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Cosmic string interactions induced by gauge and scalar fields

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We study the interaction between two parallel cosmic strings induced by gauge fields and by scalar fields with nonminimal couplings to curvature. For small deficit angles the gauge field behaves like a collection of nonminimal scalars with a specific value for the nonminimal coupling. We check this equivalence by computing the interaction energy between strings at first order in the deficit angles. This result provides another physical context for the "contact terms" which play an important role in the renormalization of black hole entropy due to a spin-1 field.

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I. INTRODUCTION

For a single cosmic string in four Euclidean dimensions, the metric is [1,2]

$$ds^2 = dr^2 + r^2 d\psi^2 + d\tau^2 + dz^2. \tag{1}$$

The string tension produces a deficit angle, $\psi \approx \psi + \beta$, where

$$\beta = 2\pi - 8\pi\lambda. \tag{2}$$

Here $\lambda = G\mu$, where G is Newton's constant and μ is the mass per unit length of the string.

We will be interested in the interaction between two parallel cosmic strings. At the classical level there is no force between strings,[1] but (as in the Casimir effect) an interaction potential can be generated at one loop by a quantum field propagating on this background. For simplicity we will take a perturbative approach and calculate the interaction energy at first order in the product of the two deficit angles. We consider two types of fields—scalar fields with a nonminimal coupling to curvature and Abelian gauge fields—as the main point of this paper is to highlight a relation between these two cases. Vacuum polarization in the presence of a single cosmic string has been studied before; see for example Refs. [5–7] for scalar fields and [8,9] for gauge fields. For related calculations in the presence of multiple cosmic strings, see Refs. [10,11].

We begin by recalling the argument that, to first order in the background curvature, there should be a relation between gauge fields and scalar fields with specific nonminimal couplings to curvature. To our knowledge this relation was first stated in Ref. [12], although the essence of the following argument is taken from Ref. [13]. Consider a spacetime which is a product $\mathcal{M}_n \times \mathbb{R}^{d-n}$ of a weakly curved, n-dimensional Einstein manifold $\mathcal{M}_n$ with flat space $\mathbb{R}^{d-n}$. The metric takes the form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + \delta_{ij}dx^i dx^j, \tag{3}$$

where x^α are coordinates on $\mathcal{M}_n$, and x^i are coordinates on $\mathbb{R}^{d-n}$. The Einstein manifold has Ricci curvature $R_{\alpha\beta} = \frac{1}{n}g_{\alpha\beta}R$.[2] Choose a vielbein $g_{\alpha\beta} = e^a_\alpha e^b_\beta \delta_{ab}$ and denote the corresponding spin connection ω_α.

To establish the relation between gauge and scalar fields, we compare their equations of motion. For a gauge field, the equations of motion in Feynman gauge are

$$-\nabla_\nu\nabla^\nu A_\mu + R_{\mu\nu}A^\nu = 0, \tag{4}$$

where $x^\mu = (x^\alpha, x^i)$. There are ghosts associated with this choice of gauge which behave like a pair of minimally coupled scalar fields [14]. The components of the gauge field tangent to $\mathbb{R}^{d-n}$ obey

$$-\nabla_\beta\nabla^\beta A_i - \partial_j\partial^j A_i = 0, \tag{5}$$

where the covariant derivative ∇_α treats A_i as a singlet of $SO(n)$. That is, the components A_i behave like minimally coupled scalar fields. The components of the gauge field tangent to $\mathcal{M}_n$, on the other hand, obey

$$-\nabla_\beta\nabla^\beta A_a - \partial_j\partial^j A_a + \frac{1}{n}RA_a = 0. \tag{6}$$

Here ∇_α acts on $A_a = e_a{}^\alpha A_\alpha$ in the fundamental representation of $SO(n)$, and we've made use of the fact that $R_{\alpha\beta} = \frac{1}{n}g_{\alpha\beta}R$. So the components A_a are in the fundamental representation of $SO(n)$ and have an explicit nonminimal coupling to curvature.

