City University of New York (CUNY)

CUNY Academic Works

2018

Mathematics in Contemporary Society - Chapter 11 (Spring 2018)

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Chapter 11

We have an exam next week.

Test Topics for Exam #2

There will be writing on the exam!!!

The test will consist of 6-8 short answer questions and 3 or 4 problems.
Topics include:

Range
Range Rule of Thumb
Five Number Summary

Normal Distribution Characteristics
The 68-95-99.7 Rule

Calculating z-scores, finding percentiles
Using the z-scores table to find percentages below, above and between
Finding a score that corresponds to a specific percentile

Problem Solving (Lab #8 problems)

US Customary System Characteristics
US Customary System Conversions (using chart)

Metric System Characteristics
Metric System Conversions (know your prefixes!)
Metric-USCS Conversion (using charts)

Temperature Conversions (using formulas)
Density

Problem Solving
The Four Step Process

The Mathematics of Music
$Q = Q_0 \cdot 1.05946^n$
The Mathematics of Art

The Mathematics of Music

Question 1:   What makes music mathematical? Give three examples.

There are usually many responses to this kind of question. Some students say that different pieces of music have a certain beat. Others refer to the idea of the frequency of pitches. (Frequency sounds mathematical, doesn't it?) Still others mention the different kinds of notes: whole notes, half notes, quarter notes, sixteenth notes, etc. Now we're getting into fractions!

How is music created? Sound is created by vibrating objects, but we need more than that to create music. If you slam a book on a table, that's not music. So what is?

Music is created by vibrating objects that produce sound waves that radiate into the surrounding air in all directions and are perceived as music (it's pleasing to the senses, unlike the sound of a door slamming) by the ear.

Many musical sounds are produced by either:

a) vibrating strings (as in the piano, violin, cello and guitar)
b) vibrating reeds (as in the clarinet, saxophone, oboe and bassoon)
c) vibrating columns of air (as in organ pipes, flutes and trombones)

Different musical instruments make different sounds; what makes them different?

One of the more basic qualities of sound and music is pitch. To understand pitch, try to find a rubber band at home, and stretch it between your fingers. Pluck the rubber band. You should hear a low, dull sound as the rubber band vibrates.

Now take a small piece of the entire rubber band and pull it tight between your fingers. Pluck the rubber band again. It makes a much higher, sharper sound.

We notice here that the shorter the rubber band, the higher the sound, the higher the pitch.

Different instruments are designed to produce lower or higher pitched sounds.

Low Pitched Instruments	"In The Middle"	High Pitched Instruments
Tuba	French horn	Trumpet
String Bass	Cello	Violin
Bassoon	Clarinet	Flute

Question 2:   Name three other instruments and classify them as low pitched or high pitched instruments.

The material which an instrument is made of and the size of the instrument contribute to the pitch of sounds, or notes, created by the instrument. The tuba, French horn and trumpet are all made of brass, but the smaller trumpet produces the highest pitched notes. The violin, cello and string bass may look similar in appearance, but the high pitched violin is much smaller than the human-sized string bass.

Reed instruments, such as the bassoon, clarinet and saxophone, produce sound by the musician blowing on a reed, which vibrates throughout the entire instrument.

The small, sleek shape of the metallic flute and piccolo can produce particularly high pitched notes.

Each instrument also has a range of notes that can be played with that instrument. Depending on the way a musician pushes the buttons of a tuba or trumpet, slides his or her bow across the different strings of a cello or violin, or covers the multitude of holes on a clarinet or flute, the "lower pitched" or "higher pitched" notes of the instrument's range can be reached.

Some instruments have a very great range indeed. Within a piano is a vast array of strings of different length and thickness. The long, thick strings on the left side produce the lowest notes and the short, thin strings on the right side produce the highest notes.

Can we get a more technical definition of pitch?

Suppose you pluck a string from a guitar or piano. What happens? It vibrates, of course.

One cycle corresponds to one up and down vibration of the string. Frequency is the number of cycles per second (cps) of the string when plucked.

If the frequency of the string is 100 cps, it will vibrate at 100 cycles per second when plucked or struck.

