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Binding Modalities

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Abstract

The standard first-order reading of modality does not bind individual variables, i.e., if x is free in $F(x)$, then x remains free in $\Box F(x)$. Accordingly, if \Box stands for ‘provable in arithmetic,’ $\forall x \Box F(x)$ states that $F(n)$ is provable for any given value of $n = 0, 1, 2, \dots$; this corresponds to a *de re* reading of modality. The other, *de dicto* meaning of $\Box F(x)$, suggesting that $F(x)$ is derivable as a formula with a free variable x , is not directly represented by a modality, though, semantically, it could be approximated by compound constructions, e.g., $\Box \forall x F(x)$.

We introduce the first-order logic FOS4* in which modalities can bind individual variables and, in particular, can directly represent both *de re* and *de dicto* modalities. FOS4* extends first-order S4 and is the natural forgetful projection of the first-order logic of proofs FOLP. The same method of introducing binding modalities obviously works for other modal logics as well.

1 Introduction

Let $A(x)$ be a formula with a parameter x . Then, in the standard reading of the modality in first-order language, formula $\Box A(x)$ also has x as a

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parameter. In the provability interpretation of \Box^1 , this reflects the reading of $\Box A(x)$ as

given a natural parameter $x = n$, formula $A(n)$ is provable,

whereas the alternative meaning

formula $A(x)$ with a free variable x is provable

does not have a direct modal representation.

We extend the first-order modal language with a construction that makes it possible. Namely, we replace the modality \Box by a family of modalities \Box_X , each labeled by a finite set of individual variables X . In a formula $\Box_X A$, the modality \Box_X binds all variables in A except those in X . We call \Box_X *binding modalities*.

In Section 2, we introduce the logic FOS4^* which is a version of first-order modal logic S4^2 in the richer language with binding modalities. In Section 3, we present the sequent calculus for FOS4^* that admits cut-elimination. In Section 4, we discuss the connections between FOS4^* and first-order S4 . In Section 6, we show that the first-order logic of proofs FOLP is able to realize FOS4^* which, in combination with the results of the previous sections, yields another proof of the Realization Theorem for FOS4 and FOLP . Section 5 is devoted to the Kripke-style semantics for FOS4^* .

2 Modal logic with binding modalities

By X, Y , etc., we denote finite sets of individual variables. If y is an individual variable, then we will write Xy for $X \cup \{y\}$. An additional convention: notation Xy means, in part, that $y \notin X$.

Definition 1 Let \mathcal{L}^* denote the first-order modal language that contains individual variables x_0, x_1, x_2, \dots , Boolean connectives, quantifiers over individual variables, predicate symbols Q_i^n of any arity n ($i, n = 0, 1, 2, \dots$), and the family of modal operators \Box_X where X ranges over finite sets of individual variables.

¹For informal and formal provability interpretations of modality see, e.g., [1, 2, 3, 4, 5].

²First-order modal logic S4 has postulates of first-order classical logic and modal postulates of S4 : $\Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G)$, $\Box F \rightarrow \Box \Box F$, $\Box F \rightarrow F$, and the necessitation rule ‘ $\vdash F$ yields $\vdash \Box F$.’

Formulas are defined in the standard way. The only difference is in the definition of open and closed occurrences of variables: the modality \Box_X binds all variables except those from X , namely,

- All occurrences of individual variables in an atomic formula are free.
- Boolean connectives preserve free and bound occurrences of variables.
- $\forall x$ binds all occurrences of x in $\forall xA$ and preserves free and bound occurrences of all other variables.
- Free occurrences of variables from X in A , and only they, are free in $\Box_X A$.

The set of free variables of a formula A is denoted by $FVar(A)$.

Note also that in \Box_X , variables from X are merely a syntactic label of this modality and are not considered occurrences of variables.

Definition 2 The logic $FOS4^*$ is axiomatized by the following schemas. Here A, B are formulas, X is a set of individual variables, and y is an individual variable.

