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Patrick J. Wallach
*CUNY Queensborough Community College*

Chapter 12

Symmetry

What is the meaning of the word symmetry? We may think of words like balance, equality, patterns, repetition, and visually pleasing.

In applying mathematics to art, we think of **symmetry** as a situation in which an image or object remains unchanged after some "operation" or act.

Reflection symmetry occurs when an image remains the same after being cut in half and reflected (flipped) over a straight line (called the **line of symmetry**). Consider the following images:

(*The word Symmetry was cut off*)
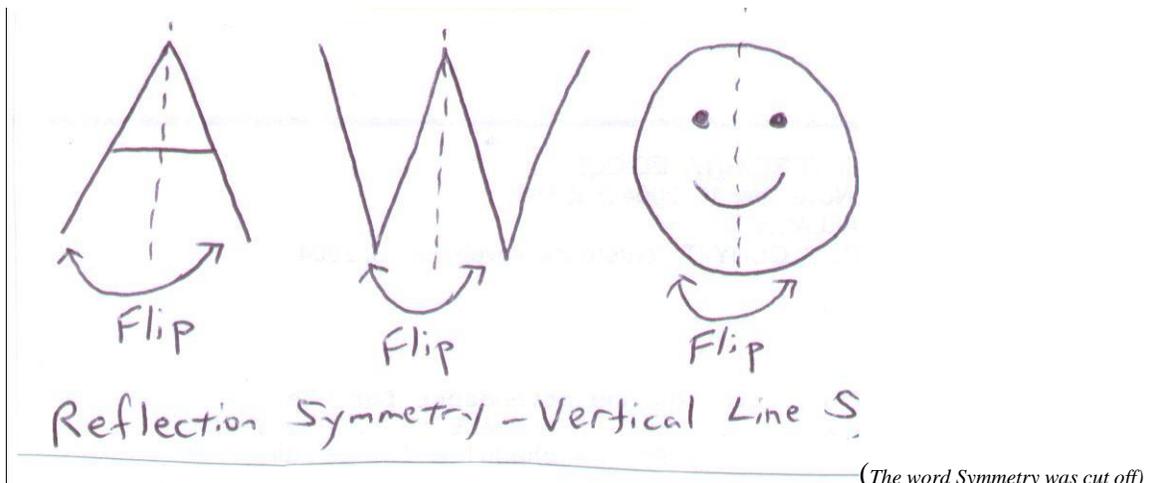
These images contain **vertical line symmetry** (a type of reflection symmetry). If you cut the letter A in half with a vertical line, you can flip the image over the line and still have the same image (A). The same thing can be done with the letter W or a smiley face.

Consider the next set of images:
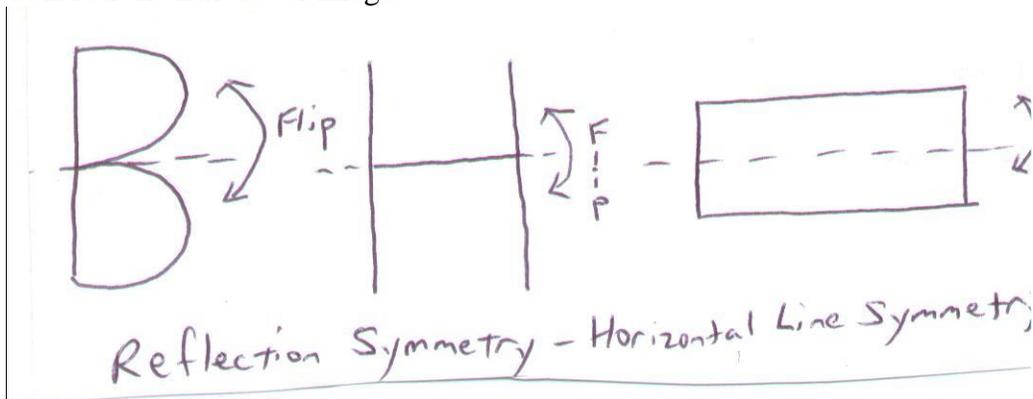
These images contain **horizontal line symmetry** (another type of reflection symmetry). If you cut the letter B in half with a horizontal line, you can flip the image over the line and still have the same image (B). The same thing can be done with the letter H or a rectangle.

