A BUSINESS APPROACH TO TEACHING QUADRATIC FUNCTIONS

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Teaching mathematics in a high school and in a community college in the New York City, I have noticed that students grasp mathematical concepts more quickly and understand them better when a connection between the concept and the world of business is established. Students were visibly more interested and involved when business applications were presented first, laying the groundwork for technological applications and for the introduction of further theoretical concepts. This observation, well known to mathematics instructors, deserves further research.

This paper aims to show how the business approach can be used in teaching quadratic functions and parabolas: a topic encountered at every level of mathematics and one with a wide range of applications. It will be shown how business information may be presented in a classroom to improve students' understanding of the topic. The paper assumes that students have had some practice in quadratic functions and graphing parabolas.

MOTIVATION: LINEAR AND QUADRATIC FUNCTIONS IN BUSINESS

Before undertaking quadratic functions, students study linear equations and functions, and as part of that process, learn to graph a line. Before introduction of a business model of the quadratic function, students should confirm they have grasped the following: the total cost of any given number of goods is a sum of a variable cost proportional to the amount purchased and of a fixed cost, independent from the amount purchased. Examples of the fixed costs are payments for packing, delivery, or postage. Buying pizza, concert tickets, or flowers, paying the electric or telephone bills are examples that depict a linear model represented by a linear function and a line graph. The first question I usually ask my students is this one: "How and where do quadratic, rather than linear functions, appear in the business world?" One of the possible answers is buying items on a sale.

CASE STUDY: MANAGING A SALE

The holiday season signals a dynamic period during which consumers buy presents, and stores try to move as much of their inventory as possible. To accomplish this, a store typically announces a sale, which is in essence, a call out for customers to buy goods at lower prices. Often, in the hopes of outselling the competition, retailers modify the discounts further, in the hope that customers will purchase more items. Investigation of a case study presented below leads us to a mathematical model using the quadratic function.

A store announces a sale with the following conditions: The buyer must purchase a sales coupon that costs \$9. However, he receives \$1 as cash back when purchasing a single item, \$2 cash back for each item when purchasing two, \$3 off on each item when purchasing three and so on. This means that the bonus per item is proportional to the number of item purchased. For simplicity, it is assumed that the listed price of every item is \$8 – a simplification that allows us to better investigate the case study.

The sale conditions mean that the customer pays \$8 - 1 = \$7 when buying one item. Since the coupon costs \$9, the total adds up to \$7 + \$9 = \$16 for the first item. After buying two items, the customer pays 2·(\$8 - \$1·2) = \$12 as well as the cost of the coupon. Thus, two items total is \$12 + \$9 = \$21, and the cost per item is now \$21/2 = \$10.5. This is still more than \$16 for two single items without the sales coupon - an enticing incentive to buy more (recall that customers need a lot of presents for the holidays). Students are asked to calculate further and note that the cost per item drops as the number of purchased items increases.

This scenario paves the way for a discussion of the driving forces of sale pricing and on the occurrence of quadratic functions in business. In the course of the discussion, a question may arise: why would a customer be interested in purchasing several or more goods in the same store? One explanation would be that they usually want to spend as little time on shopping for presents as will allow, because of the sheer number of items needed. It is advantageous to buy all of the presents in the same location. An added bonus is that the reward per item increases proportionately with the number of items bought.

Let's now consider the store management's position. What might be some of their goals and what conditions should be in place to properly meet them? Once again, to simplify the real world, we decide that the store has an unlimited supply of goods, which will depreciate post-holidays, such as Halloween costumes, Christmas trees, New Year's greeting cards etc. We can assume that one of store management objectives is to sell the most goods possible to each and every customer. The scenario that is shaping up can be likened to a game with two types of players: buyers who wish to buy more while minimizing their spending per item, and store management, which wants to attract more buyers while maximizing revenue per buyer.

In order to depict how the different objectives shape the game, some questions can again be brought before the students. Is it possible, given the present conditions, that the cash-back bonus may exceed the total sum of the sales coupon and payment for the purchased goods? If so, how to prevent that?

To help answer these, students are asked to calculate cost per item, total cost (equal to the revenue of the store), and savings for 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 purchased items. By doing so, students fill in Table 1 and a box in Figure 1.

Table 1. Revenue from the sale. Listed price is $8, sales coupon cost, $9.