- A0 classical axioms of first-order logic
- A1 $\Box_{Xy}A \rightarrow \Box_X A, \quad y \notin FVar(A)$
- A2 $\Box_X A \rightarrow \Box_{Xy}A$
- A3 $\Box_X A \rightarrow \Box_X \forall xA, \quad x \notin X$
- B1 $\Box_X(A \rightarrow B) \rightarrow (\Box_X A \rightarrow \Box_X B)$
- B2 $\Box_X A \rightarrow \Box_X \Box_X A$
- B3 $\Box_X A \rightarrow A$

$FOS4^*$ has the following inference rules:

- R1 $\vdash A, A \rightarrow B \Rightarrow \vdash B \quad \textit{Modus Ponens}$
- R2 $\vdash A \Rightarrow \vdash \forall xA \quad \textit{generalization}$
- R3 $\vdash A \Rightarrow \vdash \Box_\emptyset A \quad \textit{necessitation.}$

We define derivations in $FOS4^*$ and derivations from the hypothesis in the standard manner. In particular, in a derivation from the set of hypotheses Γ , the generalization rule may not be applied to variables that are free in Γ .

Note that each \Box_X is a normal propositional modality. In particular, the following standard lemma holds:

Lemma 1 1. $\text{FOS4}^* \vdash \Box_X(A \wedge B) \leftrightarrow (\Box_X A \wedge \Box_X B)$.

2. If $\text{FOS4}^* \vdash A \leftrightarrow B$ then $\text{FOS4}^* \vdash \Box_X A \leftrightarrow \Box_X B$.

Modality \Box_\emptyset , which considers all individual variables as local and admits unlimited quantification under its scope, represents *de dicto* modality.

The following lemma shows that the principle

$$\Box_X A \rightarrow \Box_X \forall x A \quad \text{with } x \notin X,$$

(e.g., axiom A3) is derivable from the rest of FOS4^* . However, we decide to keep it in the list of FOS4^* postulates to underline the significance of this principle. Furthermore, the proof of Lemma 2 relies on transitivity axiom B2 and hence does not work for logics without transitivity, e.g., binding modalities versions of first-order logics **K** or **T** which, however, should contain A3.

Lemma 2 $\text{FOS4}^* \vdash \Box_X A \leftrightarrow \Box_X \forall x A$, provided $x \notin X$.

Proof. The left-to-right direction.

1. $\Box_X A \rightarrow A$ axiom B3;
2. $\forall x \Box_X A \rightarrow \forall x A$ from 1 by first-order logic;
3. $\Box_X A \rightarrow \forall x \Box_X A$ by first-order logic, since $x \notin FVar(\Box_X A)$;
4. $\Box_X A \rightarrow \forall x A$ from 2,3 by first-order logic;
5. $\Box_\emptyset(\Box_X A \rightarrow \forall x A)$ from 4 by R3;
6. $\Box_X(\Box_X A \rightarrow \forall x A)$ from 5 in view of A2;
7. $\Box_X \Box_X A \rightarrow \Box_X \forall x A$ from 6 by B1 and propositional logic;
8. $\Box_X A \rightarrow \Box_X \Box_X A$ axiom B2;
9. $\Box_X A \rightarrow \Box_X \forall x A$ from 7, 8 by propositional logic.

The right-to-left direction.

1. $\forall x A \rightarrow A$ by first-order logic;
2. $\Box_\emptyset(\forall x A \rightarrow A)$ from 1 by R3;
3. $\Box_X(\forall x A \rightarrow A)$ from 2 in view of A2;
4. $\Box_X \forall x A \rightarrow \Box_X A$ from 3 in view of B1. □

A straightforward iteration of Lemma 2 yields

Lemma 3 Let A be a formula in the language FOS4^* . By A^c we denote a new formula obtained from A by replacing each subformula of A of the form $\Box_X B$ by $\Box_X \forall y_1 \dots \forall y_n B$ where $Y = \{y_1, \dots, y_n\} = FVar(B) \setminus X$. Then $\text{FOS4}^* \vdash A \leftrightarrow A^c$.

