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Abstract

We outline an intuitionistic view of knowledge which maintains the original Brouwer-Heyting-Kolmogorov semantics of intuitionism and is consistent with Williamson's suggestion that intuitionistic knowledge be regarded as the result of verification. We argue that on this view co-reflection $A \to \mathbf{K}A$ is valid and reflection $\mathbf{K}A \to A$ is not; the latter is a distinctly classical principle, too strong as the intuitionistic truth condition for knowledge which can be more adequately expressed by other modal means, e.g. $\neg A \to \neg\mathbf{K}A$ "false is not known." We introduce a system of intuitionistic epistemic logic, IEL, codifying this view of knowledge, and support it with an explanatory possible worlds semantics. From this it follows that previous outlines of intuitionistic knowledge are insufficiently intuitionistic: by endorsing $\mathbf{K}A \to A$ they implicitly adopt a classical view of knowledge, by rejecting $A \to \mathbf{K}A$ they reject the constructivity of truth. Within the framework of IEL, the knowability paradox is resolved in a constructive manner which, as we hope, reflects its intrinsic meaning.

1 Introduction

Our goal is to outline an intuitionistic view of knowledge which is faithful to the intrinsic semantics of intuitionistic logic: the Brouwer-Heyting-Kolmogorov (BHK) semantics. This view regards **knowledge as the product of verification**, as suggested by Williamson [77]. While the standard domain of our theory is the same as that of BHK – mathematical statements, proofs and verifications – we aim to show that BHK and the resulting intuitionistic epistemic logic, IEL, also yields principles of constructive epistemic reasoning which apply in more general settings; specifically, in settings where the notion of a conclusive and checkable verification makes sense. The resulting framework also offers a natural constructive/intuitionistic resolution of the Church-Fitch knowability paradox.

According to the BHK semantics a proposition is true if proved, yet we allow that justifications weaker than proof can be adequate for knowledge, e.g. verification by trusted means which do not necessarily produce explicit proofs of what is verified. This distinguishes knowledge from constructive truth, since the requirements for the latter are more stringent than for the former. Whereas the classical truth of a proposition is only necessary, but not sufficient, for it to be known, e.g. it must also be believed on the basis of adequate justification, the intuitionistic truth of a proposition is sufficient for knowledge, because every proof is also a verification. Consequently, we have the following contrast between the classical and the intuitionistic universes: in the classical case the following relation between knowledge and truth holds,

$$\textit{Classical Knowledge} \;\Rightarrow\; \textit{Classical Truth}$$

while intuitionistically,

$$\textit{Intuitionistic Truth} \;\Rightarrow\; \textit{Intuitionistic Knowledge.}$$

This insight is fundamental to the nature of intuitionistic reasoning about knowledge, and how it differs from classical epistemic reasoning. Intuitionistically the principle of the *constructivity of truth*, a.k.a *co-reflection*:

$$A \to \mathbf{K}A \tag{CT}$$

is a truism about intuitionistic knowledge – for the aforementioned reason that all proofs are verifications. Classically, of course, it is invalid because it asserts a form of omniscience, that all classical truths are classically known.

Classically the principle of the *factivity of knowledge*, a.k.a *reflection*

$$\mathbf{K}A \to A \tag{FK}$$

is constitutive of the conception of knowledge. Intuitionistically, however, it is too strong and is not a valid epistemic principle, because not every verification yields a BHK-compliant proof.

Extending the BHK semantics with the notion of verification, and conceiving of intuitionistic knowledge as a result of it, yields an intuitionistic epistemic logic, IEL, which validates CT and falsifies FK.

We begin with a general discussion of these epistemic principles within the BHK and classical frameworks and give an account of intuitionistic, verification-based, knowledge (section 2). Since, classically, FK expresses the truth condition on knowledge – that falsehoods cannot be known – our discussion also involves considering intuitionistically valid alternative expressions of the truth condition, (section 2.2, appendix B). On this basis we construct the system of intuitionistic epistemic logic, IEL, formalizing this view of knowledge and epistemic reasoning (sections 3 and 4). We prove soundness and completeness, and derive some notable epistemic principles (section 5).

Another justification for the correctness of our approach is given by the fact that IEL can be embedded in classical modal logic via the Gödel translation. Just like intuitionistic logic embeds into S4 considered as a classical provability calculus, IEL embeds into S4 extended with a verification modality, S4V (section 6).

Finally, we compare IEL to other approaches to intuitionistic knowledge, specifically in relation to intuitionistic responses to the knowability paradox (section 7). We argue that previous formulations of intuitionistic epistemic reasoning have not been intuitionistic enough, and that IEL offers a natural constructive resolution of the knowability paradox.

2 The Brouwer-Heyting-Kolmogorov Semantics and Knowledge

The Brouwer-Heyting-Kolmogorov semantics for intuitionistic logic (cf. [14]) holds that a proposition, A, is true if there is a proof of it, and false if we can show that the assumption that there is a proof of A leads to a contradiction. Truth for the logical connectives is defined by the following clauses:

- a proof of $A \wedge B$ consists in a proof of A and a proof of B;
- a proof of $A \vee B$ consists in giving either a proof of A or a proof B;
- a proof of $A \to B$ consists in a construction which given a proof of A returns a proof of B;
- $\neg A$ is an abbreviation for $A \to \bot$, and $\bot$ is a proposition that has no proof.

Our question is: if we add an epistemic (knowledge) operator $\mathbf{K}$ to our language, what should be the intended semantics of a proposition of the form $\mathbf{K}A$? To answer we need to clarify what counts as intuitionistic knowledge, and its relation to proof.

That intuitionistic knowledge can be gained from strict proofs is obvious, the question is whether it can be gained by less strict means also? We propose that **intuitionistic knowledge is the result of verification.** We can think of intuitionistic verifications as a generalisation of the notion of proof.[1] Intuitionistically the conception of proof has two salient features.

1. Proofs are conclusive of the truth they establish;

[1] In particular, but not exclusively, canonical verification as a generalization of canonical proof, see e.g. [12, 15, 20, 22, 23, 24, 26, 52, 56, 60, 62, 63, 70, 71, 73, 74]. See [78] for some discussion of the nature of verification and its relation to a generalised intuitionism. Note that a verification in our sense does not have to be canonical or even a means for acquiring a canonical verification, consider the examples in section 2.1.2. In a more formal intuitionistic setting Williamson reads $\mathbf{K}$ in this fashion in his proposal for an intuitionistic epistemic logic, see [77].

2. Proofs should be checkable – that something is a proof is itself capable of proof.

The appropriate generalisation of this idea, hence, holds that

1. verifications are procedures that establish as conclusively as possible the truth of the proposition in question, and

2. verifications are checkable in the weak sense of being public and repeatable, i.e. are available to anyone appropriately situated, and support counter-factuals that were the verification repeated it would yield the same result.[2]

Structurally, proof-based knowledge behaves differently from verification-based knowledge. For example, with strict proof-based knowledge, knowing (having a proof of) $A \vee B$ yields knowing A or knowing B, whereas for verification-based knowledge, $\mathbf{K}(A \vee B)$ does not necessarily yield that $\mathbf{K}A$ holds or $\mathbf{K}B$ holds (this will be shown below, Theorem 10, once the model theory of IEL is developed). A possible (but not the only) way of thinking about this example is as follows: $\mathbf{K}(A \vee B)$ states that there is a proof of disjunction $A \vee B$, but does not actually produce such a proof, or even a means for constructing such a proof. In contrast, to state that $A \vee B$ is constructively true, one has to assume a specific proof of $A \vee B$, which should yield a proof of one of the disjuncts.

Intuitionists, hence, are faced with a choice to either,

- allow knowledge as the result of conclusive reliable verifications that do not necessarily provide comprehensive constructive proofs, or

- to deny in principle the possibility of such knowledge.

We see no principled reasons for such an *a priori* denial. Indeed, such a denial yields too strict a view of constructive knowledge where a proposition is known only when it has been proved. This view of knowledge is too restrictive; it validates $A \leftrightarrow \mathbf{K}A$ and hence trivializes the notion of intuitionistic knowledge. Moreover, as the examples in 2.1 show, non-proof verifications are around and they are used to obtain knowledge; intuitionistic epistemic reasoning would not be complete without an account incorporating them.

A further question to consider is whether a proposition is intuitionistically true only if an agent is aware of a proof, or whether the possibility of such awareness is enough? Traditionally, intuitionism assumes that proofs are available to the agent. For Brouwer and Heyting proofs are mental constructions,[3] and so the existence of a proof requires its actual construction. This position is the traditional one adopted by verificationists, see e.g. [22, 24, 27].

[2] E.g. verification by perception; one cannot perceive that a perception is indeed a perception, all one can do to check a perception is to repeat it and ask others to have it.

[3] Brouwer [6] considered intuitionistic mathematics to be “an essentially languageless activity of the mind.” Heyting [42, p.2] says “In the study of mental mathematical constructions ‘to exist’ must be synonymous with ‘to be constructed.’” See also [40, 41]

On the other hand some verificationists e.g. Prawitz [56, 57, 58, 59, 60, 61] and Martin-Löf [50, 51, 52], consider proofs to be timeless entities, and that intuitionistic truth consists in the existence of such proofs, and their potential to be constructed.

The principles of intuitionistic knowledge we discuss below are compatible with either of these positions. Hence, if BHK proofs are assumed to be available to the agent, then $\mathbf{K}A$ can be read as "A is known." If proofs are platonic entities, not necessarily available to the knower, then $\mathbf{K}A$ is read as "A can be known under appropriate conditions." To keep things simple, in our exposition we follow the former, more traditional, understanding. So to claim $\mathbf{K}A$ is true is to claim that the agent is aware of a verification of A.

2.1 Principles of Intuitionistic Knowledge

Given the assumption that intuitionistic knowledge is the result of verification, CT and FK may be seen as expressing two informal principles about the relationship between truth and verification-based knowledge:

1. proof yields verification-based knowledge;

2. verification-based knowledge yields proof.

A BHK-compliant view of knowledge accepts 1 and rejects 2.

2.1.1 Proof yields knowledge

The principle that proof yields verification is practically constitutive of the concept of proof,[4] precisely because **proofs are a special and most strict kind of verification**, and immediately justifies the validity of the formal principle CT.