How can we change this frequency? If you halve the string and pluck it again, you will notice that the frequency doubles! If the original frequency was 100 cps, the halved string will vibrate at 200 cps when plucked.

We call this doubling of frequency the raising of pitch by an **octave**, which we will visit again in the next set of notes.

Examine the following line from a song:

Now The Green Blade Rises

Voice

Now   the   green   blade   ris - es   from   the __ bur - ied   grain.

Where's the mathematics? All over the place!

The time signature (the numbers on the left) indicates the timing of the music.

The **2** on top indicates that every measure contains two beats. Each measure is separated by vertical lines. A person singing or playing this music should count 1-2,1-2,1-2,… as he or she sings or plays.

The **4** on the bottom indicates that every quarter note counts as one beat.
That means that every measure could have:

2 ♩ quarter notes, as in the second measure.

4 ♪ eighth notes, as in the first measure

1 𝅗𝅥 half note, as seen in the fourth measure

8 𝅘𝅥𝅯 sixteenth notes

(Note: We can't see any 𝅝 whole notes because they are four times as long as quarter notes.)

or some combination of these notes; we can see the third measures has 3 eighth notes and 2 sixteenth notes (1/8 + 1/8 + 1/8 + 1/16 + 1/16 = 2/4)

With all that in consideration, how do we sing this music?

The positioning of the notes in each measure indicates how high or low the notes are to be sung. Some of you may remember "Every Good Boy Does Fine" to represent E-G-B-D-F, the horizontal lines of the staff. The first note (for "Now") is an E, which is sung low. The second note (for "the") is B, which is sung higher and so on throughout the music.

The person singing counts 1-2 as they sing through each measure, going up and down note by note.

1   2   1   2   1   2   1   2

Now the green blade   riiiiiiiiiiii sesssssssss   from the-e burrr-iiiied graaaaaaaaaain

It may seem confusing to the non-musician, but to a seasoned musician it's just like breathing.

In the last set of notes we saw that doubling the frequency of a tone raises the pitch by an **octave**. If a string vibrates at a speed of 100 cycles per second (cps) when struck or plucked, we can cut the string in half and change its pitch. The half string will vibrate at a speed of 200 cps (twice as much) when struck or plucked. The tone of the vibrating string has been raised in pitch by an octave.

Note: Singers can do something similar with their voices. A bass (low) singer and a soprano (high) singer may be singing the "same" note, but the soprano will be singing it an octave higher.

We can also represent this idea with a piano. The keys of the piano represent a 12-tone (half-step) scale. The notes from one C on the piano to the next C:
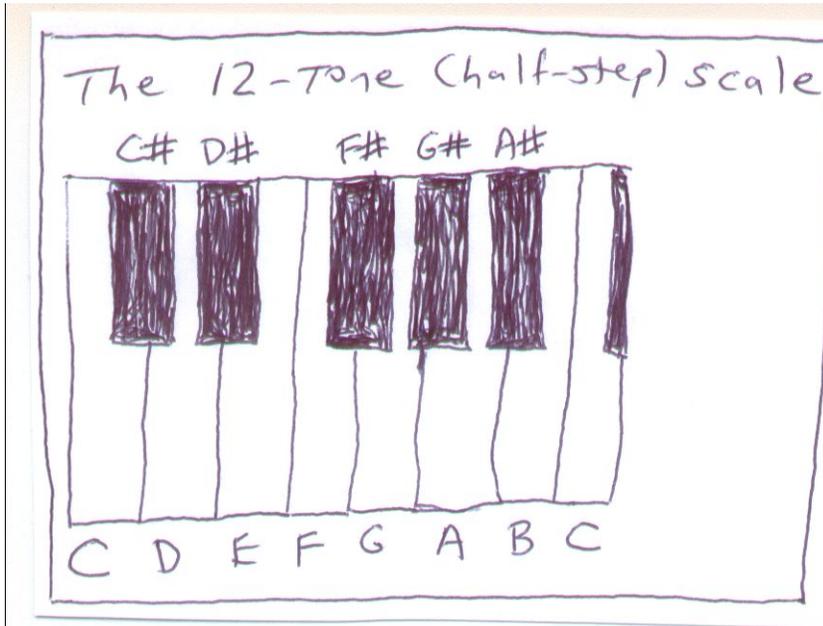
C→C#→D→D#→E→F→F#→G→G#→A→A#→B→C
represent an octave. Therefore, if the frequency of the C on the left is 260 cycles per second (cps), the next C on the right has a doubled frequency, or 520 cps.
(The C with a frequency of 260 cps is often called middle C.)
That's not all. We can see from the following table:

Note	Frequency (cps)	Frequency Ratio to Previous Note
C	260.0	1.05946
C#	275.5	1.05946
D	291.8	1.05946
D#	309.2	1.05946
E	327.6	1.05946
F	347.1	1.05946
F#	367.7	1.05946
G	389.6	1.05946
G#	412.7	1.05946
A	437.3	1.05946
A#	463.3	1.05946
B	490.8	1.05946
C	520.0	1.05946

that each note has a frequency that is 1.05946 times the previous note. For example, if I start with middle C (260 cps), the next note (up a half-step) of C# will have a frequency of $260 \cdot 1.05946 = 275.5$ cps. The next note (up a half-step) of D will have a frequency of $275.5 \cdot 1.05946 = 291.8$. The next note (up a half-step) of D# will have a frequency of $291.8 \cdot 1.05946 = 309.2$. And so on and so on.

The increasing frequencies of notes is an example of exponential growth, where each note frequency is a fixed percentage higher (in this case 0.05946 or 5.9%) than the previous note's frequency. We can create a general formula to calculate the frequency of any note for any number of half-steps above a starting point:

$$Q = Q_0 \cdot 1.05946^n$$

Where:

Q_0 = the frequency of the starting note (usually middle C =260 cps)
n = the number of half-steps above the starting note
Q = the frequency of the note we're looking for

Example:

Suppose we want to know the frequency of the note that is 8 half-steps above middle C. Using $Q_0 = 260$ cps and $n = 8$, we can calculate:

$Q = Q_0 \cdot 1.05946^n$
$Q = 260 \cdot 1.05946^8$
$Q = 260 \cdot 1.5874$ (Use the power key y^x or ^ on your calculator)
$Q = 412.7$ cps

(Note: This answer corresponds to the frequency of G# on the chart.)

Example:

Suppose we want to know the frequency of the note that is 20 half-steps above middle C. Using $Q_0 = 260$ cps and $n = 20$, we can calculate:

$Q = Q_0 \cdot 1.05946^n$
$Q = 260 \cdot 1.05946^{20}$
$Q = 260 \cdot 3.1746$
$Q = 825.4$ cps

(Note: If we were to extend the original chart for an additional octave, this answer would correspond to G# in the second octave.)

Example:

Suppose we want to know the frequency of the note that is 36 half-steps above middle C. Using $Q_0 = 260$ cps and $n = 36$, we can calculate:

$$Q = Q_0 \cdot 1.05946^n$$
$$Q = 260 \cdot 1.05946^{36}$$
$$Q = 260 \cdot 7.9992$$
$$Q = 2079.8 \text{ cps}$$

(In this example, we could have used the fact that every 12 half-steps is an octave, which doubles the frequency. To go up 36 half-steps is to go up $12+12+12=36$ three octaves, which would cause the frequency to double three times $260 \rightarrow 520 \rightarrow 1040 \rightarrow 2080$ cps. It's not a surprise that 1.05946^{36} is almost 8.)

Question 3: **Find the frequency of the note that is 10 half-steps above middle C.**

Question 4: **Find the frequency of the note that is 15 half-steps above middle C.**

Question 5: **Find the frequency of the note that is 48 half-steps above middle C.**

Question 6: **Find the frequency of the note that is *2 octaves* above middle C.**

Question 7: **Find the frequency of the note that is 8 half-steps above middle A, which has a frequency of 437 cps.**

Question 8: **Find the frequency of the note that is 6 half-steps below middle C. Explain how you got the result.**

Notes:
   1) *The piano does have its limits, of course. The key furthest to the left on the piano has a frequency of about 27 cps; the key furthest to the right has a frequency of about 4200 cps.*
   2) *Other scales are possible besides the 12-step scale. The Greeks used a 7-step scale; 3-tone scales are used in African music; 19-step scales are also possible.*

Analog to Digital

Recordings of music were once stored in an analog mode corresponding to the sound waves of the music itself. For example, the grooves in the vinyl tracks of a record (back when they made such things) correspond to the original sound waves of the music.