3 Cut-elimination

We define a sequent calculus for FOS4^* similar to that of the sequent calculus for FOS4 (see [7], Section 9.1.3). In addition to structural rules (including the cut-rule), the sequential calculus for FOS4^* denoted by $\mathcal{GFOS4}^*$ contains the following axioms:

$$\perp \Rightarrow \quad \text{and} \quad P(\vec{x}) \Rightarrow P(\vec{x})$$

and logical rules:

$$\begin{array}{c} \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \quad (R \rightarrow), \quad \frac{\Gamma \Rightarrow \Delta, A, \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \quad (L \rightarrow), \\ \\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \forall x A} \quad (R\forall), \quad \frac{\Gamma, A(y/x) \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta} \quad (L\forall), \\ \\ \frac{\Box_{X_1} A_1, \dots, \Box_{X_n} A_n \Rightarrow A}{\Box_{X_1} A_1, \dots, \Box_{X_n} A_n \Rightarrow \Box_X A} \quad (R\Box), \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \quad (L\Box). \end{array}$$

In $R\forall$, we suppose that $x \notin FVar(\Gamma, \Delta)$. In $R\Box$, we assume that

$$\left(\bigcup_{i=1}^n X_i \right) \cap FVar(A) \subseteq X.$$

The following connection between FOS4^* and its Gentzen-style version $\mathcal{GFOS4}^*$ takes place:

Lemma 4

$$\mathcal{GFOS4}^* \vdash \Gamma \Rightarrow \Delta \quad \text{iff} \quad \text{FOS4}^* \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta.$$

Proof. In order to prove left-to-right implication, reason by induction on the derivation in $\mathcal{GFOS4}^*$. The only nontrivial cases here are the modal rules $R\Box$ and $L\Box$.

For $L\Box$, by the induction hypothesis, $\text{FOS4}^* \vdash A \wedge (\bigwedge \Gamma) \rightarrow \bigvee \Delta$. Since $\text{FOS4}^* \vdash \Box_X A \rightarrow A$, we obtain $\text{FOS4}^* \vdash \Box_X A \wedge (\bigwedge \Gamma) \rightarrow \bigvee \Delta$.

For $R\Box$, the induction hypothesis yields that $\text{FOS4}^* \vdash (\bigwedge \Box_{X_i} A_i) \rightarrow B$. Then for $X' = \bigcup_i X_i$ we have

1. $\Box_{\emptyset}((\bigwedge \Box_{X_i} A_i) \rightarrow B)$ by R3;
2. $\Box_{X'}(\bigwedge \Box_{X_i} A_i) \rightarrow B$ from 1 in view of A2;
3. $\Box_{X'}(\bigwedge \Box_{X_i} A_i) \rightarrow \Box_{X'} B$ from 2 in view of B1;
4. $\Box_{X'}(\bigwedge \Box_{X_i} A_i) \leftrightarrow \bigwedge (\Box_{X'} \Box_{X_i} A_i)$ by Lemma 1;
5. $\Box_{X_i} A_i \rightarrow \Box_{X_i} \Box_{X_i} A_i$ by B2;
6. $\Box_{X_i} \Box_{X_i} A_i \rightarrow \Box_{X'} \Box_{X_i} A_i$ by A2;
7. $\Box_{X_i} A_i \rightarrow \Box_{X'} \Box_{X_i} A_i$ from 5, 6;
8. $\bigwedge_i \Box_{X_i} A_i \rightarrow \Box_{X'}(\bigwedge \Box_{X_i} A_i)$ from 4, 7;
9. $\Box_{X'} B \rightarrow \Box_X B$ by A1;
10. $\bigwedge_i \Box_{X_i} A_i \rightarrow \Box_X B$ by 3, 8, 9.