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[1]In classical gravity there is, however, a nontrivial scattering amplitude which results from the conical boundary conditions [3,4].

[2]By Einstein manifold, we mean a manifold with Ricci curvature locally proportional to the metric, $R_{\alpha\beta}(x) = f(x)g_{\alpha\beta}(x)$. In two dimensions all manifolds are Einstein. In higher dimensions the contracted Bianchi identity $\nabla^\mu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0$ requires that f be a constant. In either case it follows from the definition that $R_{\alpha\beta} = \frac{1}{n}g_{\alpha\beta}R$.

Physical quantities can be computed perturbatively, as an expansion in powers of the background curvature. As a concrete example, imagine computing the effective action for the background which results from integrating out A_μ. The spin connection can appear in the effective action, but only through its field strength $F = d\omega + \omega^2$. In fact the field strength can first appear in the effective action in terms such as $F_{\alpha\beta}F^{\alpha\beta}$ that are quadratic in the curvature. So to first order in the background curvature, we can forget about the spin connection and treat A_a as a collection of n scalar fields with a nonminimal coupling to curvature. The equation of motion for a nonminimal scalar is

$$-\nabla_\beta\nabla^\beta\phi - \partial_j\partial^j\phi + \xi R\phi = 0, \tag{7}$$

and comparing to (6) we identify the effective nonminimal coupling parameter $\xi = 1/n$. Thus to first order in the background curvature, a gauge field is equivalent to n scalar fields with $\xi = 1/n$, plus $d - n$ minimally coupled scalars.

This discussion is relevant to parallel cosmic strings because in two dimensions every manifold is an Einstein manifold. The argument suggests that, to first order in the product of the deficit angles, the interaction between two cosmic strings induced by a gauge field should be the same as the interaction induced by an appropriate collection of nonminimal scalars.

In the remainder of this paper, we verify this claim by computing the interaction energy between cosmic strings perturbatively. In Sec. II, we compute the interaction energy for a scalar field, and in Sec. III we carry out the corresponding computation for a gauge field. We conclude in Sec. IV, where we comment on our results and point out the relation to studies of black hole entropy.

II. NONMINIMAL SCALAR ENERGY

The Euclidean action is

$$S = \int d^4x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\xi R\phi^2\right).$$

For the conical geometry (1), the scalar curvature is[3]

$$R = 16\pi\lambda\delta^2(x)/\sqrt{g}. \tag{8}$$

The action on a cone can be split into three pieces,

$$S_{\rm cone} = S_0 + S_{\rm int}, \qquad S_{\rm int} = S_{\rm wedge} + S_{\rm tip}, \tag{9}$$

where

$$S_0 = \int d^4x\frac{1}{2}\delta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{10}$$

is the action in flat space,

[3]The easiest way to see this is to note that a truncated cone, i.e., a disc with a conical singularity at the center, has Euler characteristic $\chi = \frac{1}{4\pi}\int d^2x\sqrt{g}R + \frac{1}{2\pi}\beta = 1$.

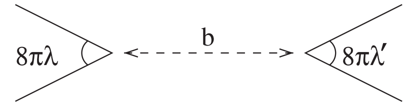

FIG. 1. Two parallel cosmic strings, separated by a distance b.

$$S_{\rm wedge} = -\int d\tau dz\int_0^\infty rdr\int_{-4\pi\lambda}^{4\pi\lambda} d\psi\frac{1}{2}\delta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{11}$$

cancels the flat-space action in the region corresponding to the deficit angle, and

$$S_{\rm tip} = \int d\tau dz 8\pi\lambda\xi\phi^2 \tag{12}$$

arises from the nonminimal coupling to curvature. It is straightforward to extend this to a pair of cosmic strings, just by putting the deficit angles in opposite directions as shown in Fig. 1.