These days, music is stored digitally. A computer can take an analog version of music, break it into very short time intervals, measure the frequencies of the sound waves, and store them as numbers. The music is therefore stored as a long list of numbers corresponding to short time intervals (and no longer sound waves at all). The music can then be reproduced by the computer reconverting the numbers back to their corresponding musical sounds.

If the time intervals are short enough, the computer version sounds just as good as the analog version. Additionally, computers can be used to modify the digital version, removing unnecessary background noise and adding additional features (perhaps even digital instruments that weren't originally there!) to create a new, digitized form of music. Is this better? I don't know how Mozart would have felt about it.

Mathematics in Art

Suppose I ask you to draw a picture of a car on a road. Some of you may draw a picture like this:
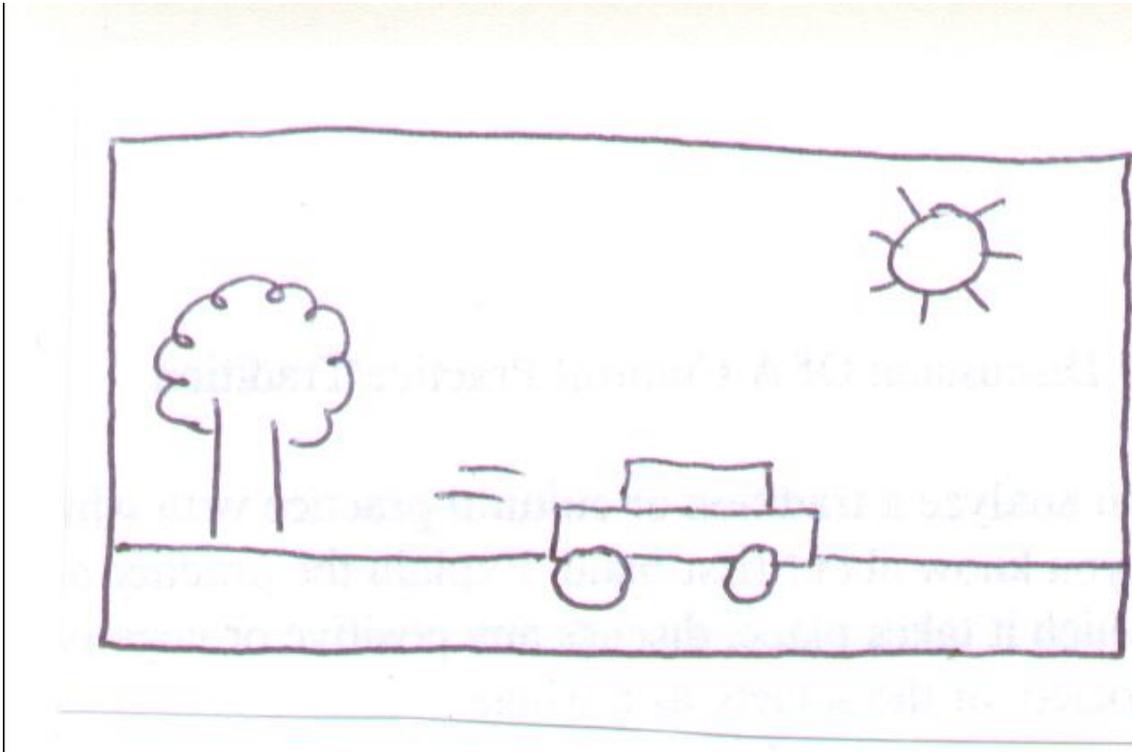
Some of you may attempt to draw a more realistic picture and have something like this:
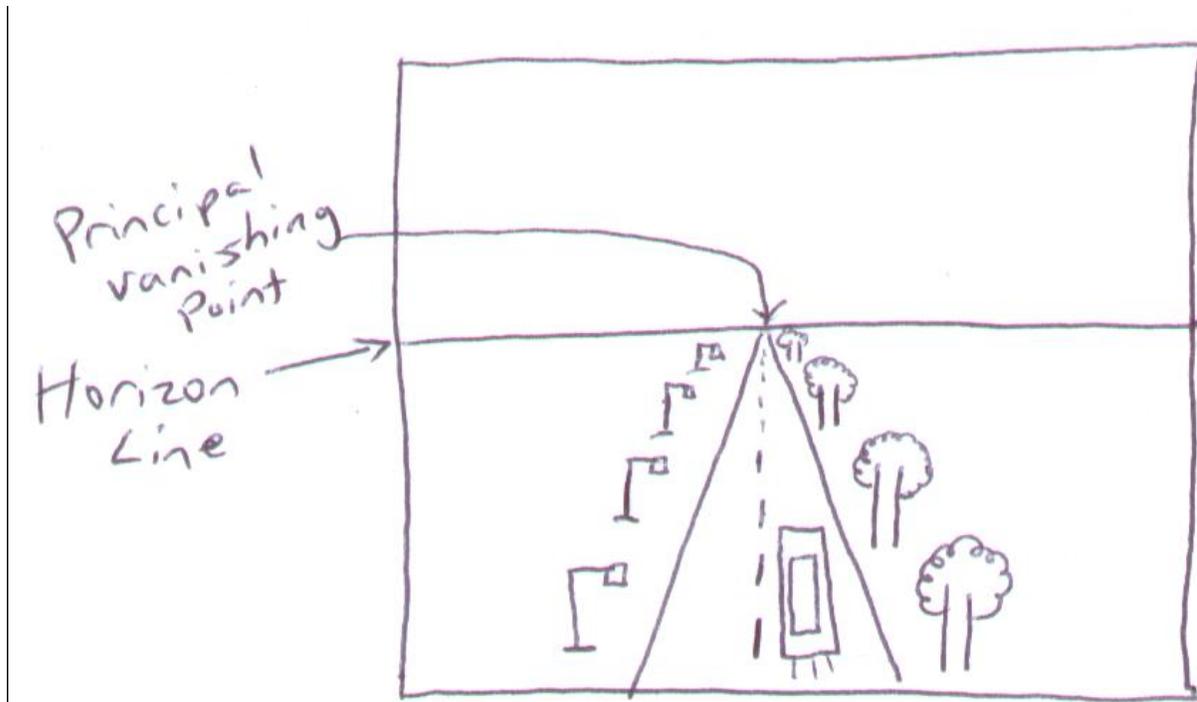
In the second drawing, perspective is being used.

