2010

TR-2010003: The Impossibility of Definitive Solutions for Some Games

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April 22, 2010

Abstract

In his dissertation of 1950, Nash based his concept of solution to a game on the principles that "a rational prediction should be unique, that the players should be able to deduce and make use of it." In this paper, we address the issue of when such definitive solutions are possible.

We assume player rationality at least as strong as Aumann's rationality. By formalizing Nash's reasoning, we show that any justified definitive solution to a game is a Nash equilibrium, hence games without Nash equilibria cannot have definitive solutions under any notion of rationality. However, each strategic game with Nash equilibria admits a justified definitive solution under some notion of rationality. For Aumann's rationality, no game with two or more Nash equilibria can have a definitive solution whereas some games with a unique Nash equilibrium have definitive solutions and some do not.

1 Introduction

Some classical games, such as Prisoner's Dilemma or Centipede (cf. [10]), have definitive solutions which follow logically from game description and plausible principles of rationality. These solutions are justified by rigorous reasoning involving an epistemic notion of knowledge[1]. Here is a quote from Nash's dissertation [9][2] which raises the issue of a deductive approach to solving games:

[1]Such reasoning could in principle be carried out in an appropriate formal system of the logic of knowledge. By the same token, the Pythagorean theorem, which is usually proven rigorously but informally, could be completely formalized and derived in an axiomatic geometry.

[2]We are indebted to Adam Brandenburger for this quote.

We proceed by investigating the question: what would be a rational prediction of the behavior to be expected of rational[ly] playing the game in question? By using the principles that a rational prediction should be unique, that the players should be able to deduce and make use of it, and that such knowledge on the part of each player of what to expect the others to do should not lead him to act out of conformity with the prediction, one is led to the concept of a solution defined before.

Another quote from [9] explains this issue even further:

... we need to assume the players know the full structure of the game in order to be able to deduce the prediction for themselves.

The main goal of this paper is to investigate when "a unique rational prediction" that "the players should be able to deduce and make use of" is possible. Such definitive solutions exist for **extensive** games: by Aumann's Rationality Theorem ([2]), each generic[3] extensive game with the standard game-theoretical assumptions of **common knowledge of the game and rationality** (*CKGR*) has a unique Aumann-rational solution: the backward induction solution. Moreover, this solution logically follows from the game rules: [7] provides an example showing in full detail how to derive a definitive solution to a perfect information game from game description and plausible principles of rationality.

We consider games from the point of view of epistemic logic – a well-established mathematical foundation of reasoning involving notions such as knowledge. For the reader's sake, the exposition is not formal, but can be completely formalized in an appropriate class of logical languages.

2 Definitions, assumptions, preliminary remarks

We consider games[4] with n players $1, 2, \ldots, n$. A strategy profile

$$\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$$

is a collection of strategies σ_i for players $i = 1, 2, \ldots, n$. Each strategy profile σ uniquely determines the *outcome* in which each move is made according to σ. We assume that everyone who knows the game can calculate i's payoff as determined by σ.

A strategy profile σ is a **Nash equilibrium** if, given strategies of the other players, no player can profitably deviate (cf. [10] for rigorous definitions).

We assume that the rules of the game and player rationality can be described in some formal mathematical language which includes an appropriate amount of logic and let

GAME RULES

[3] A game is generic if for any given player, payoffs of different outcomes are all different.

[4] Either in strategic or extensive form.

be such a description. In principle, *GAME RULES* may be an infinite set of assumptions, but for simplicity we assume that *GAME RULES* is one, possibly very long, sentence which contains a comprehensive game description.

The use of **knowledge operators**[5] (cf. [6])

$$\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_n$$

for players $1, 2, \ldots, n$ in this paper is mostly typographical and is intended to make informal reasoning about knowledge more visual. We assume **the knowledge of the game**, i.e., *each player knows the rules of the game*:

$$\mathbf{K}_i[\mathit{GAME\ RULES}] \quad \text{for each } i = 1, 2, \ldots, n.$$

An informal account of Aumann's notion of rationality can be found in ([2]). Aumann states that for a rational player i,

> *there is no strategy that i knows would have yielded him a conditional payoff ... larger than that which in fact he gets.*

A formal logical account of Aumann's rationality can be found, e.g., in [5, 7] and we adopt this approach. Strategy σ_i for player i is **Aumann-rational** if there is no other strategy σ_i' which i knows to strictly dominate σ_i on all other players' strategies deemed possible by i. Player i is Aumann-rational at the strategy profile $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ if σ_i is Aumann-rational for i at σ. Technically, rationality of player i is a proposition r_i which either holds or does not hold at a strategy profile $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$.