Some images have multiple line of symmetry:

Multiple Lines of Symmetry
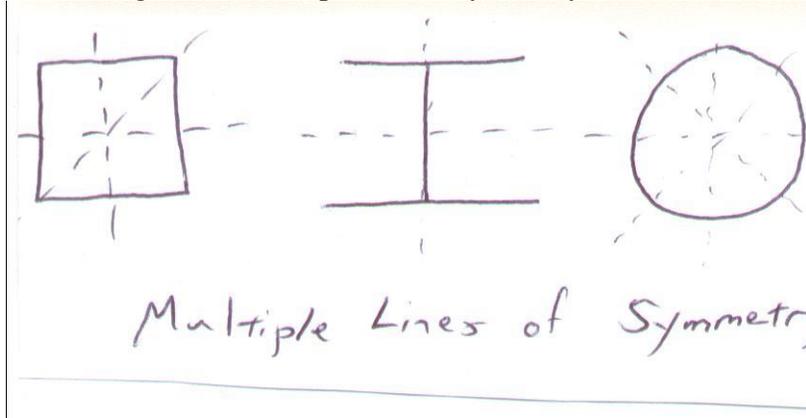
A square can be cut vertically and horizontally and along its diagonals, creating mirror images on both sides. Letters like I and H have both vertical and horizontal line symmetry. A circle can be cut an infinite number of ways into two equal halves.

Rotation Symmetry

There are letters like N and S that do not have reflection symmetry, but if you turn them over (using a 180º half turn):

turn 180°   turn 180°   turn 90°
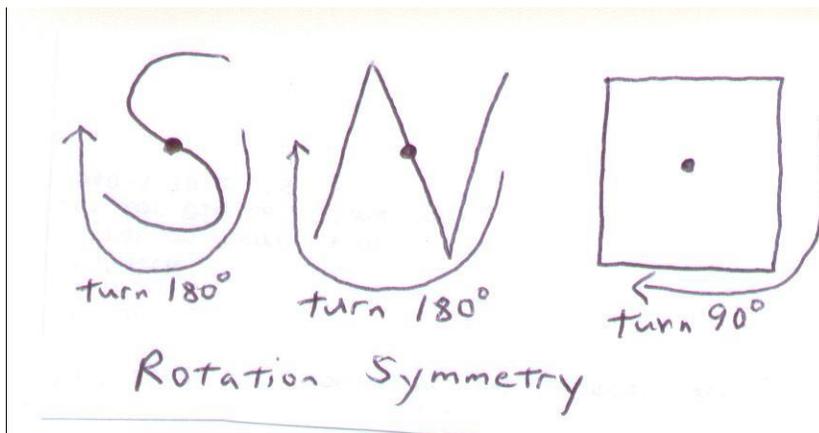
Rotation Symmetry

We see that the letters S and N look exactly the same after a 180º half turn. We also notice above that a square looks the same after a 90º turn (or a half turn or a 270º turn).

Objects with rotation symmetry appear the same after a rotation of some degree through the central point of the object. A circle can be rotated any number of degrees and will always appear the same.

Note: Everything looks the same after a full 360° turn. That's not rotation symmetry.

Translation Symmetry

Consider the following image:
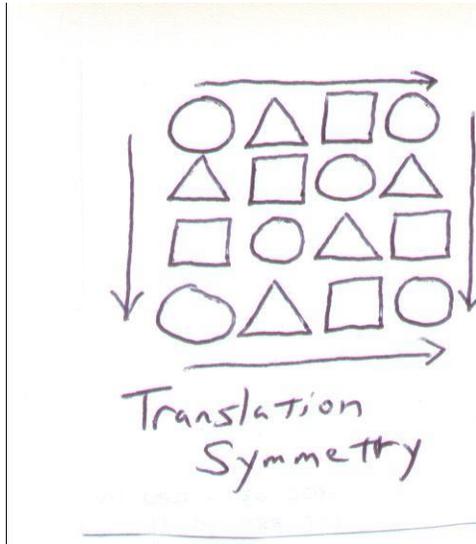
It's not a work of art, but we can see a repeating pattern of shapes as we travel up and down or left and right. If we continued going down, we know the fifth row would be Triangle-Square-Circle-Triangle. If we put in a fifth column, it would be Triangle-Square-Circle-Triangle. And so on.

Translation symmetry occurs when a repeating patterns remains the same when shifted in different directions. Even SSSSSSSSSSSSSSSS has translation symmetry.