Perspective

The ancient Greeks were known to have developed strong ties between art and mathematics, but the ideas of linking the two were largely lost within western civilization by the Middle Ages. This can be seen in the artwork of the time, which focuses more on images and storytelling without being concerned about realism and mathematical or scientific correctness.

Consider this work of art (***The Marriage of Cana*** by Jacopo Torriti) from the 13th century:
Source https://www.gettyimages.com/detail/photo/marriage-of-cana-fresco-by-jacopo-torriti-high-res-stock-photography/158648042
*(If the link gives you trouble, search for the artist and "The Marriage of Cana")*

The artist was not concerned about the realism of the image—it seems as if the objects on the table should fall to the ground. The religious significance of the artwork is of greater importance here.

During the time of the Renaissance (c. 1450-1600), there was a renewed interest in the sciences, and the need to paint with realism resurfaced. There was a strong desire to paint realistic three-dimensional images on two-dimensional canvases.

Two artists, Filippo Brunelleschi (1377-1446) and Leon Alberti (1404-1472), are recognized as the founders of painting with **perspective**. When using perspective, an artist paints or draws so as to give his or her canvas a realistic appearance, as if any viewer was looking thought a window and not on a canvas or paper.

How does one go about painting with perspective? Consider the figure below:
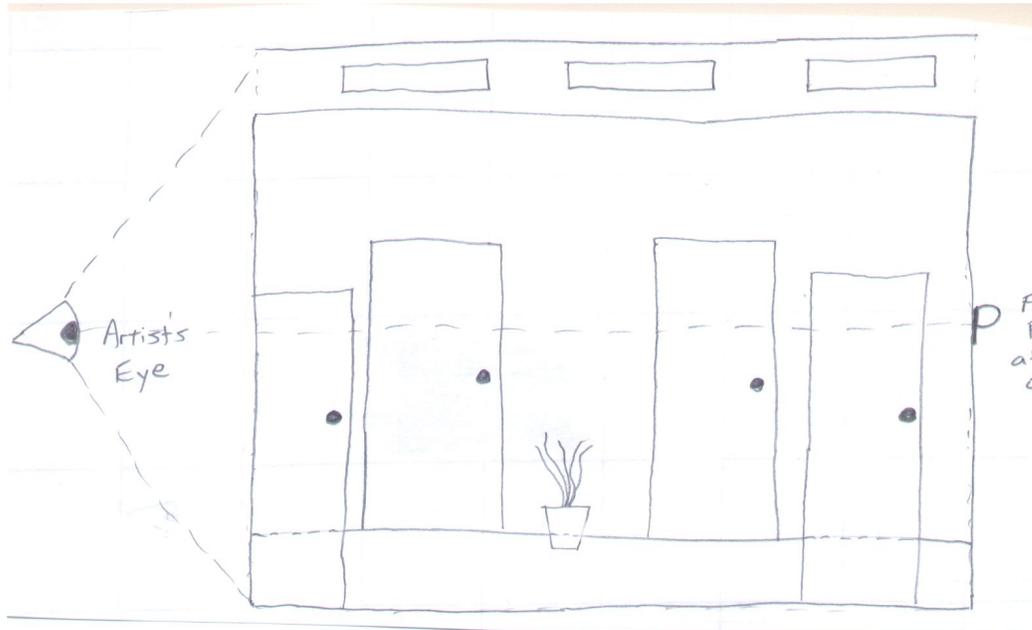

P (on the right) is the focus point at the end of the hallway

The artist gazes upon a hallway, bounded by four walls that intersect at the lines between the walls. The hallway contains four doors, three lights on the ceiling and a plant on the floor. The artist's eye is focused on a point P at the center of the end of the hallway.

To create the painting, the artist follows the rules below.

1) The point P becomes the **principal vanishing point** at the center of the painting.
2) All lines that are perpendicular to the canvas (meaning they go straight into the canvas) disappear at the principal vanishing point.
3) All other lines that move away from the canvas should be drawn to disappear on the **horizon line**, which is a horizontal line intersecting the principal vanishing point.
4) All shapes that are the same size should decrease by a constant percentage as they get closer to the principal vanishing point.
5) All other shapes should decrease to an appropriate size according to their position in the painting with respect to other objects.