In order to prove right-to-left implication it suffices to show that

$$\mathcal{GFOS4}^* \vdash \Rightarrow A$$

where A is any axiom of $\mathcal{FOS4}^*$. Below are the proofs for axioms A1 and A2; axioms B1–B3 are derived similarly with the standard $\mathcal{FOS4}$.

Axiom A1. If $y \notin FVar(A)$, then the tree below is a valid derivation in $\mathcal{GFOS4}^*$:

$$\frac{\frac{A \Rightarrow A}{\Box_{Xy} A \Rightarrow A}}{\Box_{Xy} A \Rightarrow \Box_X A}.$$

Axiom A2.

$$\frac{\frac{A \Rightarrow A}{\Box_X A \Rightarrow A}}{\Box_X A \Rightarrow \Box_{Xy} A}.$$

□

Theorem 1 (Cut-elimination) *If $\mathcal{GFOS4}^* \vdash \Gamma \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$ can be derived in $\mathcal{GFOS4}^*$ without using the cut-rule.*

Proof. We repeat the standard proof of the cut-elimination theorem for $\mathcal{GFOS4}$ by joint induction on the complexity of the cut and its rank (see [7]).

□

4 Connection to first-order modal logic

In this section, we study the interconnection between FOS4* and the standard modal logic FOS4.

Definition 3 Let A be a first-order modal formula. By A^* we mean a formula of the language of FOS4* that is obtained from A by replacing all occurrences of subformulas of A of the form $\Box B$ by $\Box_X B$ where $X = FVar(B)$.

Definition 4 Let \vec{x} be a finite string of individual variables $\{x_1, \dots, x_n\}$. By \vec{x}, \vec{y} we denote the concatenation of \vec{x} and \vec{y} . By $\forall \vec{x} A$ and $\exists \vec{x} A$ we mean the formulas

$$\forall x_1, \dots, \forall x_n A \quad \text{and} \quad \exists x_1, \dots, \exists x_n A$$

respectively.

Definition 5 For each FOS4*-formula F , we define its translation to the language of FOS4, denoted by F^\sharp , by induction. For atomic formulas, we stipulate $F^\sharp = F$, \sharp commutes with Boolean connectives and quantifiers, and

$$(\Box_X F)^\sharp = \Box \forall \vec{y} F^\sharp,$$

where $\vec{y} = \{y_0, \dots, y_k\} = FVar(F) \setminus X$.

Note that $FVar(A^*) = FVar A$ and $FVar(F^\sharp) = FVar(F)$.

Lemma 5 *If FOS4 $\vdash F$, then F^* is derivable in FOS4*.*

Proof. By induction on a derivation in FOS4. For the base of induction consider axioms of FOS4.

- First-order axioms remain the same axioms.
- $(\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B))^* = \Box_Z(A^* \rightarrow B^*) \rightarrow (\Box_X A^* \rightarrow \Box_Y B^*)$ where $X = FVar(A)$, $Y = FVar(B)$, and $Z = X \cup Y$; then in FOS4* we reason as follows.
 1. $\Box_Z(A^* \rightarrow B^*) \rightarrow (\Box_Z A^* \rightarrow \Box_Z B^*)$ axiom B1
 2. $\Box_Z A^* \leftrightarrow \Box_X A^*$ axioms A1 and A2
 3. $\Box_Z B^* \leftrightarrow \Box_Y B^*$ axioms A1 and A2
 4. $\Box_Z(A^* \rightarrow B^*) \rightarrow (\Box_X A^* \rightarrow \Box_Y B^*)$ from 1–3 by propositional logic

- $(\Box A \rightarrow \Box \Box A)^* = \Box_X A^* \rightarrow \Box_X \Box_X A^*$ for $X = FVar A$; axiom B2.
- $(\Box A \rightarrow A)^* = \Box_X A^* \rightarrow A^*$ for $X = FVar A$; as the axiom B3.

For the induction step, we note that the translation $(\cdot)^*$ preserves *Modus Ponens* and the generalization rule.

