Condition Estimation by Means of Power Method

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Abstract

We employ the Power Method (that is essentially a sequence of matrix-by-vector multiplications) to estimate the condition number of a matrix.

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Assume a real symmetric nonnegative definite $n \times n$ matrix $S$ and apply the Power Iteration

$$v_k = S v = S v_{k-1}, \quad k = 1, 2, \ldots$$

for a random vector $v = v_0$ to approximate the largest eigenvalue $\lambda = \lambda(S)$ of the matrix $S$ by the Rayleigh quotients $q_i = v_k^T S v_k / v_k^T v_k$. The paper [2] has proposed this technique and proved that $q_k \leq \lambda \theta q_k$ with a probability at least $1 - 0.8\theta^{-k/2} n^{1/2}$ for any scalar $\theta > 1$. This estimate defines a stopping criterion for the iteration, and heuristically one can also stop where $q_i / q_{i-1} \approx 1$ or $||S v_i - q_i v_i|| / ||v_i|| \leq t$ for a fixed tolerance $t$. Instead of the Rayleigh quotients one can use the simple quotients $s_i = e_i^T S v_k / e_i^T v_k$ for the $i$th coordinate vectors $e_i$ and fixed or random integers $i = i(k), 1 \leq i \leq n$ (cf. [1], [3], [4]). Now assume an $m \times n$ matrix $A$ for $m \geq n$, let $\sigma_j(A)$ denote its $j$th largest singular value, and seek a crude estimates for $\sigma_1(A)$ and $\sigma_n(A)$, e.g., to decide whether the matrix is well conditioned. Apply the power iteration (1) to the matrix $S = A^T A$ to computed a close upper bound $\sigma_2^2$ on $\lambda(S) = ||A||^2 = \sigma_1^2(A)$. Then apply the power iteration (1) to the matrix $B = \sigma_2^2 I - A^T A$ to compute an approximation $\lambda_+$ to its largest eigenvalue and then obtain $\sigma_2^2 - \lambda_+ \approx \sigma_2^2(A)$. For $m \leq n$ apply the same techniques to the matrix $AA^T$.

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References