With these rules, the artist creates the following:
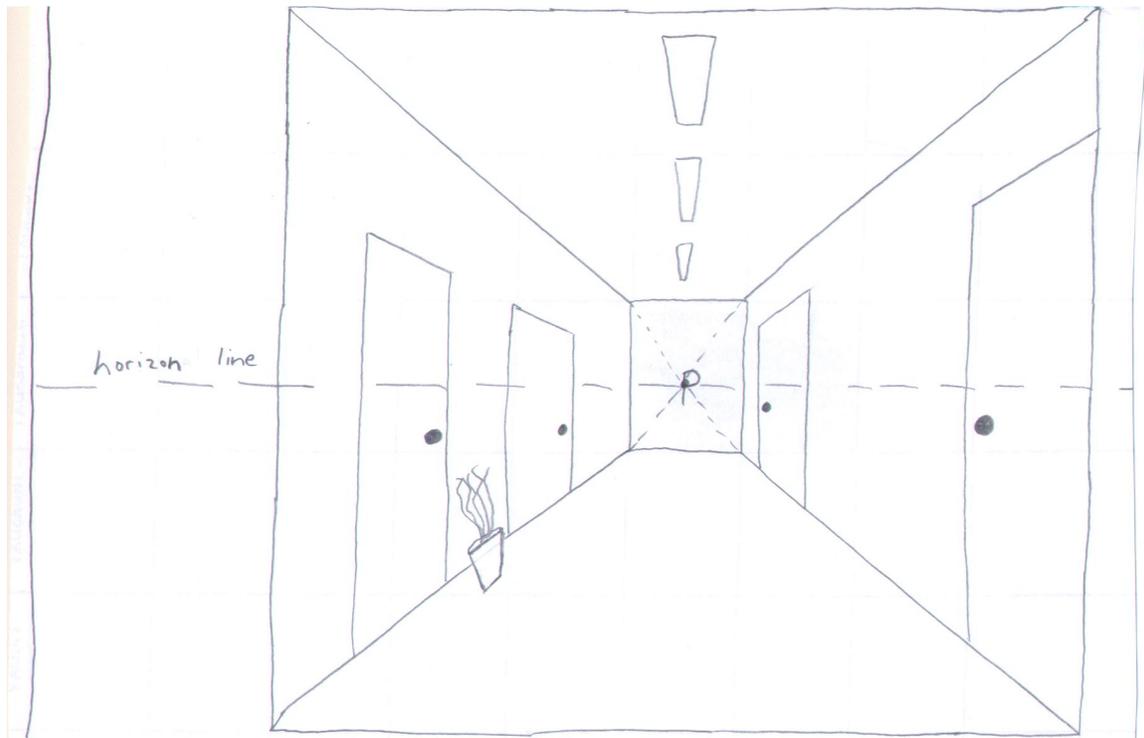
We can see that the lines of the walls, ceiling, floor, lights, and horizontal door lines (if extended) are all drawn to the principal vanishing point **(P).** If the hallway were longer, it would be drawn to disappear at the principal vanishing point.

The lights should decrease by a constant amount. If the first light has a initial width of 1 cm and the second light a width of 0.8 cm (80% of 1 cm), we would expect the third light to have a width of 0.64 cm (80% of 0.8 cm) and the fourth light (if there were one) to have a width of 0.512 cm (80% of 0.64 cm).

(In our primitive drawing of the car on the road earlier, we see similar techniques at work. The road disappears at the principal vanishing point. The horizon line represents the entire visible horizon. The trees are meant to decrease in size by a constant percentage, as are the lampposts.)

Perspective in Works of Art

We see the use of perspective in well known works of art. Consider the painting *The Last Supper* by Leonardo da Vinci (1452-1519):

http://www.artchive.com/artchive/l/leonardo/lastsupp.jpg

You can see how the lines of the ceiling tiles and the line of the tops of the doorways on the left and right side all move toward the principal vanishing point, where the central figure of Christ is seated. Along with the realism of the image is attached a story of strong religious significance.

Consider the next image, a sketch by da Vinci for his work *Adoration of the Magi*:

http://en.wikipedia.org/wiki/File:Leonardo_da_Vinci_-_Adorazione_dei_Magi_-_Google_Art_Project.jpg

Note the techniques of perspective (horizon line, vanishing point) being used to give a realistic view of the structures in the drawing.

Go to
http://www.webexhibits.org/sciartperspective/index.html
and
https://www.history.com/topics/renaissance-art

to observe other works of art of the Renaissance. Check them out! Wander through some of the pages and check out links. Try to find another example of one work of art you find that uses perspective. How is perspective used?

"Abuse" of Perspective

Artists who are aware of the rules of perspective can bend the rules to create unrealistic or impossible images.

Consider William Hogarth's *False Perspective*

http://annepercoco.wordpress.com/2011/05/01/hogarths-satire-on-false-perspective/

What are three ways that perspective is used to create physical impossibilities in False Perspective?

Consider **Belvedere**, by M.C. Escher:

http://www.artchive.com/artchive/E/escher/escher_belvedere.jpg.html
(Click on it to get a larger image.)

Identify one way that perspective is used to create physical impossibilities in **Belvedere**.

M.C. Escher is well known for his many works art of that blend realism, perspective, impossibility, imagination, fantasy and symmetry. Check some of his work out at www.worldofescher.com or www.mcescher.com