In the case of the necessitation rule, $\Box A$ is derived from A in FOS4, then by the induction hypothesis $FOS4^* \vdash A^*$; therefore in FOS4*

$$\begin{array}{l} \Box_{\emptyset} A^* \\ \Box_X A^* \end{array} \quad \begin{array}{l} \text{by R3;} \\ \text{by A2 and } Modus Ponens \text{ for } X = FVar(A). \end{array} \quad \square$$

Theorem 2 *If $FOS4^* \vdash A$, then $FOS4 \vdash A^\sharp$.*

Proof. Induction on the derivation of A in FOS4*. For the base of induction, consider axioms of FOS4*.

- Translation $(\cdot)^*$ respects axioms of first-order logic.
- Axiom A1. If $y \notin FVar(A)$, then $(\Box_{Xy} A)^\sharp$ coincides with $(\Box_X A)^\sharp$.
- Axiom A2. If $y \notin FVar(A)$, then $(\Box_{Xy} A)^\sharp$ coincides with $(\Box_X A)^\sharp$ as above. If $y \in FVar(A)$, then $(\Box_X A)^\sharp = \Box(\forall y, \vec{y} A^\sharp)$ and $(\Box_{Xy} A)^\sharp = \Box(\forall \vec{y} A^\sharp)$; it is clear that the latter formula follows from the former one.
- Axiom B1. Let $\vec{y}, \vec{u} = \{y_1, \dots, y_k, u_1, \dots, u_l\}$ be $FVar(A) \setminus X$, and $\vec{z}, \vec{u} = \{z_1, \dots, z_m, u_1, \dots, u_l\}$ be $FVar(B) \setminus X$, and $\vec{u} = \{u_1, \dots, u_l\}$ be $(FVar(B) \cap FVar(A)) \setminus X$. Then $FVar(A \rightarrow B) \setminus X$ is $\vec{y}, \vec{z}, \vec{u}$. Therefore $(\Box_X(A \rightarrow B) \rightarrow (\Box_X A \rightarrow \Box_X B))^\sharp$ is

$$\Box(\forall \vec{y}, \vec{z}, \vec{u}(A^\sharp \rightarrow B^\sharp)) \rightarrow \Box(\forall \vec{y}, \vec{u} A^\sharp) \rightarrow \Box(\forall \vec{z}, \vec{u} B^\sharp),$$

which is clearly provable in FOS4 since no y_i occurs in B^\sharp and no z_i occurs in A^\sharp .

- Axiom B2. Let $\vec{y} = \{y_1, \dots, y_k\}$ be $FVar(A) \setminus X$. Since $FVar(\Box_X A) = X$, we obtain $(\Box_X A)^\sharp = \Box \forall \vec{y} A^\sharp$ and $(\Box_X \Box_X A)^\sharp = \Box \Box \forall \vec{y} A^\sharp$; therefore the translation of B2 is an instance of the transitivity axiom in S4.
- Axiom B3. $(\Box_X A \rightarrow A)^\sharp = (\Box \vec{y} A^\sharp \rightarrow A^\sharp)$; the last formula is clearly derivable in FOS4.

□

Corollary 1 $\text{FOS4} \vdash A$ if and only if $\text{FOS4}^* \vdash A^*$.

Proof. The left-to-right direction is proven by Lemma 5. To prove the right-to-left direction, we assume that $\text{FOS4}^* \vdash A^*$. Since for each subformula of A^* of the form $\Box_X B$ one has $X = FVar(B)$, we conclude that $(A^*)^\sharp$ coincides with A . By Theorem 2, from $\text{FOS4}^* \vdash A^*$ we obtain $\text{FOS4} \vdash (A^*)^\sharp$, thus $\text{FOS4} \vdash A$.

□

Corollary 2 $\text{FOS4}^* \vdash A$ if and only if $\text{FOS4} \vdash A^\sharp$.

