}) = p_{i-1}^j$ and with $t(\mathbb{A}) = q_{i-1}^j$. Notice that $t(\mathbb{D}_i) = p_{i-1}^m$ and $\mathbb{D}_i \in \mathcal{N}_{\mathbb{W}'}$.

If $i = 1$, then there are only two variants for $\pi(\mathbb{D}_i)$: one is $p_0^{m(l)}$ and the other one is $\mathbb{D}_1 p_0^{m(l)}$, where $l = 2$ or $l = 4$. Therefore, since the φ -join of \mathbb{D}_1 and $\mathbb{D}_1 p_0^{m(l)}$ is $\mathbb{F}_1^1 \in \mathcal{N}_{\mathbb{W}'}$, $\pi(\mathbb{D}_1) = p_0^{m(l)}$ and $[p_0^{m(l)}, \mathbb{D}_1]_{\sqsubset}$ is π -closed.

If $i > 1$, then there are four variants for $\pi(\mathbb{D}_i)$: $\mathbb{F}_{i-1}^1 p_{i-1}^{m(l)}$, $\mathbb{D}_i p_{i-1}^{m(l)}$, where $l = 2$ or $l = 4$, $\mathbb{H}_{i-1} p_{i-1}^{m(2)}$, and $\mathbb{F}_{i-1}^0 p_{i-1}^{m(-2)}$. The second variant is ruled out. If $\pi(\mathbb{D}_i) = \mathbb{H}_{i-1} p_{i-1}^{m(2)}$, then $\pi(\mathbb{D}_{i-1}) = \mathbb{D}_{i-1} p_{i-2}^{m(l)}$, where $l = 2$ or $l = 4$, and the φ -join of \mathbb{D}_{i-1} and $\mathbb{D}_{i-1} p_{i-2}^{m(l)}$ is $\mathbb{F}_{i-1}^1 \in \mathcal{N}_{\mathbb{W}'}$. If $\pi(\mathbb{D}_i) = \mathbb{F}_{i-1}^0 p_{i-1}^{m(-2)}$ then since the segment $(\mathbb{F}_{i-1}^0, \mathbb{D}_i]_{\sqsubset}$ is φ -closed and π -closed, $\mathbb{G} \not\prec_{\psi} \mathbb{F}_{i-1}^0 p_{i-1}^{m(-2)}$ for all $\mathbb{G} \notin (\mathbb{F}_{i-1}^0, \mathbb{D}_i]_{\sqsubset}$. But $\psi^2(\mathbb{D}_i) = \varphi(\pi(\mathbb{D}_i)) = \varphi(\mathbb{F}_{i-1}^0 p_{i-1}^{m(-2)}) = \mathbb{F}_{i-1}^0 \notin (\mathbb{F}_{i-1}^0, \mathbb{D}_i]_{\sqsubset}$. Therefore $\mathbb{D}_i \not\prec_{\psi} \varphi(\mathbb{D}_i) = \mathbb{H}_i p_i^{m(2)}$ and proof condition (4) is not satisfied. Therefore $\pi(\mathbb{D}_i) = \mathbb{F}_{i-1}^1 p_{i-1}^{m(l)}$ and $(\mathbb{F}_{i-1}^1, \mathbb{D}_i]_{\sqsubset}$ is π -closed.

Therefore since $(\mathbb{F}_{i-1}^1, \mathbb{D}_i]_{\sqsubset}$ is π -closed and φ -closed, by Lemma 3.3 for \mathbb{W}'_1 there is π' satisfying proof conditions (1)-(4) and

$$L^*(\setminus, /) \vdash \Pi_1 \dots \Pi_{i-1} F_i(1) \dots F_n(t_n) \rightarrow G.$$

Suppose that $\pi(\mathbb{P}_1) = \mathbb{P}_6$, $\pi(\mathbb{P}_3) = \mathbb{P}_4$, and $\pi(\mathbb{P}_5) = \mathbb{P}_2$.

Here in $(\mathbb{F}_i^0, \mathbb{F}_i^1]_{\sqsubset}$ the only elements for which we had not determined $\pi(\mathbb{A})$ are elements in $(\mathbb{D}_i, \mathbb{F}_i^1]_{\sqsubset}$ and in $(\mathbb{F}_i^0, \mathbb{A}_i]_{\sqsubset}$ with $t(\mathbb{A}) = p_{i+1}^j$ and with $t(\mathbb{A}) = q_{i+1}^j$. Let $\mathbb{E} = \mathbb{F}_i^0 p_{i+1}^{m(-2)} \in \mathcal{P}_{\mathbb{W}'}$.

There are only two variants for $\pi(\mathbb{E})$: one is \mathbb{F}_i^0 and the other one is \mathbb{F}_i^1 . The φ -join of \mathbb{E} and \mathbb{F}_i^0 is $\mathbb{F}_i^0 \in \mathcal{N}_{\mathbb{W}}$. Therefore $\pi(\mathbb{E}) = \mathbb{F}_i^1$ and $(\mathbb{F}_i^0, \mathbb{F}_i^1]_{\sqsubset}$ is π -closed.

Therefore since $(\mathbb{F}_i^0, \mathbb{F}_i^1]_{\sqsubset}$ is π -closed and φ -closed, by Lemma 3.3 for \mathbb{W}'_0 there is π' satisfying proof conditions (1)-(4) and

$$L^*(\backslash, /) \vdash \Pi_1 \dots \Pi_{i-1} F_i(0) \dots F_n(t_n) \rightarrow G.$$

□

Lemma 4.8. *If $L^*(\backslash, /) \vdash \Pi_1 \dots \Pi_n \rightarrow G$, then the formula $c_1 \wedge \dots \wedge c_m$ is satisfiable.*

Proof. Applying n times Lemma 4.7, we get that there exists $\langle t_1, \dots, t_n \rangle \in \{0, 1\}^n$ such that $L^*(\backslash, /) \vdash F_1(t_1) \dots F_n(t_n) \rightarrow G$. By Lemma 4.4 this means that $\langle t_1, \dots, t_n \rangle$ is a satisfying assignment for $c_1 \wedge \dots \wedge c_m$. □

Since for all sequents $L(\backslash, /) \vdash \Pi \rightarrow A \Rightarrow L^*(\backslash, /) \vdash \Pi \rightarrow A$, Lemma 4.6 and Lemma 4.8 together give us Theorem 2.1.

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References

- [1] E. Aarts and K. Trautwein, Non-associative Lambek categorial grammar in polynomial time, *Mathematical logic Quarterly* **41** (1995) pp. 476–484.
- [2] Ph. de Groote, The non-associative Lambek calculus with product in polynomial time, in: *Automated Reasoning with Analytic Tableaux and Related Methods*, (N. V. Murray, ed.), LLNC vol. **1617**, Springer (1999), pp. 128–139.
- [3] J. Lambek, The mathematics of sentence structure, *American Mathematical Monthly* **65** (3) (1958) pp. 154–170.
- [4] M. Pentus, Lambek calculus is NP-complete, *Theoretical Computer Science* **357**, no. 1–3 (2006) pp. 186–201.
- [5] Y. Savateev, Lambek grammars with one division are decidable in polynomial time, in: *Computer Science — Theory and Applications*, (E.A. Hirsch et al. eds.), LLNC vol. **5010**, Springer (2008), pp. 273–282.