That proofs are taken to be verifications is a matter of the ordinary usage of the term which understands a proof as "an argument that establishes the validity of a proposition" [65]. It is also a fairly universal view in mathematics (cf. [7, 66, 72]). Within computer science this concept is the cornerstone of a big and vibrant area in which one of the key purposes of computer-aided proofs is for the verification of the propositions in question,[10, 11]. Amongst intuitionists the idea of a constructive proof is often treated as simply synonymous with verification [21, 25, 44]. Hence CT should be read as expressing the constructive nature of intuitionistic truth, which itself being a strict verification yields verification-based knowledge.[5]

[4]Though not common in mainstream epistemology there are, or have been, mathematical skeptics. Perhaps the best known mathematical skeptical argument is the one Descartes puts forward in [18], see also [30, 31, 32]. See also [33] and [45] who both discuss the skeptical consequences of empiricism regarding mathematical knowledge.

[5]Martino and Usberti seem to have the informal principle that proof yields knowledge in mind when they say, [53]: "...[CT] can be interpreted only according to the intuitionistic meaning of implication, so that it expresses the trivial observation that, as soon as a proof of A is given, A becomes known."

According to the BHK reading of intuitionistic implication CT states that **given a proof of A one can always construct a proof of $\mathbf{K}A$.** Is such a construction always possible? Indeed, it is well established that proof-checking is a valid operation on proofs,[6] so if x is a proof of A then it can be proof-checked and hence produce a proof $p(x)$ of 'x is a proof of A.' Having checked a proof we have a proof that the proposition is proved, hence verified, hence known. In whatever sense we consider a proof to be possible, or to exist, CT states that the proof-checking of this proof is always possible, or exists, in the same sense. So, by the principle that proof yields verification we have that a proof of A yields knowledge of A, and by proof-checking we obtain a proof of $\mathbf{K}A$.

We are not, of course, the first to outline arguments that an intuitionistic conception of truth validates CT, see for instance [17, 37, 46, 53, 54, 55, 75, 76, 79]. Our contention is that this principle, *when properly understood in line with the intended BHK semantics*, is a fairly immediate consequence of uncontroversially intuitionistic views about truth, and should therefore be endorsed as part of a properly intuitionistic conception of knowledge.

2.1.2 Knowledge does not yield proof

If not all verifications are BHK-compliant proofs then it follows that verification-based knowledge does not yield proof, and consequently that FK is not a valid intuitionistic epistemic principle. It is possible to have knowledge of a proposition without it being intuitionistically true, i.e. proved.[7]

FK is a distinctly classical epistemic principle, because classical truth is "verification-transcendent." Since a proposition can be true even if it is not in principle possible to know it, if it is classically known then it must be classically true. On the other hand, for FK to be BHK valid there should be a uniform procedure which given a proof of $\mathbf{K}A$ returns a proof of A itself. Since we allow that $\mathbf{K}A$ does not necessarily produce specific proofs this requirement cannot be met. What uniform procedure is there that can take any adequate, non-proof, verification of A and return a proof of A? There is no such construction. Consider the following counter-examples:

Zero-knowledge protocols A class of cryptographic protocols, normally probabilistic, by which the prover can prove to the verifier that a given statement is true, without conveying any additional information apart from the fact that that statement is true.

Testimony of an authority Even concerning mathematical knowledge FK fails. Take Fermat's Last Theorem. For the educated mathematician it is credible to claim that it is known, but most mathematicians could not produce a proof of it. Indeed, more generally,

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[6] [1, 36, 43, 48]. Moreover, proof-checking is generally a feasible operation, routinely implemented in a standard computer-aided proof package.

[7] "...**K** requires more than warranted assertion. However, it does not follow that **K** requires strict proof; that would not be a reasonable requirement when **K** is applied to empirical statements...", [77, p.68].

any claim to mathematical knowledge based on the authority of mathematical experts is not intuitionistically factive. It is legitimate to claim to know a theorem when one understands its content, and can use it in one's reasoning, without being in a position to produce or recite the proof.

Highly probable truth Suppose there is a computerized probabilistic verification procedure, which is constructive in nature, that supports a proposition A with a cosmologically small probability of error, so its result satisfies the strictest conceivable criteria for truth. Then any reasonable agent accepts this certification as adequate justification of A, hence A is known. Moreover, observing the computer program to terminate with success, we have a proof that $\mathbf{K}A$. However, we do not have a proof of A in the sense required by the BHK clause for implication.

Existential generalisation Somebody stole your wallet in the Rome subway. You have all the evidence for this: the wallet is gone, your backpack has a cut in the corresponding pocket, but you have no idea who did it. You definitely know that "there is a person who stole my wallet" (in logical form, $\exists x S(x)$, where $S(x)$ stands for "x stole my wallet") so you have a justification p of $\mathbf{K}(\exists x S(x))$. If $\mathbf{K}(\exists x S(x)) \to \exists x S(x)$ held intuitionistically, you would have a constructive proof q of $\exists x S(x)$. However, a constructive proof of the existential sentence $\exists x S(x)$ requires a witness a for x and a proof b that $S(a)$ holds. You are nowhere near meeting this requirement. So, $\mathbf{K}(\exists x S(x)) \to \exists x S(x)$ does not hold intuitionistically.

Classified sources In a social situation, imagine a statement of A coming from a most reliable source but with a classified origin. So, there is no access to the "strict proof" of A. Should we abstain from reasoning about A as something known unless we gain full access to the strict proof. This is not how society works. We treat $\mathbf{K}A$ as weaker than A, and keep reasoning constructively without drawing the conclusion that A.

Perceptual Knowledge Under optimal lighting conditions with all my cognitive faculties working normally I hold up a hand before me and perceive that there is a hand before me. For all practical purposes this perception is a verification of this fact, so **K**(there is a hand before me) holds, but there is no reason to claim having a natural BHK-compliant proof of this.

If we allow that knowledge may be gained by any of the methods above then FK is not valid according to the BHK semantics.

2.2 Knowledge and Falsity

Nevertheless FK is taken to be practically definitive of knowledge, especially from a constructive standpoint. For instance, Williamson [77], in outlining his system of intuitionistic epistemic logic affirms that $\mathbf{K}A \to A$ holds. Similarly Proietti, [64], argues that knowledge is factive in his system of intuitionistic epistemic logic. Wright states that an operator could not be a knowledge operator if it were not factive [81].[8] More generally still the principle $\mathbf{K}A \to A$ is probably the only principle about knowledge that has not been seriously contested.[9] And, of course, it is implied by virtually every extant definition of knowledge. Must not our arguments above be wrong in some fashion? Are we not arguing the intuitionist is committed to holding that false propositions can be known? No, such a predicament would hold only if FK failed classically.

Every analysis of knowledge agrees that **only true propositions can be known** and that **false propositions cannot be known**. Apart from FK, there are natural logical ways to express these:

1. $\neg(\mathbf{K}A \wedge \neg A)$;
2. $\neg A \to \neg\mathbf{K}A$;
3. $\mathbf{K}A \to \neg\neg A$;
4. $\neg\neg(\mathbf{K}A \to A)$;
5. $\neg\mathbf{K}\bot$.

Intuitionistically 1, 2 and 5 can be considered as saying that knowledge of falsehood is impossible. 3 expresses that knowledge is stronger than classical truth, and 4 that FK is a distinctly classical principle.[10]

Principles 1 – 4 are classically equivalent to FK, and all 1 – 5 are intuitionistically strictly weaker (see section 3). In this way the intuitionist maintains the impossibility of knowing false propositions, while holding that knowledge is not factive in the sense of FK.

Which of principles 1 – 5 best expresses the truth condition on knowledge? Any of them will do, as in the presence of CT all 1 – 5 are equivalent, (see theorem 2); so we pick 5 as being the simplest.

[8] "I take this to be a non-negotiable feature of the concept of knowledge. If a theory takes a view of something which it purports to regard as knowledge, but which lacks this feature, it is not a theory of knowledge" [81, p.242].

[9] Hazlett's [38, 39] would seem to be the only such challenge. However Hazlett challenges the view that utterances of 'S knows A' imply A. Put another way he challenges the idea that the truth of A is necessary for the truth of the utterance 'S knows A'; utterances of 'S knows A' may be true even if A is false. He is careful to distinguish this challenge from the claim that it is possible to know false propositions – that he does not challenge. Hazlett's arguments do not appear to be relevant to our concerns since we are not occupied with the truth conditions of utterances of knowledge ascriptions, but the logical analysis of the epistemic operator.

[10] This follows from the double negation translation of intuitionistic logic into classical logic and Glivenko's Theorem.

3 Intuitionistic Epistemic Logic (IEL)

We are now in a position to define a system of intuitionistic epistemic logic which, we argue, respects the intended BHK meaning of intuitionism and which incorporates a reasonable verification-based knowledge operator.

IEL is the system of Intuitionistic Epistemic Logic. The language is that of intuitionistic propositional logic augmented with the propositional operator $\mathbf{K}$.

Axioms

1. *Axioms of propositional intuitionistic logic.*
2. $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$
3. $A \to \mathbf{K}A$
4. $\neg\mathbf{K}\bot$

Rules

Modus Ponens

Proposition 1 *In* IEL

1. *The rule of* $\mathbf{K}$-necessitation, $\vdash A \ / \ \vdash \mathbf{K}A$, *is derivable.*
2. IEL *is a normal modal logic.*
3. *The Deduction Theorem holds.*
4. *Uniform Substitution holds.*
5. *Positive and Negative Introspection hold;* $\vdash \mathbf{K}P \to \mathbf{KK}P$, $\vdash \neg\mathbf{K}P \to \mathbf{K}\neg\mathbf{K}P$.

Proof.

1. Assume $\vdash A$. By axiom $A \to \mathbf{K}A$ it follows that $\vdash \mathbf{K}A$.
2. By definition IEL contains all propositional intuitionistic validities and all instances of $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$, and is closed under *modus ponens*; by 1, IEL is closed under $\mathbf{K}$-*necessitation.*
3. From 1, and the fact that intuitionistic propositional logic validates the deduction theorem.
4. By induction on the complexity of formulas.

5. Both are instances of axiom $A \to \mathbf{K}A$, with $\mathbf{K}P$ and $\neg\mathbf{K}P$ for A respectively.

□

Theorem 1 *The following are all theorems of* IEL

1. $\vdash \neg(\mathbf{K}A \wedge \neg A)$*;*

2. $\vdash \mathbf{K}A \to \neg\neg A$*;*[11]

3. $\vdash \neg\neg(\mathbf{K}A \to A)$*;*

4. $\vdash \neg A \to \neg\mathbf{K}A$*.*

Proof.