We will treat $S_{\rm int}$ as a perturbation.[4] To find the interaction energy per unit length along the strings $\mathcal{H}_{\rm int}$, we use [15]

$$\int d\tau dz\mathcal{H}_{\rm int} = \langle 1 - e^{-S_{\rm int}}\rangle_{C,0}, \tag{13}$$

where the subscript C, 0 denotes a connected correlation function computed in the unperturbed theory (10). Expanding in powers of $S_{\rm int}$, the leading $\mathcal{O}(\lambda\lambda')$ interaction between the strings comes from

$$\int d\tau dz\mathcal{H}_{\rm int} \approx -\langle S_{\rm int}^{(1)}S_{\rm int}^{(2)}\rangle_{C,0}, \tag{14}$$

where the superscripts (1), (2) refer to the first and second cosmic string, respectively. Some useful unperturbed correlators are

$$\langle\phi(x)\phi(x')\rangle = \frac{1}{4\pi^2}\frac{1}{(x-x')^2} \tag{15}$$

and

$$\begin{aligned}\langle(\partial\phi)^2(x)(\partial\phi)^2(x')\rangle &= \frac{6}{\pi^4}\frac{1}{(x-x')^8}\\ \langle(\partial\phi)^2(x)\phi^2(x')\rangle &= \frac{1}{2\pi^4}\frac{1}{(x-x')^6}\\ \langle\phi^2(x)\phi^2(x')\rangle &= \frac{1}{8\pi^4}\frac{1}{(x-x')^4}.\end{aligned}$$

There are three types of interactions. For generality we can imagine that the two strings have different nonminimal couplings ξ, ξ'.

[4]This is somewhat subtle, since it is not manifest that perturbation theory in $S_{\rm int}$ will enforce the proper conical boundary condition $\phi(r,\psi) = \phi(r,\psi+\beta)$. Fortunately, the boundary conditions are controlled by the spin connection on the cone, which as we argued in the introduction can only enter at second order in the deficit angle.

wedge-wedge

To first order in λ and λ', the wedges can be treated as very narrow, so that

$$\mathcal{H}_{\rm int} = -16\pi^2\lambda\lambda' \int d\tau dz \int_0^\infty x dx \int_0^\infty x' dx' \times \frac{6}{\pi^4}\frac{1}{(\tau^2+z^2+(x+x'+b)^2)^4} = -\frac{4\lambda\lambda'}{15\pi b^2}.$$

wedge-tip

For wedge 1 with tip 2 we have

$$\mathcal{H}_{\rm int} = 32\pi^2\lambda\lambda'\xi' \int d\tau dz \int_0^\infty x dx \times \frac{1}{2\pi^4}\frac{1}{(\tau^2+z^2+(x+b)^2)^3} = \frac{4\lambda\lambda'\xi'}{3\pi b^2}.$$

tip-tip

The interaction between the two tips is

$$\mathcal{H}_{\rm int} = -64\pi^2\lambda\lambda'\xi\xi' \int d\tau dz \frac{1}{8\pi^4}\frac{1}{(\tau^2+z^2+b^2)^2} = -\frac{8\lambda\lambda'\xi\xi'}{\pi b^2}.$$

Assembling these results, to first order in λ and λ' the interaction energy per unit length due to a nonminimally coupled scalar field is

$$\mathcal{H}_{\rm int} = \frac{\lambda\lambda'}{\pi b^2}\left(-\frac{4}{15}+\frac{4}{3}(\xi+\xi')-8\xi\xi'\right). \tag{16}$$

To check the validity of our perturbative approach, consider computing $\langle\phi^2\rangle$ for a minimally coupled scalar field in the presence of a single cosmic string. At first order in perturbation theory, after subtracting the divergence which is present in flat space, we have

$$\langle\phi^2\rangle = -\langle\phi^2 S_{\rm wedge}\rangle_{C,0} = \frac{\lambda}{6\pi^2 r^2}, \tag{17}$$

where r is the distance from the tip of the cone. On the other hand $\langle\phi^2\rangle$ can be computed exactly,

$$\langle\phi^2\rangle = \int_0^\infty ds K(s,x,x), \tag{18}$$

where the scalar heat kernel on a cone is[5]

$$K(s,x,x) = -\frac{1}{2\beta}\frac{1}{(4\pi s)^2}\int_{-\infty}^{\infty} dy e^{-\frac{r^2}{s}\cosh^2(y/2)} \times\left(\cot\frac{\pi}{\beta}(\pi+iy)+\cot\frac{\pi}{\beta}(\pi-iy)\right). \tag{19}$$

Expanding the heat kernel to first order in the deficit angle and integrating over s reproduces (17).