Proof. The left-to-right direction is proven by Theorem 2. To prove the right-to-left direction let us assume $\text{FOS4} \vdash A^\sharp$. Then by Lemma 5, we obtain $\text{FOS4}^* \vdash (A^\sharp)^*$. By induction on A it can be easily proven that $\text{FOS4}^* \vdash (A^\sharp)^* \leftrightarrow A$, therefore $\text{FOS4}^* \vdash A$.

□

5 Semantics

Definition 6 An FOS4^* -model is a standard Kripke model for FOS4 . Namely, a frame $\mathcal{F} = (W, R, D)$ for FOS4^* consists of a nonempty set of possible worlds $W \neq \emptyset$, a transitive reflexive accessibility relation $R \subseteq W^2$, and a domain function D which assigns a nonempty set $D(u)$ to each element u of W monotonically, i.e., uRv yields $D(u) \subseteq D(v)$ for all $u, v \in W$.

A model $\mathcal{M} = (\mathcal{F}, V)$ is a frame \mathcal{F} supplied with the validity function V which assigns each n -ary predicate letter P and each possible world u a subset of $D(u)^n$ (on which P is true at node u).³

For each formula A of the language FOS4^* , each substitution of σ of the values from

$$U = \bigcup_{u \in W} D(u)$$

for free variables, and each $u \in W$, we define $\mathcal{M}, u \models A\sigma$ by induction on the complexity of A :

³For comparison, Kripke-style semantics for FOLP in [6] treats both variable binding operators and proof terms.

- if A is atomic, that is, $A = P(x_1, \dots, x_n)$ for a predicate letter P , then $\mathcal{M}, u \models A\sigma$ iff $(x_1\sigma, \dots, x_n\sigma) \in V(P, u)$;
- quantifiers and Boolean connectives are treated in the usual way;
- if $A = \Box_X B$, then $\mathcal{M}, u \models A\sigma$ iff $\mathcal{M}, v \models B\sigma'$ for each v accessible from u and each substitution σ' which coincides with σ on variables from X and assigns arbitrary elements from $D(v)$ to the remaining free variables of B .⁴

We say that A is valid in a model \mathcal{M} ($\mathcal{M} \models A$) if $\mathcal{M}, u \models A$ for each $u \in W$.

Theorem 3 $\text{FOS4}^* \vdash A$ iff A is valid in all models.

Proof. Soundness follows easily by induction on derivations in FOS4^* . Completeness follows immediately from the connection to FOS4 established in Section 4. Indeed, suppose $\text{FOS4}^* \not\vdash F$. By Corollary 2, $\text{FOS4} \not\vdash A^\sharp$. By Completeness, there is an FOS4 -model \mathcal{M} in which A^\sharp does not hold in the sense of FOS4 , hence A^\sharp does not hold in \mathcal{M} in the sense of FOS4^* . \square

6 Connection to the first-order logic of proofs

Definition 7 (see [4] for details). The language of the first-order logic of proofs FOLP in the extension of the first-order language with a countable set of predicate symbols of any arity, without functional symbols and equality by special means to represent proofs and proof assertions, namely,

- proof variables and constants;
- functional symbols for operations on proofs: binary $+$, \cdot and unary $!$ and gen_x for each individual variable x ;
- an operational symbol $(\cdot):_X(\cdot)$ for each finite set X of individual variables.

⁴It is easy to see that $\mathcal{M}, u \models A$ if and only if $\mathcal{M}, u \models A^\sharp$ in the usual sense of FOS4 .

Proofs are represented by *proof terms*, constructed from proof constants and variables by means of functional symbols. *Formulas* are defined in the standard way with an additional clause for proof assertions:

- if t is a proof term, A is a formula, and X is a finite set of individual variables, then $t:XA$ is a formula.

The intended semantics of $t:XA$ is

t is a proof of A with parameters X

has been made precise in [4].

In addition to axioms and rules of first-order logic, FOLP has the following axioms and rules.