For 1:

1. $\mathbf{K}A \wedge \neg A$ - assumption;
2. $\neg A \to \mathbf{K}(\neg A)$ - axiom 3;
3. $\mathbf{K}A \wedge \mathbf{K}(\neg A)$ - from 1 and 2;
4. $\mathbf{K}(A \wedge \neg A)$ - from 3;
5. $\mathbf{K}\bot$ - from 4;
6. $\mathbf{K}\bot \to \bot$ - axiom 4;
7. $\bot$ 5, 6 MP;
8. $(\mathbf{K}A \wedge \neg A) \to \bot$ i.e. $\neg(\mathbf{K}A \wedge \neg A)$.

For 2, continue with:

9. $\mathbf{K}A \to (\neg A \to \bot)$ - from 8;
10. $\mathbf{K}A \to (\neg\neg A)$.

For 3, continue with:

11. $\neg\neg\neg A \to \neg\mathbf{K}A$ contrapositive of 10;
12. $\neg A \to \neg\mathbf{K}A$ by $\neg X \leftrightarrow \neg\neg\neg X$;
13. $\neg\neg\mathbf{K}A \to \neg\neg A$ contrapositive of 12;
14. $\neg\neg(\mathbf{K}A \to A)$ by $(\neg\neg X \to \neg\neg Y) \leftrightarrow \neg\neg(X \to Y)$.

For 4:

1. $\neg(\mathbf{K}A \wedge \neg A)$ Part 1;
2. $\neg(\neg A \wedge \mathbf{K}A)$;
3. $\neg A \to \neg\mathbf{K}A$.

□

[11] With Axiom 2 this gives us $A \to \mathbf{K}A \to \neg\neg A$ in IEL, i.e. intuitionistic truth is stronger than knowledge is stronger than classical truth.

Consider the system IEL^0 which is IEL without axiom $\neg\mathbf{K}\bot$. It turns out that IEL^0 with each of $\neg(\mathbf{K}A \wedge \neg A)$, $\mathbf{K}A \to \neg\neg A$, $\neg\neg(\mathbf{K}A \to A)$, $\neg A \to \neg\mathbf{K}A$ as additional axioms is equivalent to IEL.

Theorem 2 *Each of*

$\mathit{IEL}^0 + \neg(\mathbf{K}A \wedge \neg A)$,
$\mathit{IEL}^0 + \mathbf{K}A \to \neg\neg A$,
$\mathit{IEL}^0 + \neg\neg(\mathbf{K}A \to A)$ *and*
$\mathit{IEL}^0 + \neg A \to \neg\mathbf{K}A$

proves $\neg\mathbf{K}\bot$*, hence each is equivalent to* IEL.

Proof.

$\mathsf{IEL}^0 + \neg(\mathbf{K}A \wedge \neg A) \vdash \neg\mathbf{K}\bot$

1. $\neg(\mathbf{K}\bot \wedge \neg\bot)$, instance of $\neg(\mathbf{K}A \wedge \neg A)$;
2. $\neg(\mathbf{K}\bot \wedge (\bot \to \bot))$;
3. $(\mathbf{K}\bot \wedge (\bot \to \bot)) \to \bot$;
4. $\mathbf{K}\bot \to ((\bot \to \bot) \to \bot)$;
5. $\mathbf{K}\bot \to \bot$, i.e. $\neg\mathbf{K}\bot$, by $((\bot \to \bot) \to \bot) \leftrightarrow \bot$.

$\mathsf{IEL}^0 + \mathbf{K}A \to \neg\neg A \vdash \neg\mathbf{K}\bot$

1. $\mathbf{K}\bot \to \neg\neg\bot$, instance of $\mathbf{K}A \to \neg\neg A$;
2. $\mathbf{K}\bot \to ((\bot \to \bot) \to \bot)$;
3. $\mathbf{K}\bot \to \bot$, i.e. $\neg\mathbf{K}\bot$.

$\mathsf{IEL}^0 + \neg\neg(\mathbf{K}A \to A) \vdash \neg\mathbf{K}\bot$

1. $\neg\neg(\mathbf{K}\bot \to \bot)$ instance of $\neg\neg(\mathbf{K}A \to A)$;
2. $\neg\neg\neg\mathbf{K}\bot$ equivalent to 1;
3. $\neg\mathbf{K}\bot$ by $\neg\neg\neg X \leftrightarrow \neg X$.

$\mathsf{IEL}^0 + \neg A \to \neg\mathbf{K}A \vdash \neg\mathbf{K}\bot$

1. $\neg\bot \to \neg\mathbf{K}\bot$ instance of $\neg A \to \neg\mathbf{K}A$;
2. $\neg\bot$ tautology;
3. $\neg\mathbf{K}\bot$.

□

Since each of $\neg(\mathbf{K}A \wedge \neg A)$, $\mathbf{K}A \to \neg\neg A$, $\neg\neg(\mathbf{K}A \to A)$ and $\neg A \to \neg\mathbf{K}A$ can be regarded as expressing the truth condition on knowledge, we see that the axiom $\neg\mathbf{K}\bot$ is an adequate intuitionistic expression of this idea. (For further discussion of the intuitionistic truth condition and the justification for adopting $\neg\mathbf{K}\bot$ as the proper intuitionistic formulation of the condition see Appendix B).

3.1 K as ¬¬

Došen [19] proposes an intuitionistic modal logic, Hdn$\Box$, in which $\Box$ is read as intuitionistic ¬¬, i.e.

$$\Box A \leftrightarrow \neg\neg A.$$

Hdn$\Box$ validates $A \to \Box A$ and invalidates $\Box A \to A$. Could Dosen's $\Box$ be an intuitionistic epistemic operator?

We argue not. If it were it would follow that all classical theorems are known intuitionistically. By Glivenko's Theorem, if $\mathsf{Cl} \vdash A$ then $\mathsf{Int} \vdash \neg\neg A$, and hence $\mathsf{Hdn}\Box \vdash \mathbf{K}A$. Such a **K** is not intuitionistic knowledge but rather a simulation of classical knowledge within Int[4].

4 Models for IEL

Definition 1 (IEL Model) A model for IEL is a quadruple $< W, R, E, \Vdash>$ such that:

1. W is a non-empty set of states;
2. R is a transitive and reflexive binary relation on W, a standard intuitionistic 'cognition relation';
3. E is a binary 'knowledge' relation s.t.
   - $E(u)$ is non-empty;[12]
   - $E \subseteq R$, i.e. $E(u) \subseteq R(u)$ for any state u;
   - $R \circ E \subseteq E$, i.e. uRv yields $E(v) \subseteq E(u)$;
4. $\Vdash$ is an evaluation function such that
   - $u \nVdash \bot$;
   - for atomic p if $u \Vdash p$ and uRv then $v \Vdash p$.

Given a model $< W, R, E, \Vdash>$, the evaluation '$\Vdash$' extends to all IEL-formulas by the standard Kripke conditions for intuitionistic models:

- $u \Vdash A \wedge B$ iff $u \Vdash A$ and $u \Vdash B$;
- $u \Vdash A \vee B$ iff $u \Vdash A$ or $u \Vdash B$;
- $u \Vdash A \to B$ iff for each $v \in R(u)$, $v \Vdash B$ or $v \nVdash A$;

[12] Let $R(u)$ and $E(u)$ denote the R-successors and the E-successors, respectively, of some state u

along with the Kripkean knowledge condition with respect to the 'knowledge' relation E:

- $u \Vdash \mathbf{K}A$ iff $v \Vdash A$ for all $v \in E(u)$.

A formula F is true in a model, if F holds at each world of this model, $\mathsf{IEL} \Vdash F$, or $\Vdash F$ for short, means that F holds in each IEL-model.

In Kripke model-theoretic terms the intuitionistic truth of A is represented as the impossibility of a situation in which A does not hold. To represent $\mathbf{K}$ in the same model-theoretic terms we suggest the following: in a given world u, there is an "audit" set of possible worlds $E(u)$, the set of states E-accessible from u, in which verifications, though not necessarily strict proofs, could possibly occur. An R-successor of a state u can be thought of as an "in principle (logically) possible" cognition state given u, and an E-successor can be thought of as a "possible" state of verification. Knowledge, hence, is "truth in any audit set," i.e. no matter when and how an audit occurs, it should confirm A.

That such audits are correct is reflected in the condition that the audit set for any given u, $E(u)$, cannot be empty since in such u's $\mathbf{K}\bot$ would hold vacuously, hence $\neg\mathbf{K}\bot$, no verification can certify a false statement.

Note that $E(u)$ does not necessarily contain u, hence the truth of $\mathbf{K}A$ at u does not guarantee that A holds at u. Therefore, $\mathbf{K}A \to A$ does not necessarily hold. In the extreme $E(u)$ can coincide with $R(u)$, hence an audit can take place in any situation, in which case $\mathbf{K}A \to A$ would hold. Furthermore, the condition $E \subseteq R$ coupled with the monotonicity of R ensures the validity of $A \to \mathbf{K}A$.

The condition $R \circ E \subseteq E$ expresses the set-theoretical monotonicity of audit sets with respect to intuitionistic accessibility. This corresponds to the Kripkean ideology that R denotes the discovery process (of the ideal researcher), and that things become more and more certain in the process of discovery. As the set of intuitionistic possibilities, $R(u)$, shrink, audit sets, $E(u)$, shrink as well. In the limit case where $R(u) = \{u\}$, the audit set $E(u)$ is also $\{u\}$, and hence coincides with $R(u)$. Note that at such "leaf" worlds, intuitionistic evaluation behaves classically; at such a u, $u \Vdash \mathbf{K}A \to A$ for all A's. In the epistemic case, at leaf worlds the reflexivity of $\mathbf{K}$ – a typical classical epistemic principle – holds.

As for intuitionistic logic, we can think of IEL models as representing the states of information of an ideal researcher, with each state representing the stock of propositions which are verified at that state. The monotonicity of truth represents the idealization of the researcher's memory; once a proposition becomes true, its truth at that state is retained forever.[13]

[13]Our language cannot express this, but we can think of the objects of the ideal researcher's beliefs as eternal sentences, stating that 'A holds at state x', a model models when the researcher discovers its truth, and their subsequent retention of it.