We can find symmetry in so many works of art, particularly fantasy images or works of art made of different shapes.

Check out *The Vision of the Empyrean*, by Gustave Dore.

http://www.astrogeographia.org/star_of_the_magi/galactic_center_central_sun/
(Scroll down to the middle of the page)

The entire image is not symmetric, but contains an overwhelming image of reflection and rotation symmetry.

If you go to:

http://www.mcescher.com

and click on **Picture Gallery** and click on the **Symmetry** link (or other links), you will find all kinds of works of art with different symmetries.

Tiling

Tiling is the art of covering a flat service with geometrical shapes, usually polygons (a shape made of straight lines) and very often regular polygons (where all sides are equal).

Here are some examples of regular polygon tilings made of equilateral triangles, squares and hexagons. Tilings are made by joining the polygons at the corners.
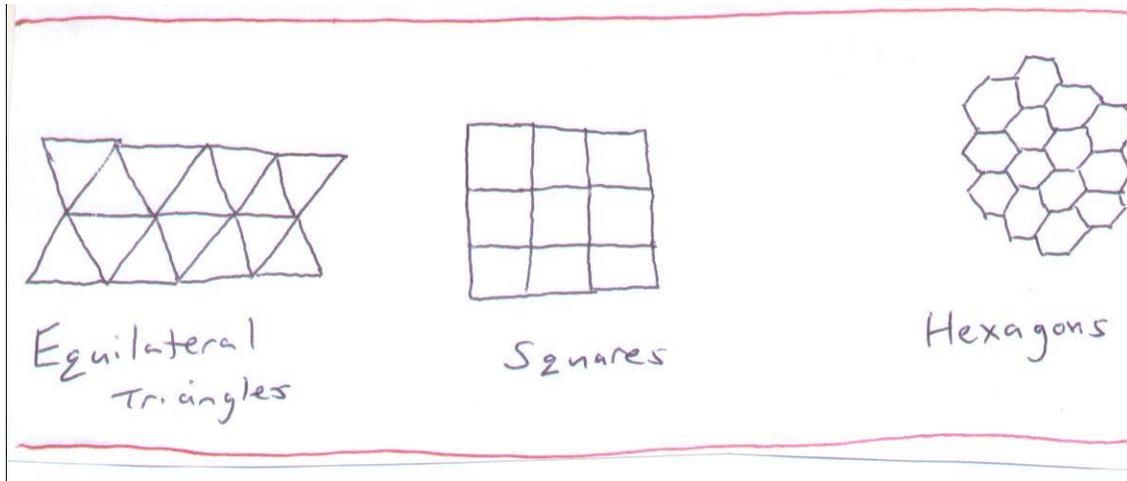
However, not all regular polygons can make a tiling; you can see in the following example that joining regular pentagons at the corners creates a gap that can't be filled.
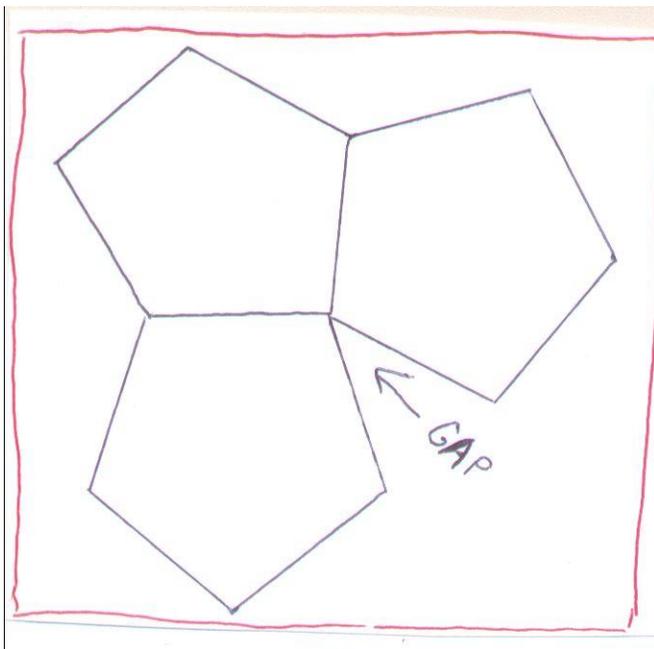
There are all sorts of tilings that can be made with more than one regular polygon. As you can see in the following examples, there are tilings of hexagons, squares and triangles, hexagons, triangles and squares, octagons and squares, and so on.