There are stronger notions of rationality, including refinement methods (cf. [10]). By **rationality**, we mean a predicate r_i which is at least as strong as Aumann's rationality at any strategy profile $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$:

$$\mathit{Rationality} \quad \Rightarrow \quad \mathit{Aumann's\ rationality.}$$

Naturally, it is assumed to be commonly known that a rational player i does not choose irrational strategies, hence for a game with rational players, outcomes marked as irrational by at least one player are impossible.

As a convenient formalization feature, we assume that each strategy σ_i is completely specified by a corresponding logical sentence S_i stating that

$$\textit{player } i \textit{ has committed to strategy } \sigma_i.$$

We assume that sentences S_i and S_i' corresponding to different strategies σ_i and σ_i' are known to be incompatible: each player knows that *GAME RULES* yields that S_i and S_i' cannot occur together:

$$\mathbf{K}_i[\mathit{GAME\ RULES} \rightarrow \neg(S_i \wedge S_i')] \quad \text{for each } i = 1, 2, \ldots, n.$$

[5]We assume the standard models of knowledge: Aumann structures or S5 Kripke models, cf. [6].

A strategy profile $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ is a **definitive solution** of the game if it logically follows from the description of the game and rationality, i.e.,

$$GAME\ RULES \rightarrow (S_1 \wedge S_2 \wedge \ldots \wedge S_n) \tag{1}$$

for sentences $S_1, S_2 \ldots S_n$ corresponding to $\sigma_1, \sigma_2, \ldots, \sigma_n$. Since different strategy profiles are incompatible, the definitive solution, if it exists, is unique.

A definitive solution $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ is **justified** if each player knows that the choice of $\sigma_1, \sigma_2, \ldots, \sigma_n$ logically follows from the description of the game. We formalize the justified solution requirement by assuming

$$\mathbf{K}_i[GAME\ RULES \rightarrow (S_1 \wedge S_2 \wedge \ldots \wedge S_n)] \quad \text{for each } i = 1, 2, \ldots, n. \tag{2}$$

As yet another example, consider the following **War and Peace Dilemma**, *WPD*, introduced in [1].

> *Imagine two neighboring countries: a big powerful B, and a small S. Each player can choose to wage war or keep the peace. The best outcome for both countries is peace. However, if both countries wage war, B wins easily and S loses everything, which is the second-best outcome for B and the worst for S. In situation $(war_B, peace_S)$, B loses internationally, which is the second-best outcome for S. In $(peace_B, war_S)$, B's government loses national support, which is the worst outcome for B and the second-worst for S.*

The ordinal payoff matrix of this game is then

	war_S	$peace_S$
war_B	**2**,**0**	**1**,**2**
$peace_B$	**0**,**1**	**3**,**3** .

There is one Nash equilibrium,

$$\{peace_B, peace_S\}. \tag{3}$$

Let us assume Aumann's rationality and *CKGR*. We claim that strategy profile (3) is the justified definitive solution to *WPD*. Indeed, S has a dominant strategy $peace_S$ and as a rational player, has to commit to this strategy. This is known to B, since B knows the game and is aware of S's rationality. Therefore, as a rational player, B chooses $peace_B$. This reasoning can be carried out by any intelligent player (cf. Section 5.1). Hence both players know that the solution $\{peace_B, peace_S\}$ logically follows from the game description (which includes *CKGR*).