5 Properties of IEL models

Lemma 1 (Monotonicity) *For each formula A, if $u \Vdash A$ and uRv then $v \Vdash A$.*

Proof. Monotonicity holds for the propositional connectives, we show this just for $\mathbf{K}$. Assume $u \Vdash \mathbf{K}p$, then $x \Vdash p$ for each $x \in E(u)$. Take an arbitrary v such that uRv and arbitrary w such that vEw. By the condition on $R \circ E$ in the model, $uRvEw$ yields uEw, hence $w \in E(u)$. Therefore, $w \Vdash p$ and hence $v \Vdash \mathbf{K}p$. □

Theorem 3 (Soundness) *If* $\mathsf{IEL} \vdash A$ *then* $\mathsf{IEL} \Vdash A$.

Proof. By induction on derivations in IEL. We check the epistemic clauses only.

1) $A \to \mathbf{K}A$. Assume $x \Vdash A$ for some arbitrary state in an IEL model. Hence, by monotonicity of $\Vdash$, for all $y \in R(x)$, $y \Vdash A$. Since $E(x) \subseteq R(x)$, for any $z \in E(x)$, $z \Vdash A$, but then $x \Vdash \mathbf{K}A$ as well.

2) $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$. Due to monotonicity, it suffices to check that $x \Vdash \mathbf{K}(A \to B)$ and $x \Vdash \mathbf{K}A$ yields $x \Vdash \mathbf{K}B$. Assume $x \Vdash \mathbf{K}(A \to B)$ and $x \Vdash \mathbf{K}A$, then all $y \in E(x)$, $y \Vdash A \to B$ and $y \Vdash A$, hence $y \Vdash B$. By definition, this means that $x \Vdash \mathbf{K}B$.

3) $\neg\mathbf{K}\bot$. Take a state x and $y \in E(x)$. This can be done since $E(x)$ is not empty. Since $y \nVdash \bot$, $x \nVdash \mathbf{K}\bot$. □

Theorem 4 (Completeness) *If* $\mathsf{IEL} \Vdash A$ *then* $\mathsf{IEL} \vdash A$.

Proof. See Appendix A. □

Theorem 5 $\mathsf{IEL} \nvdash \mathbf{K}A \to A$

Proof. Consider the following model: $1R2$, R is reflexive (and vacuously transitive), $E(1) = E(2) = \{2\}$, p is atomic and $2 \Vdash p$. Clearly, $1 \Vdash \mathbf{K}p$ and $1 \nVdash p$.

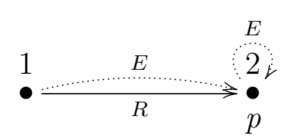

Figure 1: Model $\mathcal{M}_1$

□

Though reflection does not hold generally it does hold for negative formulas.[14]

[14] This is no surprise since a corollary of Glivenko's Theorem is that $\mathsf{Cl} \vdash \neg A \Leftrightarrow \mathsf{Int} \vdash \neg A$, see [8, 35], and FK holds of classical truths.

Theorem 6 $\mathsf{IEL} \vdash \mathbf{K}\neg A \to \neg A$.

Proof.

1. $\mathbf{K}\neg A \wedge A$, assumption;
2. $A \to \mathbf{K}A$, Axiom;
3. $\mathbf{K}A$, from 1, 2;
4. $\mathbf{K}A \wedge \mathbf{K}\neg A$, from 1, 3;
5. $\mathbf{K}(A \wedge \neg A)$, from 4;
6. $\mathbf{K}\bot$, from 5;
7. $\neg\mathbf{K}\bot$, Axiom;
8. $\bot$, from 6, 7;
9. $\neg(\mathbf{K}\neg A \wedge A)$, from 1–8;
10. $\mathbf{K}\neg A \to \neg A$, from 9, by standard intuitionistic reasoning. □

In IEL knowledge and negation commute: the impossibility of verifying A is equivalent to verifying that A cannot possibly hold.

Theorem 7 $\mathsf{IEL} \vdash \neg\mathbf{K}A \leftrightarrow \mathbf{K}\neg A$

Proof. '$\leftarrow$' follows by Theorem 6 and Theorem 1 part 4. Let us check '$\rightarrow$':

1. $\neg\mathbf{K}A$, hypothesis;
2. $A \to \mathbf{K}A$, Axiom 3;
3. $\neg\mathbf{K}A \to \neg A$, from 2;
4. $\neg A$, from 2 and 3;
5. $\neg A \to \mathbf{K}\neg A$, Axiom 3;
6. $\mathbf{K}\neg A$, from 4 and 5. □

In IEL the impossibility of verification is equivalent to the impossibility of proof, see section 7.4 for discussion.

Theorem 8 $\mathsf{IEL} \vdash \neg\mathbf{K}A \leftrightarrow \neg A$

Proof.

1. $\neg\mathbf{K}A$, assumption;
2. $\mathbf{K}\neg A$, from 1 and Theorem 7;
3. $\neg A$, from 2 and Theorem 6;

1. $\neg A$, assumption;
2. $\mathbf{K}\neg A$, from 1 and Axiom 1;
3. $\neg\mathbf{K}A$, from 2 and Theorem 7. □

Within the IEL-framework, no truth is unverifiable, see section 7.4 for discussion.

Theorem 9 $\mathsf{IEL} \vdash \neg(\neg\mathbf{K}A \wedge \neg\mathbf{K}\neg A)$

Proof.
1. $\neg\mathbf{K}A \wedge \neg\mathbf{K}\neg A$, assumption;
2. $\mathbf{K}\neg A \wedge \mathbf{K}\neg\neg A$, by Theorem 7;
3. $\neg A \wedge \neg\neg A$, by Theorem 6;
4. $\perp$ 3;
5. $\neg(\neg\mathbf{K}A \wedge \neg\mathbf{K}\neg A)$, 1–4. □

Intuitionistic verifications do not have the disjunction property.

Theorem 10 $\mathsf{IEL} \nvdash \mathbf{K}(A \vee B) \to (\mathbf{K}A \vee \mathbf{K}B)$

Proof. Consider the following model. $1R2, 1R3$ (R is reflexive); $1E2, 1E3, 2E2, 3E3$; p is atomic and $3 \Vdash p$.

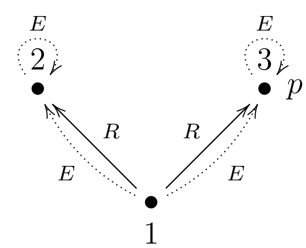

Figure 2: Model $\mathcal{M}_2$

Since $2 \nVdash p$, $2 \Vdash \neg p$, hence $2 \Vdash p \vee \neg p$. Since $3 \Vdash p$, $3 \Vdash p \vee \neg p$. Hence $1 \Vdash \mathbf{K}(p \vee \neg p)$. However, $1 \nVdash \mathbf{K}p$, and $1 \nVdash \mathbf{K}\neg p$. □

While verifying a disjunction does not require verification of one of the disjuncts, it does require that verification of one of the disjuncts be possible.

Theorem 11 $\mathbf{K}(A \vee B) \wedge (\neg\mathbf{K}A \wedge \neg\mathbf{K}B)$ *is inconsistent in* IEL.

Proof.
1. $\mathbf{K}(A \vee B) \wedge (\neg\mathbf{K}A \wedge \neg\mathbf{K}B)$, assumption;
2. $\neg\mathbf{K}A$, from 1;
3. $\neg\mathbf{K}B$, from 1;
4. $\neg\mathbf{K}A \to \mathbf{K}\neg A$, Theorem 7;
5. $\neg\mathbf{K}B \to \mathbf{K}\neg B$, Theorem 7;
6. $\mathbf{K}\neg A$, from 1 and 4;
7. $\mathbf{K}\neg B$, from 2 and 5;
8. $\mathbf{K}(\neg A \wedge \neg B)$, from 6 and 7;

9. $\mathbf{K}\neg(A \vee B)$, intuitionistic De Morgan;
10 $\neg\mathbf{K}(A \vee B)$, Theorem 7;
11. $\perp$, from 1 and 7. □

Theorem 12 *The rule* $\vdash \mathbf{K}A \ / \vdash A$ *is admissible.*

Proof. Suppose $\nvdash A$, hence, by completeness, there is a model $\mathcal{M} =< W, R, E, \Vdash>$ with a node $x \in W$ s.t. $x \nVdash A$. Construct a new model, $\mathcal{N} =< W', R', E', \Vdash'>$ such that

- $W' = W \cup \{x_0\}$ (x_0 is a new node);
- $x_0 R' u$ and $x_0 E' u$ for all $u \in W'$, R' coincides with R and E' coincides with E on W;
- $x_0 \nVdash' p$ for each atomic sentence p and $\Vdash'$ coincides with $\Vdash$ on W.

Clearly $\mathcal{N}$ is an IEL-model. Moreover, $\mathcal{M}$ is a generated submodel of $\mathcal{N}$, hence $\Vdash'$ coincides with $\Vdash$ on W. Furthermore, $x_0 \nVdash' \mathbf{K}A$, since $x \nVdash' A$ and $x_0 E x$. Therefore, IEL $\nvdash \mathbf{K}A$ also. □

Theorem 13 (Disjunction Property) *If* IEL $\vdash A \vee B$ *then either* IEL $\vdash A$ *or* IEL $\vdash B$.

Proof. Assume $\nvdash A$ and $\nvdash B$. By completeness, $\nVdash A$ and $\nVdash B$. Hence there are models $\mathcal{M}_1 =< W_1, R_1, E_1, \Vdash_1>$ and $\mathcal{M}_2 =< W_2, R_2, E_2, \Vdash_2>$ with nodes $x_1 \in W_1$ and $x_2 \in W_2$ such that $x_1 \nVdash_1 A$ and $x_2 \nVdash_2 B$. We define a new model $\mathcal{M} =< W, R, E, \Vdash>$ such that

- $W = W_1 \cup W_2 \cup \{x_0\}$ where $x_0 \notin W_1$ and $x_0 \notin W_2$.
- $x_0 R u$ and $x_0 E u$ for all $u \in W$, R coincides with R_i on W_i, and E coincides with E_i on $i = 1, 2$.
- $x_0 \nVdash p$ for each atomic sentence p, $\Vdash$ coincides with $\Vdash_i$ on W_i, $i = 1, 2$.