http://en.wikipedia.org/wiki/Tiling_by_regular_polygons
http://en.wikipedia.org/wiki/Great_rhombitrihexagonal_tiling

We can also make tilings out of irregular polygons. In the following example, we see how you can take make a tiling of one triangle. By translating and placing triangles down along the one side of the triangle and translating and placing triangles along another side of the initial triangle you create gaps that are themselves the identically sized triangle (some are upside down, some are positioned the same way). This creates a tiling from one triangle.
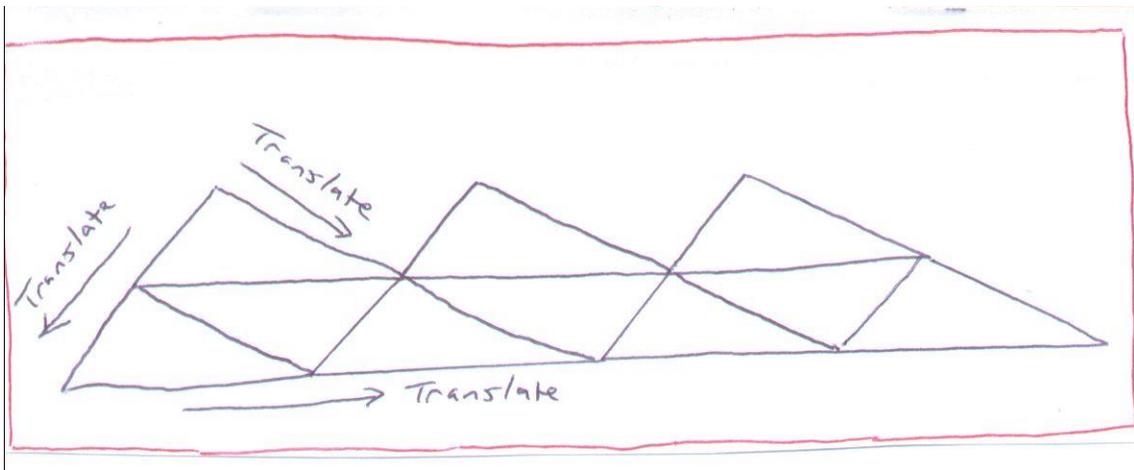
We also see an example of a quadrilateral tiling created by shifting along lines created by joining opposite corners of the quadrilateral and placing the same quadrilateral along the lines. Again, the gaps created are the identically sized quadrilateral.
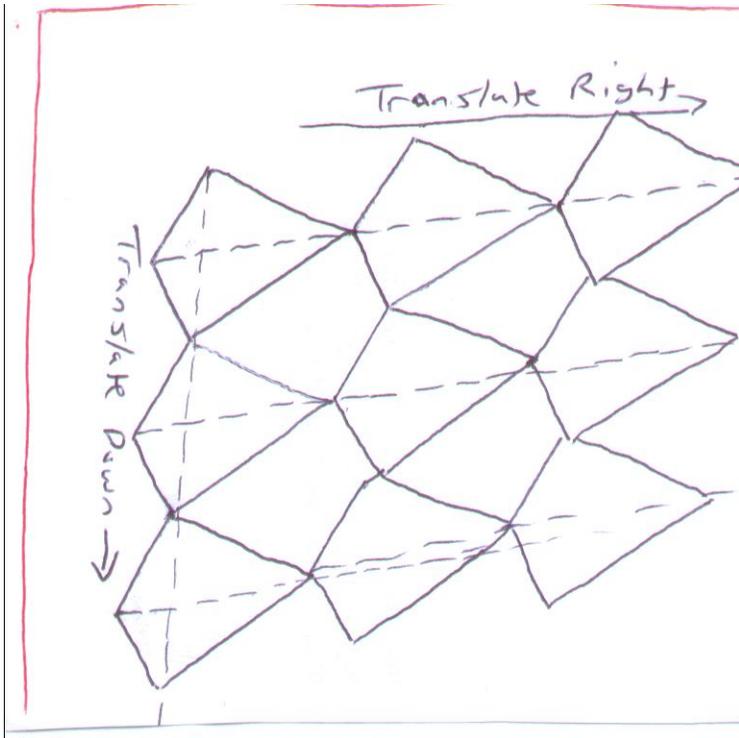
You can find many examples of tiling at:
http://www.scienceu.com/geometry/articles/tiling/
http://www.math.dartmouth.edu/~matc/math5.geometry/unit5/unit5.html#Tilings
http://mathworld.wolfram.com/Tiling.html