3 Formalizing Nash reasoning

Theorem 1 *Any justified definitive solution is a Nash equilibrium.*

Proof. Let $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be a justified definitive solution of the game, and let sentences $S_1, S_2, \ldots, S_n$ correspond to strategies $\sigma_1, \sigma_2, \ldots, \sigma_n$. Suppose σ is not a Nash equilibrium, hence for some player i, the choice of his strategy in σ is less preferable to some other choice given the other player's strategies. Without loss of generality, assume that $i = 1$. By the justification assumption,

$$\mathbf{K}_1[\mathit{GAME\ RULES} \rightarrow (S_1 \wedge S_2 \wedge \ldots \wedge S_n)].$$

By standard epistemic reasoning,

$$\mathbf{K}_1(\mathit{GAME\ RULES}) \rightarrow \mathbf{K_1}(S_1 \wedge S_2 \wedge \ldots \wedge S_n)$$

and

$$\mathbf{K}_1(\mathit{GAME\ RULES}) \rightarrow \mathbf{K_1}(S_1) \wedge \mathbf{K}_1(S_2) \wedge \ldots \wedge \mathbf{K}_1(S_n).$$

Since the game is known to each player,

$$\mathbf{K}_1(\mathit{GAME\ RULES})$$

holds, hence player 1 knows the strategies of all other players:

$$\mathbf{K_1}(S_1) \wedge \mathbf{K}_1(S_2) \wedge \ldots \wedge \mathbf{K}_1(S_n).$$

Since σ is not a Nash profile, for player 1 there is a strategy σ_1' such that the strategy profile $\{\sigma_1', \sigma_2, \ldots, \sigma_n\}$ is, for 1, strictly preferable to $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. We claim that the choice of strategy σ_1 for player 1 is not Aumann-rational and hence cannot be rational in any stronger sense. Indeed, player 1 knows that σ_1' is his possible choice, since he knows the game and all his choices. Moreover, 1 knows the unique choices by other players

$$\mathbf{K}_1(S_2) \wedge \ldots \wedge \mathbf{K}_1(S_n).$$

Moreover, 1 knows his payoffs for outcomes $\{\sigma_1', \sigma_2, \ldots, \sigma_n\}$ and $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ and knows that the former is strictly higher. Therefore, 1 knows that his strategy σ_1' yields a higher payoff than his strategy σ_1 **for all other players' strategies deemed possible by** 1. So the choice of σ_1 by 1 is not Aumann-rational. □

Corollary 1 *If a game has no Nash equilibria, this game does not have a definitive justified solution.*

As an example of why knowledge of epistemic conditions of the game is important in Theorem 1, consider a version of the War and Peace Dilemma *WPD* in which players follow knowledge-based rationality *KBR* ([1]) that uses Aumann's rationality to delete strictly dominated strategies and then applies Harsanyi's Maximin Postulate to make a definitive choice. Assume that the payoff matrix is mutually known but players are not aware of each other's rationality. Then S chooses $peace_S$ as his dominant strategy. Since B considers both moves by S possible, Aumann's rationality alone does not provide a definitive solution. Then B should follow the maximin strategy, hence choosing war_B. The resultant strategy profile

$$\{war_B, peace_S\} \tag{4}$$

is a justified definitive solution of this game:

$$\mathbf{K}_i[\mathit{GAME\ RULES} \rightarrow war_B \wedge peace_S] \quad i \in \{B, S\}.$$

However, (4) is not a Nash equilibrium. There is no contradiction with Theorem 1, since none of the players know the epistemic conditions of the game in full, e.g., B does not know that S is rational:

$$\neg\mathbf{K}_i[\mathit{GAME\ RULES}] \quad i \in \{B, S\}.$$

As a result, each player does not know his opponent's choice:

$$\neg\mathbf{K}_S[war_B] \quad \text{and} \quad \neg\mathbf{K}_B[peace_S].$$

4 Converse of Theorem 1

The converse of Theorem 1 does not necessarily hold **for Aumann's rationality**. In particular, the following game

$$\begin{pmatrix} \mathbf{1,2} & \mathbf{1,0} & \mathbf{0,1} \\ \\ \mathbf{0,1} & \mathbf{0,2} & \mathbf{1,0} \end{pmatrix}$$

has a unique Nash equilibrium **(1**,**2)**, but no definitive solution within the scope of Aumann's rationality, even if the game and rationality are commonly known. Indeed, each strategy in this game is Aumann-rational and hence cannot be ruled out. However, the Nash equilibrium **(1**,**2)** is a definitive solution of a game with the same payoff matrix for **an appropriate extension** of Aumann's rationality, cf. Section 4.1.