It is easy to check that for each $i = 1, 2$ and each $x \in W_i$,

$$x \Vdash F \quad \textit{iff} \quad x \Vdash_i F.$$

We claim that $x_0 \nVdash A \vee B$, hence $\nvdash A \vee B$. Indeed, if $x_0 \Vdash A \vee B$, then $x_0 \Vdash A$ or $x_0 \Vdash B$. If $x_0 \Vdash A$ then, by monotonicity, $x_1 \Vdash A$, hence $x_1 \Vdash_1 A$ which contradicts our assumptions. Case $x_0 \Vdash B$ is symmetric. □

Despite Theorem 10, IEL has a weak disjunction property for verifications.

Corollary 1 *If* $\vdash \mathbf{K}(A \vee B)$ *then either* $\vdash \mathbf{K}A$ *or* $\vdash \mathbf{K}B$.

Proof. Assume IEL $\vdash \mathbf{K}(A \vee B)$ then, by Theorem 12, $\vdash A \vee B$, hence $\vdash A$ or $\vdash B$. In which case $\vdash \mathbf{K}A$ or $\vdash \mathbf{K}B$ by $\mathbf{K}$-necessitation. □

6 Modeling IEL via Provability and Verification.

In [36], Gödel offered a provability semantics for intuitionistic logic IPC by means of a syntactical embedding of IPC into the modal logic S4 which he considered a calculus for classical provability. Gödel's embedding later became a key ingredient of the BHK-style semantics of classical proofs for intuitionistic logic via the Logic of Proofs, cf. [1]. Though a faithful intuitionist may question the merits of the very goal of finding a classical semantics for intuitionistic systems, others, e.g. those who stay within the position of classical mathematics, could find useful insights from such semantics.[15]

We will establish the embedding/interpretation of IEL into a natural version of Gödel's provability logic S4 augmented by a verification modality, which we suggest calling S4V. Basically, we will explain IEL via classical provability and verification coded in S4V the way Gödel explained intuitionistic logic by interpreting it in the logic of provability S4 [36].

System S4V is a classical logic with two modalities, $\Box$ for the S4-type provability modality, and $\mathbf{K}$ for a verification modality. The logical principles of S4V include the axioms and rules of modal logic S4 for $\Box$ and of the modal logic D for $\mathbf{K}$: $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$, $\neg\mathbf{K}\bot$. The latter principle reflects our assumption that the verification procedure is sound, i.e. does not verify false statement. Naturally, S4V includes the connection axiom

$$\Box A \to \mathbf{K}A,$$

stating that a proof of A counts as a verification of A.

Since we are in a classical setting why not assume $\mathbf{K}A \to A$ as an axiom? Why only just consistency, $\neg\mathbf{K}\bot$? This turns out to be a delicate issue. If we assume $\mathbf{K}$-reflection, then we have $\mathbf{K}\Box A \to \Box A$, $\Box(\mathbf{K}\Box A \to \Box A)$, by $\Box$-necessitation, and then, by straightforward modal reasoning,

$$\Box\mathbf{K}\Box A \to \Box A,$$

which is a classical version (supported by Gödel's translation) of

if A is intuitionistically known, then A is intuitionistically true.

So, adopting $\Box$-necessitation of $\mathbf{K}$-reflection leads to the collapse of intuitionistic knowledge into intuitionistic truth, which is not acceptable within our framework.

One way to save $\mathbf{K}$-reflection would be to remove it from the scope of $\Box$-necessitation. This would lead to a non-normal modal logic and will open many new natural questions, interesting in themselves but which we leave to future work. As a first step we begin with a minimalistic, "vanilla", logic of provability and verification, S4V, all the principles of which are non-controversial, and which is sufficient both for Gödel's embedding of IEL and for distinguishing reflection from co-reflection of intuitionistic knowledge. Certainly we have no *a priori* objection to the study of other bi-modal logics of classical provability and

[15] After all, the paradigmatic Kripke semantics for intuitionistic logic is itself classical [14], which does not make it any less useful and insightful as a tool for studying intuitionistic logic.

knowledge (which may or may not incorporate reflection) both as a topic interesting in itself and in virtue of its connections to intuitionistic epistemology.

Definition 2 The list of postulates of S4V consists of

- *axioms and rules of modal logic S4 for* $\Box$;
- *axioms and rules of modal logic* D *for* $\mathbf{K}$;
- $\Box A \to \mathbf{K}A$.

Example 1 The following is derivable in S4V: $\Box A \to \Box\mathbf{K}A \to \mathbf{K}A$. Indeed,

1. $\Box A \to \Box\Box A$, $\Box$-transitivity;
2. $\Box A \to \mathbf{K}A$, connection;
3. $\Box\Box A \to \Box\mathbf{K}A$, $\Box$-necessitation and distributivity;
4. $\Box A \to \Box\mathbf{K}A$, from 1,3 by propositional reasoning;
5. $\Box\mathbf{K}A \to \mathbf{K}A$, $\Box$-reflection.

Now we focus on spelling out how exactly IEL can be interpreted in S4V. Gödel in [36] offered translating intuitionistic formulas into the classical language with a provability modality, essentially equivalent in S4 to the rule "box each subformula." Since $\Box$ denotes provability, this was for Gödel an elegant syntactical way of representing the *truth is provability* paradigm. Gödel's translation suggests reading $\mathbf{K}p$ in S4V as $\Box\mathbf{K}\Box p$ "there is a proof that provability of p is verified." Since "provability of X" is the classical approximation to "X holds intuitionistically," Gödel's translation supports IEL's reading of $\mathbf{K}p$ as

"it is intuitionistically true that intuitionistic truth of p has been verified."

Definition 3 $tr(F)$ is the result of prefixing each subformula in F with a $\Box$.

Example 2 Here are some examples of the translation. For brevity, we do not distinguish equivalent formulas $\bot$ and $\Box\bot$:

- $tr(p) = \Box p$, for an atomic p;
- $tr(\mathbf{K}p) = \Box\mathbf{K}\Box p$;
- $tr(\neg\mathbf{K}\bot) = \Box\neg\Box\mathbf{K}\bot$;
- $tr(p \to \mathbf{K}p) = \Box(\Box p \to \Box\mathbf{K}\Box p)$.
- $tr(\mathbf{K}p \to \mathbf{K}q) = \Box(\Box\mathbf{K}\Box p \to \Box\mathbf{K}\Box q)$.

Theorem 14 (Provability/verification soundness of IEL).

$$\mathsf{IEL} \vdash F \quad \Rightarrow \quad \mathsf{S4V} \vdash tr(F).$$

Proof. By induction on derivations of F in IEL.

Case 1. F is an axiom of IPC – this has been checked by Gödel, since our translation coincides with Gödel's on intuitionistic atoms and connectives.

Case 2. F is $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$. Without loss of generality, we can make a simplifying assumption that A and B are propositional letters. Then $tr(F)$ is

$$tr(F) = \Box(\Box\mathbf{K}\Box(\Box A \to \Box B) \to \Box(\Box\mathbf{K}\Box A \to \Box\mathbf{K}\Box B))$$

which is derivable in S4V:

1. $\Box(\Box A \to \Box B) \to (\Box\Box A \to \Box\Box B)$, S4 theorem;
2. $\mathbf{K}\Box(\Box A \to \Box B) \to \mathbf{K}(\Box\Box A \to \Box\Box B)$, $\mathbf{K}$-necessitation and distribution;
3. $\mathbf{K}\Box(\Box A \to \Box B) \to \mathbf{K}(\Box A \to \Box B)$, by $\Box A \leftrightarrow \Box\Box A$;
4. $\mathbf{K}\Box(\Box A \to \Box B) \to (\mathbf{K}\Box A \to \mathbf{K}\Box B)$, from 3, $\mathbf{K}$-distribution;
5. $\Box\mathbf{K}\Box(\Box A \to \Box B) \to \Box(\mathbf{K}\Box A \to \mathbf{K}\Box B)$, $\Box$-necessitation and distribution;
6. $\Box\mathbf{K}\Box(\Box A \to \Box B) \to (\Box\mathbf{K}\Box A \to \Box\mathbf{K}\Box B)$, from 5, $\Box$-distribution;
7. $\Box\Box\mathbf{K}\Box(\Box A \to \Box B) \to \Box(\Box\mathbf{K}\Box A \to \Box\mathbf{K}\Box B)$, $\Box$-necessitation and distribution;
8. $\Box\mathbf{K}\Box(\Box A \to \Box B) \to \Box(\Box\mathbf{K}\Box A \to \Box\mathbf{K}\Box B)$, by $\Box X \leftrightarrow \Box\Box X$;
9. $\Box(\Box\mathbf{K}\Box(\Box A \to \Box B) \to \Box(\Box\mathbf{K}\Box A \to \Box\mathbf{K}\Box B))$.

Case 3. F is $A \to \mathbf{K}A$. Then $tr(F) = \Box(\Box A \to \Box\mathbf{K}\Box A)$. The proof is as for Example 1 with $\Box A$ for A.

Case 4. F is $\neg\mathbf{K}\bot$. Then $tr(F) = \Box(\Box\mathbf{K}\bot \to \Box\bot)$

1. $\mathbf{K}\bot \to \bot$, D Axiom;
2. $\Box\mathbf{K}\bot \to \Box\bot \to \bot$, by $\Box$-necessitation and distribution;
3. $\Box(\Box\mathbf{K}\bot \to \bot)$, by $\Box$-necessitation.

Induction step – *modus ponens*, covered by Gödel in [36]. □

As we saw, the co-reflection principle $p \to \mathbf{K}p$ in IEL is valid in the provability-verification logic S4V (Case 3 above). However, reflection in IEL, $\mathbf{K}p \to p$, is not valid in S4V.

Theorem 15 $\mathsf{S4V} \nvdash tr(\mathbf{K}p \to p)$.

Proof. First we calculate $tr(\mathbf{K}p \to p)$:

$$tr(\mathbf{K}p \to p) = \Box(\Box\mathbf{K}\Box p \to \Box p)$$

Consider again the model $\mathcal{M}_1$ from Theorem 5.

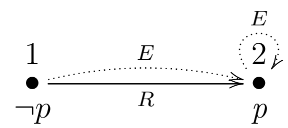

Figure 3: Model $\mathcal{M}_1$

It can be regarded as a Kripke model for a bi-modal language in which R is the accessibility relation for $\Box$ and E the accessibility relation for $\mathbf{K}$. Since R is reflexive and transitive, all $\mathsf{S4}$ axioms are valid. Since E is serial, all axioms of D are valid. Axiom $\Box A \to \mathbf{K}A$ is valid because $E \subset R$. So, any theorem of $\mathsf{S4V}$ is true at each node of $\mathcal{M}_1$.