One last thing, check out an example of aperiodic tiling, with multiple repeated patterns.

http://www.itap.physik.uni-stuttgart.de/~gaehler/tilings/oct.html

Proportion and the Golden Ratio

Suppose you have a straight line that needs to be cut into two pieces. What way would produce a balanced yet visually pleasing results?

_____

Instead of cutting the line in half, we will cut the line so that one piece has a length of 1 and the other a length of L:

_____|_____

If it is true that the ratio of the larger length to the smaller length is equal to the ratio of the entire length to the larger length, that is:

$$\frac{L}{1} = \frac{L+1}{L}$$

Then L has a special value, denoted by the Greek Letter ϕ (pronounced "fee"), such that:

$L = \phi = (1 + \text{the square root of } 5)/2 = 1.618033989\ldots \approx 1.6$

In short, this occurs when the larger piece is about 1.6 times as long as the shorter piece.

ϕ is referred to as the **golden ratio, golden proportion** or **divine proportion**. The golden ratio is a value of tremendous mathematical significance, not unlike π.

What do I do with the golden ratio?

From the golden ratio, we can create **golden rectangles**, rectangles whose length and width are in the proportion of the golden ratio:

```
 _____
|                           |
|                           |
|                           | width
|                           |
|_____|
          length
```
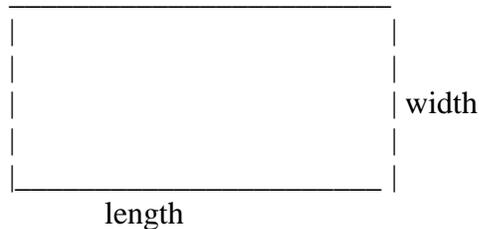
If it's true that length/width $= \phi \approx 1.6$, this is a golden rectangle.

Example   If you have a rectangle with a length of 48 and a width of 30, it could be considered a golden rectangle because length/width=48/30=1.6

What else?

The golden rectangle has historical significance; it is often seen in works of art, such the Parthenon in Athens, Greece, which was built about 2500 years ago:

http://www.ancient-greece.org/architecture/parthenon.html

The length divided by the height of the building is the golden ratio. The golden ratio is also seen in other parts of the structure.

In Leonardo da Vinci's *St. Jerome*, it is felt that the main portion of St. Jerome's body is contained within a golden rectangle:

http://www.artchive.com/artchive/L/leonardo/leonardo_jerome.jpg.html

(Click on the painting to enlarge it.)

In a painting filled with rectangles, we should be able to find many golden rectangles, as in Piet Mondrian's *Place de la Concorde*.
http://www.wikipaintings.org/en/piet-mondrian/place-de-la-concorde-1943

Golden rectangles often occur in nature. A collection of golden rectangles can used to simulate the logarithmic spiral of a nautilus shell. Look at the following example:

http://mathworld.wolfram.com/GoldenSpiral.html

Any golden rectangle can be divided into a square (with a side equal to the width of the rectangle) and another golden rectangle. This second golden rectangle can then be divided into another square and another golden rectangle. This process can be performed infinitely.

By drawing an arch through the corner of every square we can create a logarithmic spiral, as seen in the picture.

In nature, the logarithmic spiral is seen in the curve of a nautilus shell:

http://www.math.wsu.edu/faculty/kallaher/Images/small_shell.jpg

What else can we do with the golden ratio? Let's see:

The Fibonacci Sequence

Let's say you are a scientist trying to raise "baby" amoeba. You start with one amoeba (which is not male or female, but reproduces by dividing itself). The "rules" are:
1)   It takes one hour for an amoeba to become a mature "adult."
2)   Every month an adult amoeba splits to produce a baby amoeba.