4.1 Bullet Nash rationality

In this section, we will show that a non-uniform version of converse Theorem 1 holds for strategic games.

Let G be a strategic game with n players, and $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ a (not necessarily unique) Nash equilibrium of G. A **Bullet Nash rationality**[6] corresponding to σ is a predicate r_i that holds at σ and does not hold at any other strategy profile. As introduced, Bullet Nash rationality is a technical notion which, however, has a reasonable description for games in which players' moves are known to the others. In such a game, Bullet Nash rationality associated with σ yields that player i has to choose σ_i once he knows all other players' choices in σ and does not have rational choices at all other nodes.

Lemma 1 *Bullet Nash rationality yields Aumann's rationality.*

Proof. It suffices to check that each player is Aumann-rational at $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. Pick a player, for example player 1. Suppose σ_1 is not an Aumann-rational strategy for 1. Then there is another strategy σ'_1 such that 1 knows that his payoff at $\{\sigma'_1, \sigma'_2, \ldots, \sigma'_n\}$ is higher than 1's payoff at $\{\sigma_1, \sigma'_2, \ldots, \sigma'_n\}$ for all strategies $\sigma'_2, \ldots, \sigma'_n$ deemed possible by 1. In particular, 1's payoff at $\{\sigma'_1, \sigma_2, \ldots, \sigma_n\}$ should be higher than 1's payoff at $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ since 1 considers strategies $\sigma_2, \ldots, \sigma_n$ possible at σ. We now have a contradiction: since σ is a Nash equilibrium, 1's payoff at $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ is higher than or equal to 1's payoff at $\{\sigma'_1, \sigma_2, \ldots, \sigma_n\}$. $\square$

Theorem 2 *Let G be a strategic game under Bullet Nash rationality corresponding to a Nash Equilibrium σ. Then σ is a definitive solution to G. If the rules of the game are known to all players, then σ is a justified definitive solution to G.*

Proof. Since each player is Bullet Nash-rational with σ, each player i choses σ_i. If everybody knows that all players are Bullet Nash-rational with σ, everybody knows that each player i choses σ_i. $\square$

Corollary 2 *If σ is a Nash equilibrium in game G, then there is a notion of rationality extending Aumann's rationality which makes σ a definitive solution to G.*

Corollary 3 *If a game has a Nash equilibrium, then for an appropriate extension of Aumann's rationality, such a game has a definitive solution.*

5 Many Nash equilibria - no definitive solution either

In addition to what we have already learned about definitive solutions to games, in this section we will show that under Aumann's rationality, a strategic game with two or more Nash equilibria cannot have a definitive solution. The reasoning in this section is more logically involved, but is, we hope, still within reach of non-logicians. We will try to keep the exposition as informal as possible.

[6] The name is analogous to "bullet voting," in which the voter can vote for multiple candidates but votes for only one.

5.1 Regular form of strategic games

We start by defining the notion of *GAME RULES* in a more rigorous way. A **regular strategic game** is a strategic game described by the following set of data.

a. Conditions on strategy propositions S_i^j stating '*player* i *chooses strategy* j.' These conditions express that each player i chooses one and only one strategy:
$$(S_i^1 \vee \ldots \vee S_i^{n_i}) \text{ and } \neg(S_i^j \wedge S_i^l) \text{ for each } j \neq l.$$

b. A complete description of the preference relation for each player at each outcome.

c. Aumann's rationality condition:
if player i *knows that his strategy* j *is strictly dominated, then* $\neg S_i^j$.
This condition can be formulated in a straightforward way using strategy propositions, preference relation, and knowledge operators.

d. Knowledge of one's own moves: $S_i^j \rightarrow \mathbf{K}_i(S_i^j)$ and $(\neg S_i^j) \rightarrow \mathbf{K}_i(\neg S_i^j)$ for all i, j's.

e. Common knowledge of **a** – **d** above.

Epistemic conditions (**d**) and (**e**) are optional, but adding them makes the results of this section stronger, so we keep (**d**) and (**e**) without much discussion.