It suffices to check that $\Box\mathbf{K}\Box p \to \Box p$ does not hold at node 1. Indeed, $2 \Vdash p$, hence $1, 2 \Vdash \mathbf{K}\Box p$. Hence $1 \Vdash \Box\mathbf{K}\Box p$, but $1 \nVdash \Box p$. $\Box$

How faithful is this translation to IEL? What does $\mathbf{K}p$ mean in IEL? There are two senses which IEL can accommodate:

1. It is verified that p holds in some non-intuitionistic, "constructive," sense.[16]

2. It is verified that p holds intuitionistically, i.e. that p has a proof, not necessarily specified in the process of verification.

Our examples (section 2.1.2) of highly probable verification and zero-knowledge protocols suggest the first sense, while the other examples suggest the second: IEL is faithful to both.

However, the bi-modal language of provability and verification is more expressive and allows us to distinguish these readings. $\mathsf{S4V}$, clearly, incorporates the second sense above, by having $tr(\mathbf{K}p) = \Box\mathbf{K}\Box p$.

However the first sense also appears to be a natural understanding of intuitionistic verification and a constructive notion of knowledge. Following 1 then we would have that $tr(\mathbf{K}p) = \Box\mathbf{K}p$. However on this reading, Gödel's translation does not embed IEL into $\mathsf{S4V}$. The translation of the IEL axiom $\mathbf{K}(p \to q) \to (\mathbf{K}p \to \mathbf{K}q)$ is

$$\Box\mathbf{K}(\Box p \to \Box q) \to \Box(\Box\mathbf{K}p \to \Box\mathbf{K}q),$$

which is not provable in $\mathsf{S4V}$. The matter of how to capture reading 1 of $\mathbf{K}p$ in classical terms appears to require a different treatment.

The converse of the embedding also holds.

Theorem 16 $\mathsf{S4V} \vdash tr(F) \;\Rightarrow\; \mathsf{IEL} \Vdash F$.

Proof. See Appendix C $\Box$

[16]i.e. is not necessarily a BHK-compliant proof, but constructive in a more general sense.

7 IEL and Intuitionistic Responses to the Knowability Paradox

What of the knowability paradox? Does it not show the intuitionistic conception of truth to be absurd?[17] Accordingly, must not an intuitionistic conception of knowledge be committed to rejecting CT, as have intuitionist approaches thus far, e.g. [64, 75, 77]? We argue not. The proper intuitionistic response is that there is no paradox, see e.g. [28, 46, 53, 67].[18] The 'knowability paradox' is a paradox only when CT is read as 'all truths are known' – but that holds *only on a classical* reading of CT and when the idea that "all truths are knowable" is taken to be the fundamental property of constructive truth. On a proper intuitionistic view of truth and its relation to knowledge neither of these holds.

Informally the Church-Fitch [9, 29] 'knowability paradox' shows that

all truths are knowable

implies

all truths are known.

Formally,

$$A \to \Diamond \mathbf{K} A \tag{VK}$$

implies

$$A \to \mathbf{K} A \tag{CT}$$

The premise is taken to be definitive of constructive truth, so clearly this is a problem for such ideas. But only from a classical point of view. From an intuitionistic point of view the reasoning of the knowability paradox goes in the wrong direction. The fundamental property of intuitionistic truth is its constructivity, that proof yields knowledge, from which it follows that indeed all (constructive) truths are knowable. But that is completely unexceptionable. Intuitionistically the proper order of explanation is that CT implies VK, which depends on the principle that "what holds is possible," which appears to be valid on an intuitionistic reading of $\Diamond$.

The same holds for the intuitionistic response[19] to the paradox which holds that the proper intuitionistic manner to express the informal idea that all truths are knowable is

$$A \to \neg\neg \mathbf{K} A. \tag{IK}$$

As we have seen (Theorem 1) IK is a simple consequence of a fundamental property of intuitionistic truth; it is not itself the fundamental property.

[17]For an overview of the paradox and responses to it see [5], see also [47, 68, 69].

[18][53] and [46] in particular make this point by giving the intuitionistic reading of CT its due.

[19][16, 28, 67]

There is no need to show that the Church-Fitch construction is invalid, or that intuitionistically it has acceptable consequences. Indeed intuitionistically the Church-Fitch construction *is* valid, but trivially – since CT is valid it is implied by anything; but all it proves is the constructivity of intuitionistic truth.

The knowability paradox is the product of a doubly mishandled attempt to formalize in classical logic an intuitionistic idea. First by treating the informal principle "all truths are knowable" as definitive of constructive truth and its relation to knowledge, and then formalizing it as VK. When the idea is properly translated into classical logic, by taking the constructive nature of truth properly into account, see [2, 3], no unwanted conclusions follow.

This means that the intuitionistic responses to the paradox so far are insufficiently intuitionistic because they commit themselves to the classical reading of $A \to \mathbf{K}A$, and take it as their task to show it is invalid, when it is in fact central to a properly intuitionistic account of knowledge and its relation to truth. Moreover these rejections of CT are usually accompanied by explicit endorsement of the reflection principle for knowledge, which holds only of classical knowledge. We contend that these intuitionistic responses are not intuitionistic enough in their view of truth, and classical in their view of knowledge.

7.1 Hart

Hart's [37] sets the pattern for intuitionistic responses – though Hart himself thinks such a response is mistaken. On the one hand he canvases an argument that $A \to \mathbf{K}A$ is intuitionistically valid and, on the other hand, rejects the conclusion since he takes the argument to justify that every truth is known. He writes

> Incidentally, on an intuitionist reading, it just might be that every truth is known. For being in an intuitionist position to assert that $\forall x(Fx \to Gx)$ requires a method which given an object and a proof that it is F, yields a proof that it is G. In the present instance this means: suppose we are given a sentence ... and a proof that it is true. Read the proof; thereby you come to know that the sentence is true. Reflecting on your recent learning, you recognize that the sentence is now known by you; this shows that the truth is known. If this argument is intuitionistically acceptable ... then I think that fact reflects poorly on intuitionism; surely we have good inductive grounds for believing that there are truths as yet unknown [37, p.165].

Hart has all the elements for a proper intuitionistic response, and he gives an argument to the effect that proof (intuitionistic truth) yields knowledge. However he does not apply the argument to the reading of CT, instead treating his argument that "proof yields knowledge" as a justification for "all truths are known." This is unstable, since it requires reading CT classically; a reading which is intuitionistically incorrect, hence inapplicable to the intuitionist. One can reject intuitionism altogether and the argument for the validity of

CT with it, of course, but one cannot argue that a classical understanding of a principle invalidates an intuitionistic reading of it.

7.2 Williamson

Williamson, in [75, 76], develops Hart's argument supporting $A \to \mathbf{K}A$, but likewise assumes that CT is invalid, and seeks to devise a form of intuitionistic semantics which invalidates it.

To do this Williamson distinguishes between *proof-tokens* and *proof-types*. Proof-tokens are of the same type just if they have the same structure and conclusion, though they may be effected at different times. Williamson argues that CT holds for proof-tokens; in this case $A \to \mathbf{K}A$ says that any proof-token of A can be turned into a proof-token of $\mathbf{K}A$. But CT does not hold for proof-types. In the case of proof-types $A \to \mathbf{K}A$ says that there is a function which takes a proof-type of A to a proof-type of $\mathbf{K}A$, which in this context is read as 'there exists a time t such that A is will have been proved at t'. Moreover, the validity of CT requires that this function be *unitype*, meaning that if inputs, p and q, into the function are of the same type then the outputs, $f(p)$ and $f(q)$, are of the same type also. Hence "a proof of $A \to \mathbf{K}A$ is a unitype function that evidently takes any proof token of A to a proof token, for some time t, of the proposition that A is proved at t", [76, p.430]. Williamson's contention is that such a function does not exist in all cases. It exists where we already have an input for the function, i.e. a proof of A. But in the cases where we do not already have an input for the function we cannot prove $A \to \mathbf{K}A$ holds by deriving $\mathbf{K}A$ from A; in this case all we have to work with is the function f itself, which takes us from hypothetical proof-tokens of A to proof-tokens of $\mathbf{K}A$. But such a function is not unitype. Assume that p and q are token-proofs of A of the same type carried out at different times, then $f(p)$ and $f(q)$ will be proof tokens of different types. $f(p)$ is a proof that $\mathbf{K}A$ is proved at time t and $f(q)$ is a proof that $\mathbf{K}A$ is proved at time t'. Hence CT is not generally valid.[20]

In response, we point out that the BHK semantics has no temporal component. Williamson's reading of $\mathbf{K}$ and CT introduces a temporal aspect which is extraneous to the BHK semantics and hence is not one the intuitionist has to accept. It is clear that Williamson interprets CT as asserting a kind of universal proof-checking (as do we, see Section 2.1.1), but the time at which the proposition was proved is, normally, not essential to checking a proof's correctness. Williamson seems to endorse CT for the original BHK semantics (e.g. for proof-tokens) but then modifies it by adding an alien temporal component to devise a reading under which CT could fail. The latter modification is rather non-standard and consequently does not impugn the BHK understanding of CT.

CT asserts correctly that given a proof, x, of A proof-checking produces another proof,

[20] See [53] for an argument that such a function does exist; f cannot operate on hypothetical proof tokens, since they do not exist; so f can still be defined as a unitype function taking a proof of A and returning a proof of $\mathbf{K}A$. For an objection to this see [54]. On the debate about the status of hypothetical reasoning in intuitionism see [14, p.30] and the references contained therein.

y, that there exists a verification of A, namely a proof x of A. As we see CT holds independent of the time x is carried out or of whether it has already been constructed or is only hypothetical: proof-checking is a correct procedure that exists independent of any assumptions about specific proofs. Accordingly, proof-checking does not require any information about the time at which x was carried out and does not produce such information.

More fundamentally, we again find the insistence that $A \to \mathbf{K}A$ is intuitionistically invalid[21] in the presence of arguments justifying the principle that proof yields knowledge, which can be attributed to a classical understanding of CT. The commitment to a classical conception of knowledge is even more in evidence in Williamson's formulation of an intuitionistic modal epistemic logic [77] in which FK, $\mathbf{K}A \to A$, is endorsed explicitly.