Use this to track the amoeba population. See below:
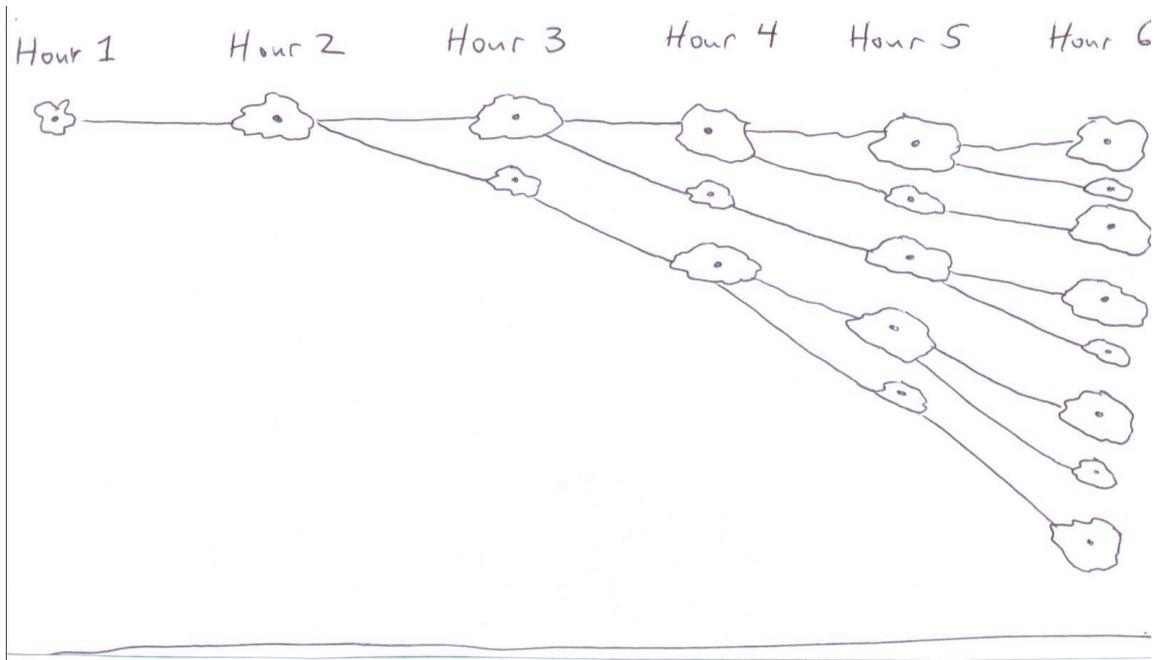
In the first hour, there is one baby amoeba.

In the second hour, the one baby grows to become an adult amoeba.

In the third hour, the amoeba splits to create one adult and one baby (1 Baby + 1 Adult = 2 amoeba).

In the fourth hour, one baby splits from the adult and the baby matures (1 Baby + 2 Adult = 3 amoeba).

In the fifth hour, two babies split from the two adults and the baby matures (2 Baby + 3 Adult = 5 amoeba).

In the sixth hour, three babies split from the three adults and two babies mature (3 Baby + 5 Adult = 8 amoeba).

We can continue this pattern hour after hour.

What will happen in Hour 7?

The five adults will split to create 5 more babies (5 babies, an increase in 5 to the total). The three babies will mature into adults (increasing the adult total to 5+3=8, the previous total).

The new total is 5 Baby + 8 Adult = 13 amoeba).

Another Way To Look At It

What is happening actually is that number of pairs in any month (after month 2) is the sum of the number of rabbits from the two previous months.

Hour 1 = 1 amoeba (We may call this F_1)

Hour 2 = 1 amoeba (F_2)

Hour 3 = 1 + 1 = 2 amoeba (F_3)
Hour 4 = 1 + 2 = 3 amoeba (F_4)
Hour 5 = 2 + 3 = 5 amoeba (F_5)
Hour 6 = 3 + 5 = 8 amoeba (F_6)

Without drawing a picture, we would see that:
Hour 7 = 5 + 8 = 13 (F_7)
Hour 8 = 8 + 13 = 21 (F_8)
Hour 9 = 13 + 21 = 34 (F_9)
Hour 10 = 21 + 34 = 55 (F_{10})
Hour 11 = 34 + 55 = 89 (F_{11})
Hour 12 = 55 + 89 = 144(F_{12})
Hour 13 = 89 + 144 = 233 (F_{13})
Hour 14 = 144 + 233 = 377 (F_{14})
Hour 15 = 233 + 377 = 610 (F_{15})
Hour 16 = 377 + 610 = 987 (F_{16})
Hour 17 = 610 + 987 = 1597 (F_{17})
Hour 18 = 987 + 1597 = 2584(F_{18})

and so on and so on.