As an example, in the regular form *WPD*, we can associate S_1^1 with war_B, S_1^2 with $peace_B$, S_2^1 with war_S, and S_2^2 with $peace_S$. Now we can show that for regular *WPD*, $\{peace_B, peace_S\}$ is a justified definitive solution. Indeed, it suffices to logically derive $peace_B \wedge peace_S$ from *GAME RULES* of *WPD* and argue that this derivation can be performed by any player, hence

$$\mathbf{K}_i[\mathit{GAME\ RULES} \rightarrow peace_B \wedge peace_S] \quad \text{for each } i \in \{B, S\}.$$

Here is a derivation of $peace_B \wedge peace_S$ from *GAME RULES* of *WPD* (an informal version of this derivation was presented in Section 2):

1. by (**b**) and (**e**), S knows that war_S is a strictly dominated strategy for S;
2. by (**c**), $\neg war_S$;
3. by (**a**) and 2, $peace_S$;
4. by (**e**), B knows 1 and (**c**), hence B knows $peace_S$;
5. from 4, B concludes that strategy war_B is dominated, hence by (**c**), $\neg war_B$;
6. by (**e**), from 5 and (**a**) B derives $peace_B$, hence $peace_B$;
7. from 3 and 6 we conclude $peace_B \wedge peace_S$.

This example was intended to illustrate that the regular form of strategic games is sufficient for accommodating the usual epistemic reasoning in games.

5.2 Knowing and playing any Nash equilibrium is consistent

Lemma 2 *A regular strategic game is consistent with the knowledge of any of its Nash equilibria: for each player i and Nash equilibrium $\{\sigma_1^{e_1}, \ldots, \sigma_n^{e_n}\}$,*

$$\mathbf{K}_i[\mathit{GAME\ RULES}] \wedge \mathbf{K}_i(S_1^{e_1} \wedge \ldots \wedge S_n^{e_n}) \tag{5}$$

is consistent.

Proof. It suffices to present an epistemic model (Aumann structure, $\mathsf{S5}$ Kripke model) $\mathcal{M}$ in which at some node u, both $\mathbf{K}_i[\mathit{GAME\ RULES}]$ and $\mathbf{K}_i(S_1^{e_1} \wedge \ldots \wedge S_n^{e_n})$ hold.

Epistemic model $\mathcal{M}$ consists of the set of worlds (nodes) W, partitions $R_1, \ldots, R_n$ corresponding to knowledge of players $1, \ldots, n$, and the truth relation '$\Vdash$' which specifies the truth value of strategy propositions at nodes from W.

- The set of worlds W is the set of all strategy profiles $\sigma = \{\sigma_1^{j_1}, \ldots, \sigma_n^{j_n}\}$ of the game. Informally, we may think of possible worlds as epistemic states of players at the moment prior to making their moves.
- All partition sets are singletons: $R_1 = \ldots = R_n = \{\{u\} \mid u \in W\}$. This corresponds to the situation in which at each node σ, all strategies from σ are known to all players.
- $\{\sigma_1^{j_1}, \ldots, \sigma_n^{j_n}\} \Vdash S_i^j$ if and only if σ_i^j occurs in $\{\sigma_1^{j_1}, \ldots, \sigma_n^{j_n}\}$, i.e., if $j = j_i$. Naturally, we assume that all 'true' preference relation statements from (**b**) hold at each world.

Let $\sigma = \{\sigma_1^{e_1}, \ldots, \sigma_n^{e_n}\}$ be a Nash equilibrium of the game. We claim that

$$\sigma \Vdash \mathbf{K}_i[\mathit{GAME\ RULES}] \wedge \mathbf{K}_i(S_1^{e_1} \wedge \ldots \wedge S_n^{e_n}) \ \text{ for each } i = 1, \ldots, n.$$

Since there are no nodes in W indistinguishable from σ, each proposition F is equivalent to $\mathbf{K}_i(F)$. It now suffices to establish that

$$\sigma \Vdash \mathit{GAME\ RULES} \wedge (S_1^{e_1} \wedge \ldots \wedge S_n^{e_n}),$$

from which

$$\sigma \Vdash S_1^{e_1} \wedge \ldots \wedge S_n^{e_n}$$

holds by definition of '$\Vdash$.' It now remains to show that

$$\sigma \Vdash \mathit{GAME\ RULES}.$$

We will check conditions (**a** – **e**) one by one.