On the other hand, Williamson acknowledges that in an intuitionistic statement $\mathbf{K}A$ need not require strict proof to hold; intuitionistic knowledge is weaker than intuitionistic truth, i.e. proof yields verification but not *vice versa*. Accordingly, this exhibits the same instability found in Hart's prototype intuitionistic response, which is attributable to working in an intuitionistic context without achieving a complete liberation from a classical conception of knowledge.

7.3 Proietti

A more recent development of the basic approach taken in [16] is found in Proietti [64], who develops an intuitionistic epistemic logic with a Kripkean semantics. Along with Williamson's [77] this is the only attempt at a formulation of an intuitionistic epistemic logic.

Proietti's basic assumptions follow the pattern for intuitionistic responses to the knowability paradox: that even under intuitionistic assumptions $A \to \mathbf{K}A$ is invalid. At the same time he assumes explicitly that $\mathbf{K}A \to A$ holds in the logic.[22] With respect to FK, Proietti asks "...whether explicit knowledge should validate other, usual axioms of classical epistemic logic such as T ... In the case of T the answer is affirmative: the explicit knowledge of an agent in some given state is likely to imply truth at the same time" [64, p.4]. We have already argued that this set of commitments is the hallmark of the classical conception of the relation between knowledge and truth.[23]

It should be noted, however, that Proietti is not trying to analyze Brouwer's original intuitionistic paradigm of "truth as provability." Proietti's starting point is rather the later Kripke semantics of intuitionistic logic, which is not ideologically and technically faithful to the original intuitionistic foundations. The relationship between truth, proof and knowledge

[21] "...the assertibility of P, being a decidable condition, seems to guarantee the assertibility of the assertibility of P, which would then make '$P \to$ it is assertible that P' itself assertible; $\mathbf{K}$ in [CT] can be interpreted as 'it is assertible that', and [CT] remains absurd on this reading" [76, p.429].

[22] Indeed, the intuitionistic epistemic logic he endorses is $\mathsf{Int}_{\mathsf{S4}}$, which is intuitionistic propositional logic with the $\mathsf{S4}$ axioms for $\mathbf{K}$.

[23] "Even for an ideal reasoner, one cannot equate explicit knowledge and truth" [p.3].

does not arise in a Kripkean semantic context, since proof is not part of the picture, but for this very reason the intuitionistic considerations in favor of CT and against FK cannot come up.

7.4 Percival

Another intuitionistic response, [75], has been to point out that in intuitionistic logic the Church-Fitch proof yields only $A \to \neg\neg\mathbf{K}A$. This is acceptable on an intuitionistic reading, indeed as mentioned some argue it serves as a better intuitionistic expression of the constructivist's view than does the classical VK, see [16, 28, 67]. Percival argues that IK is bad enough since it intuitionistically implies $\neg\mathbf{K}A \leftrightarrow \neg A$ and $\neg(\neg\mathbf{K}A \wedge \neg\mathbf{K}\neg A)$, both of which, he argues, are intuitionistically unacceptable [55, p.183].

The first Percival reads as claiming that the falsehood of A and ignorance of A are logically equivalent in Int. But, he argues, this cannot be. Assume that $\neg A$ is a mathematical proposition, hence necessarily true. Whether A is not known, $\neg\mathbf{K}A$, is a contingent matter. Hence there must be some state of some model where $\neg A$ holds and $\neg\mathbf{K}A$ does not. The second Percival reads as claiming that no statement is forever undecided. This is the 'undecidedness paradox of knowability' [5]. He claims that this second consequence is just obviously false; there exists a p for which $\neg\mathbf{K}p \wedge \neg\mathbf{K}\neg p$ holds.

We argue that both Percival's 'counterexamples' are valid epistemic principles within the BHK-based IEL paradigm (see Theorems 8 and 9) and hence do not serve as decisive arguments against the intuitionistic response to the knowability paradox, or the incorporation of verification-based knowledge into an intuitionistic framework.

First, in terms of intuitionistic knowledge (IEL) $\neg\mathbf{K}A \leftrightarrow \neg A$ claims a proof of $\neg A$ is equivalent to a proof of $\neg\mathbf{K}A$. If we can show that a proof of A reduces to a contradiction then A cannot possibly hold, hence neither can $\mathbf{K}A$. Conversely, if a proof of $\mathbf{K}A$ reduces to a contradiction then there cannot be a verification of A, but every proof is also a verification, hence there cannot be a proof of A. The contingency or necessity of $\neg A$ and $\neg\mathbf{K}A$ is not relevant because intuitionistically these are statements about proofs and verifications of statements – whatever their modal status (cf. [16, p.325]).[24]

In IEL, $\neg(\neg\mathbf{K}A \wedge \neg\mathbf{K}\neg A)$ claims that no truth is unverifiable, not that no truth remains forever undecided.[25] Indeed, by the previous principle if $\neg\mathbf{K}A$ held then so does $\neg A$, i.e. there is a proof of $\neg A$, hence $\neg A$ is verified and $\mathbf{K}\neg A$. Once again, the contingency of some agent's ignorance is beside the point.

Percival's conclusion that since "[VK] has consequences that are plainly unacceptable we *do* know in advance that no intuitionistic defense ... with a specific semantics ... is going

[24] Moreover, it is not clear this argument works in its own terms. If $\neg A$ is necessarily true, then A is necessarily false, in which case *A cannot be known*, since for a necessarily false A ignorance of A is also necessary. Hence there is no state of any model where $\neg\mathbf{K}A$ does not hold. Percival's assumptions appear to be inconsistent and his argument to violate the truth condition on knowledge.

[25] Again see [16].

to work" is false. The intended intuitionistic semantics, BHK, as extended to IEL, is such a semantics, and in its terms the consequences are quite acceptable.

7.5 IEL and Non-Mathematical Propositions

But there is a further, more general, premise that Percival assumes which he uses to support his conclusion, and which seems to apply to any attempt to enunciate an intuitionistic view of knowledge. He argues that "as anti-realist sympathizers ... admit, non-mathematical statements aren't susceptible to *proof* and a proof-conditional interpretation of '→' isn't *generally* viable. So an intuitionistic defense can't appeal to it" [55, p. 183]. According to this line of reasoning an intuitionistic view of knowledge cannot be put in terms of BHK, because BHK does not apply to all kinds of propositions. A legitimate intuitionistic defense must give a semantics that holds for all kinds of propositions, not just mathematical ones, and be "independently plausible."

We respond that this is an illegitimate constraint on an intuitionistic view of knowledge. BHK is the intended semantics of intuitionistic logic, indeed the intuitionistic calculus was constructed to capture the BHK semantics not the other way around. There are, of course, many non-BHK semantics for Int, but it is acknowledged that they are more or less artificial, not true to the intentions of intuitionism, precisely because they do not represent the BHK view.[26] To demand an intuitionistic semantics which rules out BHK is, to some extent, to demand an intuitionistic theory which is not intuitionistic. An intuitionistic semantics which is acceptable to the non-intuitionist will be either a non-intuitionistic semantics or be accompanied by an argument guaranteed to convert the non-intuitionist. Neither demand is legitimate.

The point of the objection is that the BHK interpretation cannot accommodate non-mathematical truth. But IEL *can* do this: *non-mathematical propositions are a subset of the verified propositions.*

IEL allows us to distinguish two ways in which a proposition can hold: 1) being true and 2) being a fact. A proposition is intuitionistically true if there is a proof of it. A proposition is an intuitionistic fact if there is a verification of it. Being a fact is just being intuitionistically true, in a broad, more general, sense of 'true.'[27] When it comes to knowledge, the intuitionistic universe is a mirror image of the classical universe. In the classical universe the set of known propositions is a subset of the classically true propositions, because knowledge is acquired/formed by adding something to truth. The classical universe is larger than what is known. In the intuitionistic universe the set of known propositions is a superset of the true propositions, because proofs are a kind of verification and verifications determine the extent of the facts. The intuitionistic universe is as big as what is known or knowable; i.e. the set of facts is "epistemically constrained."

[26] See [1, 13]

[27] We are not supposing anything substantive about the existence or nature of facts. We could have as easily used the terms 'true_1' and 'true_2' for the distinction we are making.

Hence, *in its own terms*, a BHK conception of knowledge can make sense both of non-mathematical propositions and of the intuitionistic consequences of IK/Theorem 1. Whether those terms should be accepted by everyone, or are adequate for all purposes, is just the argument over the correctness of intuitionism. But that is a separate question from that of the nature of intuitionistic knowledge.

8 Conclusion

Since intuitionistic truth, proof, is stronger than intuitionistic knowledge based on verification, while classical knowledge is stronger than classical truth, intuitionistic knowledge is, in a sense, the 'mirror image' of classical knowledge. Intuitionistically

$$A \to \mathbf{K}A$$

is valid while

$$\mathbf{K}A \to A$$

is not. A proper understanding of the relation between intuitionistic truth and knowledge shows that the seeming absurdity of both commitments is just an artifact of a classical view, and justifies IEL as the epistemic logic reflecting the BHK view of knowledge

We would like to think that the system IEL reflects the logic of knowledge an intuitionist who accepts the possibility of non-mathematical knowledge should accept as best incorporating the intended BHK view of truth. But is this generally acceptable as a logic of knowledge? As with plain intuitionistic logic IEL involves idealizations which some may find implausibly strong, such as the total recall implicit in IEL models. That may be, but we do not seek to argue that IEL is better as an epistemic logic than some other, and undoubtedly just as refinements of basic classical epistemic logic, e.g. by introducing temporal elements or dynamics, make for more realism, refinements of IEL would too. We have sought only to outline what a thoroughgoing intuitionistic approach to knowledge looks like, and to point out that previous attempts so far were not as intuitionistic as they could have been.

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Appendix A Completeness of IEL

We show that IEL is complete with respect to IEL models. First we define what a *prime theory* is.

Definition 4 A set of formulas, Γ, is a *theory* if it is closed under $\vdash$ in IEL. That is, for any F, if $\Gamma \vdash F$ then $F \in \Gamma$.

Definition 5 A set of formulas, Γ, is *prime* if $F \vee G \in \Gamma$ implies that either $F \in \Gamma$ or $G \in \Gamma$.

The key fact concerning prime theories is the following lemma.