We call these numbers:

1,1,2,3,5,8,13,21,34,55,89,144,etc. the **Fibonacci Sequence**. The Italian mathematician Leonardo Pisano Fibonacci (c. 1170-1250) is credited with the first use of the Fibonacci sequence.

I'm still not sure why I should care about it.

Well, if you take fractions of successive Fibonacci numbers:
1/1=1
2/1=2
3/2=1.5
5/3=1.67
8/5=1.6
13/8=1.625
21/13=1.6154
34/21=1.619
55/34=1.61765
89/55=1.61818
144/89=1.61798
233/144=1.618056

…the results you get approach the golden ratio ϕ=1.618033989 very quickly!

So I take an ordinary mathematical problem, generate a series of numbers and out comes the golden ratio! Wow! It must be a special number!

Fibonacci numbers occur in nature.

Consider the following picture of logarithmic spirals:

http://mathworld.wolfram.com/GoldenSpiral.html

This pattern can be seen in the floret arrangement of a sunflower!

http://www.public-domain-image.com/studio/slides/sunflower-on-white-background.jpg

The number of florets in the various rows are Fibonacci numbers.

Biologists have also seen Fibonacci numbers in the number of petals in different flowers (3 petals for irises, 5 petals for primroses, 34 petals for daisies, etc.).

I hope this little bit of information gives you a better appreciation of the golden ratio. It's not just a pretty number. Its applications in mathematics, science and the beauty of nature make it hard to ignore.

And if you have any question about "When will I ever use this?", here are some people who have:

http://www1.whdh.com/features/articles/specialreport/DBM596

***(I intend to drop the lowest lab grade. If you are satisfied with
your first 8 labs, you can skip this one.)***

Please e-mail me this lab because I need to check your use of formulas.

In EXCEL, you can use the ^ key for powers:

1) Calculate the first twenty powers of 3: (3^1=3, 3^2=9, 3^3=27, 3^4=81, etc.).

  For example, in cells A1 to A20, place the numbers 1 through 20 (Ask me
for a shortcut method). Go to cell B1 and type in the formula =3^A1. (This
will give you the first power of 3. Copy the formula down to the next 19
cells to get a result that looks like this:

1	3
2	9
3	27
4	81
5	243
etc.	etc.

2) Revisit the problem of the business associate, who owes you money, who
made the following proposal: Pay 1 cent on November 1, 2 cents on
November 2, 4 cents on November 3, 8 cents on November 4, etc., until
November 30.

  After creating headings, you can place the values 1 through 30 in column
D in cells D2 to D31. In column E, in cell E2, use the formula
=0.01*2^(D2-1), and copy it to the remaining cells below, to determine
the payment amount of any day (Notice that each day's payment is the
corresponding power (minus 1) of 2 times 1 cent). In column F, in cell F2,
use the formula =0.01*2^D2-0.01 and copy it to the remaining cells below
to determine the total after any number of days (Notice that each day's
total is one cent times the corresponding power of 2, take away one cent).
Use EXCEL to determine each individual payment and the sum of all
payments for the month of November. You should get a result like this:

Day	Payment	Total
1	0.01	0.01
2	0.02	0.03
3	0.04	0.07
4	0.08	0.15
5	0.16	0.31
etc.	etc.	etc.

Calculate the amounts after 50 days, 100 days and 150 days. What do you think of the proposal now?

(Apply similar methods to the next two problems.)

3) Make a table (see the notes) of the frequency of **3 octaves** starting from middle C with a frequency of 260 cps. Use $260*1.05946^N$ for N half-steps over middle C. Your table will look something like this:

# of half steps above middle C	Note	Frequency
0	C	260
1	C#	275.5
2	D	291.8
3	D#	309.2
4	E	327.6
etc.	etc.	etc.

4) Make a table to represent the growth of your initial investment of $500 at 8% interest compounded annually for 40 years. Use $500*1.08^N$ for a formula. Your table will look something like this:

Year	Balance
0	$ 500.00
1	$ 540.00
2	$ 583.20
3	$ 629.86
4	$ 680.24
etc.	etc.

How long would it take for your investment to grow to $95,000?