- (**a**) and (**b**) hold at each node by definition of '$\Vdash$.'

- (**c**) holds in σ by the following reasoning. The only way Aumann rationality at σ can be violated is when some S_i^j holds at σ, whereas player i knows at σ that j is strictly dominated. By the choice of partitions, each player i at σ knows all strategies $S_1^{e_1}, \dots, S_n^{e_n}$, therefore i knows that the dominance condition for $\sigma_i^{e_i}$ reduces to comparing i's payoff at outcomes σ and

$$\{\sigma_1^{e_1}, \dots, \sigma_{i-1}^{e_{i-1}}, \sigma_i^l, \sigma_{i-1}^{e_{i-1}}, \dots, \sigma_n^{e_n}\} \tag{6}$$

for all possible l's. Since σ is a Nash equilibrium, no outcome (6) is strictly preferable to σ for player i, therefore, the only way in which Aumann's rationality could fail at σ turns out to be impossible. Hence (**c**) holds at σ as well.

- (**d**) holds at each node by the choice of partitions $R_1, \dots, R_n$.

- (**e**) holds by the partition structure: since there are no indistinguishable states in $\mathcal{M}$, all true propositions are known, and even commonly known.

□

An appropriate probabilistic version of Lemma 2 for mixed Nash equilibria was found in [3] in which "know" means "ascribe probability 1 to."

Corollary 4 *A regular strategic game is consistent with playing any of its Nash equilibria: for any Nash equilibrium* $\{\sigma_1^{e_1}, \dots, \sigma_n^{e_n}\}$,

$$\mathit{GAME\ RULES} \wedge (S_1^{e_1} \wedge \dots \wedge S_n^{e_n}) \tag{7}$$

is consistent.

Proof. By factivity, (5) implies (7). Therefore, if (7) were inconsistent, (5) would be inconsistent too. □

5.3 No definitive solutions to multi-equilibria regular games

Theorem 3 *No regular strategic game with more than one Nash equilibium can have a definitive solution.*

Proof. Suppose otherwise, i.e., that for some Nash equilibrium $\sigma = \{\sigma_1^{e_1}, \dots, \sigma_n^{e_n}\}$,

$$\mathit{GAME\ RULES} \rightarrow (S_1^{e_1} \wedge \dots \wedge S_n^{e_n}) \ .$$

By the assumptions, the game has a different Nash equilibrium $\sigma' = \{\sigma_1^{l_1}, \dots, \sigma_n^{l_n}\}$ as well. By (**a**), two different outcomes are incompatible:

$$\mathit{GAME\ RULES} \rightarrow \neg(S_1^{l_1} \wedge \dots \wedge S_n^{l_n}),$$

which yields that

$$GAME\ RULES \wedge (S_1^{l_1} \wedge \ldots \wedge S_n^{l_n})$$

is inconsistent. This contradicts Corollary 4. □

Corollary 5 *No regular strategic game with more than one Nash equilibrium can have a justified definitive solution.*

Proof. Obvious, since each justified definitive solution is a definitive solution. □

Note that some **non-regular strategic games** can single out one of multiple Nash equilibria. For example, if G is a regular game presented by $GAME\ RULES$ and $\sigma = \{\sigma_1^{e_1}, \ldots, \sigma_n^{e_n}\}$ is one of its Nash equilibria, then, by Corollary 4, a new "game" G^σ with the same payoffs and an additional condition that everybody plays σ,

$$GAME\ RULES\ \cup\ (S_1^{e_1} \wedge \ldots \wedge S_n^{e_n})\ ,$$

is consistent. It is easy to see that σ is a justified definitive solution to G^σ.

6 Discussion

Similar methods could be applied for analyzing mixed strategies as well.

7 Acknowledgments

The author is grateful to Adam Brandenburger for attracting attention to Nash's argument and supportive discussions. The author is indebted to Mel Fitting, Vladimir Krupski, and Elena Nogina for many principal comments and suggestions which helped to improve exposition. Special thanks to Karen Kletter for editing this report.

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