Lemma 2 *For a set of formulas Γ and formula F, if $\Gamma \nvdash F$ then there exists a prime theory Δ, such that $\Gamma \subseteq \Delta$ and $F \notin \Delta$.*

Proof. Since the language of IEL is countable we can list the formulas of IEL $X_1, X_2, X_3 \ldots$. Now, assume we have a set of formulas $\Delta_0 = \Gamma$, we define a sequence of sets of formulas, $\Delta_1, \Delta_2, \Delta_3 \ldots$ thus:

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{X_{n+1}\} \text{ if } \Delta_n \cup \{X_{n+1}\} \nvdash F. \\ \Delta_n \text{ otherwise }. \end{cases}$$

Since $\Gamma \nvdash F$, for all $n \geq 0$, $\Delta_n \nvdash F$. Put

$$\Delta = \bigcup \{\Delta_n | n \geqslant 0\}.$$

It is obvious that Δ extends Γ and that $\Delta \nvdash F$. Indeed, if $\Delta \vdash F$ then, by compactness, for some n, $\Delta_n \vdash F$ which is impossible. So, $\Delta \nvdash F$ hence $F \notin \Delta$.

We have to show that Δ is a prime theory.

1) Δ is a theory. Assume for some X $\Delta \vdash X$ but $X \notin \Delta$. Take k such that $X = X_k$, then $\Delta_{k-1} \cup \{X_k\} \vdash F$, since otherwise $X_k \in \Gamma_{k+1}$ and $X \in \Delta$. In which case $\Delta \cup \{X\} \vdash F$, so $\Delta \vdash X \to F$, but since $\Delta \vdash X$ it follows that $\Delta \vdash F$, which is a contradiction.

2) Δ is prime. Suppose $X \vee Y \in \Delta$ but $X \notin \Delta$ and $Y \notin \Delta$. As in (1), both $\Delta \vdash X \to F$ and $\Delta \vdash Y \to F$ hold. Since $X \vee Y \in \Delta$, $\Delta \vdash F$. Contradiction. $\Box$

We now define the canonical model.

Definition 6 The canonical model is a quadruple $< W^C, R^C, E^C \Vdash^C >$ such that:

- W^C is the set of all consistent prime theories.
- R^C: $\Gamma R^C \Delta$ iff $\Gamma \subseteq \Delta$.
- E^C: $\Gamma E^C \Delta$ iff $\Gamma_E \subseteq \Delta$ where $\Gamma_E = \{F | \mathbf{K}F \in \Gamma\}$
- $\Vdash^C$: $\Gamma \Vdash^C p$ iff $p \in \Gamma$, for a propositional letter p.

Lemma 3 *The canonical model is a model for* IEL.

Proof. Clearly, $\subseteq$ is transitive and reflexive, hence so is R^C. We need to show that E^C is a binary relation meeting the following conditions:

1. $\Gamma E^C \Delta \Rightarrow \Gamma R^C \Delta$

2. $\Gamma R^C \Delta E^C \Omega \Rightarrow \Gamma E^C \Omega$

3. $E^C(\Gamma)$ is non-empty for each Γ.

1. Assume $\Gamma E^C \Delta$ and $X \in \Gamma$, to show that $X \in \Delta$. Since Γ is closed under IEL, $X \to \mathbf{K}X \in \Gamma$, hence $\mathbf{K}X \in \Gamma$, but then $X \in \Delta$. Since X is arbitrary, $\Gamma \subseteq \Delta$.

2. Assume $\Gamma R^C \Delta E^C \Omega$ and $X \in \Gamma_E$, to show that $X \in \Omega$. Since $X \in \Gamma_E$ then $\mathbf{K}X \in \Gamma$. Since $\Gamma R^C \Delta$ $\mathbf{K}X \in \Delta$, hence $X \in \Delta_E$. Since $\Delta E^C \Omega$ holds, $\Delta_E \subseteq \Omega$, hence $X \in \Omega$.

3. Now we have to check that $E^C(\Gamma)$ is not empty for all $\Gamma \in W^C$, i.e. that for each $\Gamma \in W^C$ there is $\Delta \in W^C$ such that $\Gamma E^C \Delta$. For a given Γ, consider Γ_E. We claim that Γ_E is consistent. Indeed, suppose otherwise, i.e. that $\Gamma_E \vdash \bot$. Then $\vdash X_1 \wedge \cdots \wedge X_n \to \bot$ for some $X_1 \ldots X_n \in \Gamma_E$. By the usual modal reasoning, this yields $\vdash \mathbf{K}X_1 \wedge \cdots \wedge \mathbf{K}X_n \to \mathbf{K}\bot$. Since all $\mathbf{K}X_i$ are from Γ, $\mathbf{K}\bot \in \Gamma$ as well, which is impossible since $\neg\mathbf{K}\bot \in \Gamma$ (as an axiom of IEL) and Γ is consistent. By Lemma 2, there is a prime theory Δ such that $\Gamma_E \subseteq \Delta$ and $\Delta \not\vdash \bot$ (hence Δ is consistent). From the definition of E^C, $\Gamma E^C \Delta$. □

Lemma 4 (Truth Lemma) *For any formula* X, $\Gamma \Vdash^C X \Leftrightarrow X \in \Gamma$.

Proof. By induction on the construction of X. The propositional cases are standard, we check the epistemic case only, i.e. when X is $\mathbf{K}Y$.

$\Rightarrow$: Assume $\mathbf{K}Y \in \Gamma$, and $\Gamma E^C \Delta$, hence $Y \in \Delta$. By the induction hypothesis $\Delta \Vdash Y$. Since Δ is arbitrary this holds for any state E^C-accessible from Γ hence $\Gamma \Vdash \mathbf{K}Y$.

$\Leftarrow$: Suppose $\mathbf{K}Y \notin \Gamma$, in which case $\Gamma_E \nvdash Y$. Suppose otherwise (i.e. suppose $\Gamma_E \vdash Y$), then $\{A_1 \ldots A_n\} \vdash Y$ for some $A_i \in \Gamma_E$. By the deduction theorem $\vdash A_1 \wedge \cdots \wedge A_n \to Y$ holds. Hence $\vdash \mathbf{K}(A_1 \wedge \cdots \wedge A_n) \to \mathbf{K}Y$. Hence $\vdash (\mathbf{K}A_1 \wedge \cdots \wedge \mathbf{K}A_n) \to \mathbf{K}Y$. Now $\mathbf{K}A_1 \ldots \mathbf{K}A_n \in \Gamma$, hence $\Gamma \vdash \mathbf{K}Y$. Since Γ is a theory, $\mathbf{K}Y \in \Gamma$, which is a contradiction. Hence $\Gamma_E \nvdash Y$. By Lemma 2 there is a prime Δ such that $\Gamma_E \subseteq \Delta$ and $Y \notin \Delta$. By the induction hypothesis $\Delta \nVdash Y$ hence $\Gamma \nVdash \mathbf{K}Y$. □

Theorem 17 (Completeness) *If* $\Vdash X$ *then* $\vdash X$.

Proof. By contrapositive. Assume $\nvdash X$, which can be read as $\emptyset \nvdash X$. By Lemma 2 there is a prime Δ s.t. $X \notin \Delta$; such a Δ is consistent. By the Truth Lemma, in the canonical model $\Delta \nVdash X$, so $\nVdash X$. □

Appendix B The Truth Condition on Knowledge

We stated above (Section 2.2) that $\neg\mathbf{K}\bot$ is the minimal candidate for expressing the truth condition on knowledge in an intuitionistic manner. In the presence of CT each of the alternatives to FK are equivalent (Theorem 2). It is easy to show that in the absence of CT we get the following hierarchy, from strongest to weakest.

$$\mathbf{K}A \to A$$

$$\Downarrow$$

$$\neg(\mathbf{K}A \wedge \neg A) \quad \Leftrightarrow \quad (\mathbf{K}A \to \neg\neg A) \quad \Leftrightarrow \quad \neg\neg(\mathbf{K}A \to A) \quad \Leftrightarrow \quad (\neg A \to \neg\mathbf{K}A)$$

$$\Downarrow$$

$$\neg\mathbf{K}\bot$$

Indeed, one can easily check all these dependencies in the logic $\mathsf{Int_K}$, the intuitionistic analogue of the classical modal logic K. This is IEL with only axiom 2 and the rule of $\mathbf{K}$-Necessitation. $\mathsf{Int_K}$-models are like IEL-models with the exception that only condition 3 holds of the binary relation E: $R \circ E \subseteq E$. $\mathsf{Int_K}$ is sound and complete for this class of models, see [4, 34].

Appendix C Converse of S4V-IEL embedding

Consider an IEL model $\mathcal{M} =< W, R, E, \Vdash>$. Clearly we can consider $\mathcal{M}$ as an $\mathsf{S4V}$ model $< W, R, E, \Vdash'>$ by treating $\Vdash'$ as the classical Kripkean forcing. R is transitive and reflexive, E is serial, and $E \subseteq R$, hence all axioms and rules of $\mathsf{S4V}$ hold in $< W, R, E, \Vdash'>$.

Lemma 5 *For each* IEL*-formula* F *and each* $u \in W$, $u \Vdash F$ *iff* $u \Vdash' tr(F)$.

Proof. A straightforward induction on formula F. We check only the case of epistemic modality, F is $\mathbf{K}X$. Let $u \Vdash \mathbf{K}X$ and $w \in R(u)$. By monotonicity in IEL-models, $w \Vdash \mathbf{K}X$, hence for each $v \in E(w)$, $v \Vdash X$. By the inductive hypothesis, $v \Vdash' tr(X)$. Since v is an arbitrary element of $E(w)$, $w \Vdash' \mathbf{K}tr(X)$. Since w is an arbitrary element of $R(u)$, $u \Vdash' \Box\mathbf{K}tr(X)$, i.e. $u \Vdash' tr(\mathbf{K}X)$.

Suppose now that $u \Vdash' tr(\mathbf{K}X)$, i.e. $u \Vdash' \Box\mathbf{K}tr(X)$. By reflexivity of R, $u \Vdash' \mathbf{K}tr(X)$ hence for each $v \in E(u)$, $v \Vdash' tr(X)$. By the induction hypothesis, $v \Vdash X$ for each $v \in E(u)$, hence $u \Vdash \mathbf{K}X$. □

Theorem 18 $\mathsf{S4V} \vdash tr(F) \;\Rightarrow\; \mathsf{IEL} \vdash F$

Proof. Assume $\mathsf{IEL} \nvdash F$. By IEL-completeness, there is a model $\mathcal{M} =< W, R, E, \Vdash>$ and a world $u \in W$ such that $u \nVdash F$. By Lemma 5, $u \nVdash tr(F)$. By S4V-soundness, $\mathsf{S4V} \nvdash tr(F)$. $\square$

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