Items	Total listed cost	Bonus per item	Total bonus	Total cost = Store revenue[1)]	Actual cost of one item	Total savings = Total listed cost - Total cost
1	$8	$1	$1*1 = $1	8*1-1*1 + 9 = $16	$16.00	-$8
2	$16	$2	$2*2 = $4	8*2-2*2+ 9 = $21	$10.50	-$5
3	$24	$ 3	$3*3 = $9	8*3-3*3 + 9 = $24	$8.00	$0
4[2)]	$32	$ 4	$4*4 = $16	8*4-4*4 + 9 = $25	$6.25	$7
5	$40	$ 5	$5*5 = $25	8*5-5*5 + 9 = $24	$4.80	$16
6	$48	$6	$6*6 = $36	8*6-6*6 + 9 = $21	$3.50	$27
7	$56	$7	$7*7 = $49	8*7-7*7 + 9 = $16	$2.29	$40
8	$64	$8	$8*8 = $64	8*8-8*8 + 9 = $9	$1.13	$55
9	$72	$9	$9*9 = $81	8*9-9*9 + 9 = $0	$0.00	$72
10	$80	$10	$10*10 = $100	8*10-10*10+9 = -$11	-$1.10	$91
x	$\$8 \cdot x$	$\$1 \cdot x$	$(1x)x = x^2$	$y = 8x - x^2 + 9$	y/x	$8x-(9 + 8x - x^2) = (x^2 - 9)$

Notes:

1) Total cost equals to total listed price minus total bonus plus sales coupon cost.

2) Maximum allowable number of items. Store revenue is maximal; customers save money.

DISCUSSION OF THE RESULTS

As Table 1 shows, a customer buying nine items just gets the cost of coupon back, while buying ten he stands to gain from the purchase. It becomes clear that the store management should restrict the purchase of more than eight items per customer. Yet in that case, the store receives a measly $9 of revenue per customer. Can it do better? From Table 1, it follows that the maximum revenue of $25 per customer is generated when he purchases four items, and sensibly, the customer should be limited to this as the maximum amount of purchasable goods.

Holiday Sale.
Spend more! Save more!
It's the easiest way to shop. No complicated equations.
Just the best saving opportunity.

Buy one and get $1 bonus.
Buy two and get $2 bonus for each.
Buy three and get $3 bonus for each.
Buy four and get $4 bonus for each.

Your total savings with buying four are $

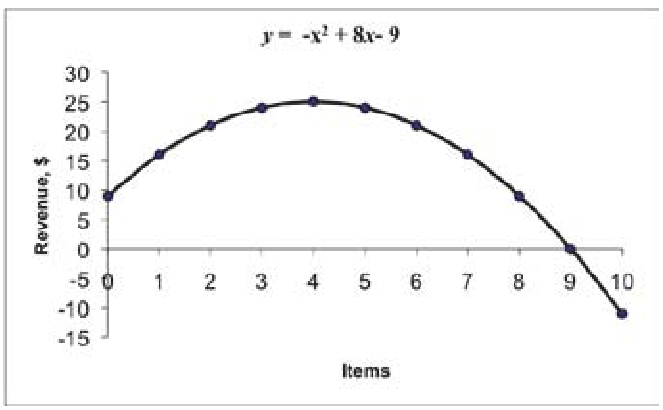
Price of each item is $8.
Coupon price is $9.
Coupon is valid for purchase up to 4 items.

Fig. 1. Sales coupon.

A header and content are compiled based on real sales coupons.
Students are asked to fill in the total savings box using data of Table 1.

This point would be an opportune time to segue way into a two-part discussion. The first part deals with gains and losses that shape the store's pricing policy. The second part addresses the derivation of a mathematical model, which would lead students to the quadratic function. Let's recall that, with a four item purchase, the customer pays \$6.25 per item, or \$62.5/\$80 = 78.125% of the listed price only. The utility of the sales coupon may also be addressed: it compels the customers to acquire four items to avoid paying for nothing. Students are encouraged to brainstorm similar offers aimed at increasing the amount purchased, such as free parking with purchase, a participation in a lottery, gifts available with purchase above a predefined sum etc.

MATHEMATICAL MODEL OF THE SALE



The previous discussion prepares the students to come up with a mathematical model. Assume that x items are purchased (following the last row of Table 1). The total listed cost is 8x dollars, with the bonus per item being 1x dollars. The price of one item with the bonus is (8 - 1x) dollars, and the total amount, including the sales coupon cost, is (8 - 1x)·x + 9 dollars. Recall that the bonus increases proportionally with the number of items purchased, so that total bonus with x items purchased is (1x)·x = 1x2.

Students were asked to write a formula of the relationship between total revenue from a single customer (y) and the number of items that have been purchased (x) and to draw a graph. Table 1 shows that the formula is y = -1x2+8x+9, and the graph is a parabola shown in Figure 2. Students were asked also to suggest different conditions of a sale and to develop Table 1 and Figures 1 and 2 correspondingly. The author noticed a positive impact on students' perceptions of the topic while performing this activity.