- $t:_{Xy}A \rightarrow t:XA, \quad y \notin FVar(A)$
- $t:XA \rightarrow t:_{Xy}A$
- $t:XA \rightarrow A$
- $s:XA \rightarrow B \rightarrow (t:XA \rightarrow [s \cdot t]:XB)$
- $t:XA \rightarrow [t + s]:XA, \quad s:XA \rightarrow [t + s]:XA$
- $t:XA \rightarrow !t:_{Xt}A$
- $t:XA \rightarrow \text{gen}_x(t):X\forall xA, \quad x \notin X$
- $\vdash c:A$, where A is an axiom, c is a proof constant.

Definition 8 Let A be an FOS4*-formula. By *realization* of a formula A , we mean a formula A^r of the language of FOLP that is obtained from A by replacing all occurrences of subformulas of A of the form $\Box_X B$ by $t:XB$ for some proof terms t . To avoid unnecessary formalism, we suggest thinking of a realization as a result of an iterated procedure which always replaces an innermost $\Box B$ by $t:XB$. A realization is *normal* if all negative occurrences of \Box are assigned proof variables.

Definition 9 The forgetful projection $(\cdot)^0$ of FOLP to the language FOS4* erases all terms in an FOLP-formula, namely, it replaces all subformulas of the form $t:XB$ by $\Box_X B$:

$$(t:XF)^0 = \Box_X F^0.$$

Lemma 6 *If $\text{FOLP} \vdash F$, then F^0 is derivable in FOS4^* .*

Proof. By straightforward induction on derivations in FOLP. □

Remark 1 Note that the forgetful projection of an FOLP–formula A to the language of FOS4 (as defined in [4]) is a superposition of the forgetful projection of A to the language of FOS4^* and the translation \sharp of FOS4^* to FOS4.

Similarly, a realization of an FOS4–formula A in FOLP is a composition of the translation $*$ of A to FOS4^* and a realization in FOLP.

Theorem 4 [Realization Theorem] *If $\text{FOS4}^* \vdash A$, then there is a normal realization A^r such that $\text{FOLP} \vdash A^r$.*

Proof. Induction on the cut-free proof in the Gentzen variant for FOS4^* similar to that in [4]. □

7 Conclusions

Binding modalities system FOS4^* naturally contains both the traditional *de re* modality $\Box F$, which is represented by $(\Box F)^*$ i.e., $\Box_X F$ with X being the set of all free variables of F , and the *de dicto* modality, which is represented by \Box_\emptyset .

This extended set of modalities allows FOS4^* to directly express and derive new principles of modal logic. For example, FOS4^* supports the Barcan Formula for \Box_\emptyset :

$$\forall x \Box_\emptyset A \rightarrow \Box_\emptyset \forall x A,$$

which does not hold for the usual S4 modality rather than \Box_\emptyset .

One could argue that FOS4^* is as expressive as traditional FOS4 since the binding modality $\Box_X F$ can be semantically encoded by $\Box \forall \vec{y} F$ where \Box is the FOS4 modality and \vec{y} is the list of all free variables of F that are not in X . However, such an encoding changes both formulas and derivations: for example, converting $\Box_\emptyset A$ to $\Box \forall x A$ renders the aforementioned Barcan Formula meaningless within the context of FOS4. In FOS4^* , however, this is a revealing and meaningful principle. We want to believe that the structural

and semantic properties of the *de dicto* modality deserve direct study and a designated formal logic system and that FOS4* is such a system.

FOS4* captures the *de dicto* modality in a coherent first-order manner, offers a natural axiom system, sequent formulation with cut-elimination, and Kripke model theory. It makes reasoning with variables in the modal context explicit and hence a subject of formal analysis which could be of interest for studies of modalities and applications.

In summary, FOS4* can be regarded as a version of FOS4 in a refined language with an explicit mechanism of binding/unbinding variables which yet enjoys the convenient features of FOS4, such as cut-elimination and Kripke completeness.

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