III. GAUGE FIELD ENERGY

We start from the Euclidean action

$$S = S_{\rm Maxwell} + S_{\rm gaugefixing} = \int d^d x\sqrt{g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}(\nabla_\mu A^\mu)^2\right).$$

There are ghosts associated with this choice of gauge that behave like a pair of minimally coupled scalars.

If we smooth out the conical singularities, so that we can freely integrate by parts, the action becomes

$$S = \int d^d x\sqrt{g}\left(\frac{1}{2}\nabla_\mu A_\nu\nabla^\mu A^\nu - \frac{1}{2}A^\mu(\nabla_\mu\nabla_\nu-\nabla_\nu\nabla_\mu)A^\nu\right) = \int d^d x\sqrt{g}\left(\frac{1}{2}\nabla_\mu A_\nu\nabla^\mu A^\nu + \frac{1}{2}R_{\mu\nu}A^\mu A^\nu\right).$$

In the second line we used $[\nabla_\mu,\nabla_\nu]A^\nu = -R_{\mu\nu}A^\nu$. We work on a space which is a product of a two-dimensional cone with coordinates x^α and a $(d-2)$-dimensional flat space with coordinates x^i,

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + \delta_{ij}dx^i dx^j.$$

In two dimensions the Ricci tensor is proportional to the metric, so from (8)

$$R_{\alpha\beta} = 8\pi\lambda g_{\alpha\beta}\delta^2(x)/\sqrt{g}, \tag{20}$$

where $8\pi\lambda$ is the deficit angle. Thus, the action for a gauge field on a cone can be decomposed into

$$S_{\rm cone} = S_0 + S_{\rm int}, \qquad S_{\rm int} = S_{\rm wedge} + S_{\rm tip}. \tag{21}$$

For example in four dimensions

$$S_0 = \int d^4x\frac{1}{2}(\partial_\mu A_\nu)^2 \tag{22}$$

is the Feynman gauge action in flat space.

$$S_{\rm wedge} = -\int d\tau dz\int_0^\infty rdr\int_{-4\pi\lambda}^{4\pi\lambda} d\psi\frac{1}{2}(\partial_\mu A_\nu)^2 \tag{23}$$

cancels the flat-space action in the region corresponding to the deficit angle, and

$$S_{\rm tip} = 4\pi\lambda\int d\tau dz g_{\alpha\beta}A^\alpha A^\beta \tag{24}$$

arises from the explicit coupling to curvature. Aside from the sums over photon polarizations, this is identical to the decomposition of the nonminimal scalar action (9).

[5] See for example Ref. [16]. We dropped the term in the heat kernel $1/(4\pi s)^2$, which is responsible for the divergence in flat space.

The interaction between two cosmic strings can be calculated perturbatively, just as for a nonminimal scalar field.[6] In fact, the two calculations are identical. There are $d-2$ polarizations transverse to the cone, which behave in perturbation theory just like minimally coupled scalars. Two of these polarizations are canceled by the ghosts, leaving no contribution in four dimensions. The two polarizations tangent to the cone behave like nonminimal scalars with $\xi = 1/2$. So the overall interaction energy coming from a gauge field in four dimensions is simply twice the scalar result (16) evaluated at $\xi = 1/2$. That is, for a gauge field in four dimensions

$$\mathcal{H}_{\rm int} = \frac{2\lambda\lambda'}{\pi b^2}\left(-\frac{14}{15}\right). \tag{25}$$

To check the validity of our perturbative approach, consider computing $\langle A_\mu A^\mu\rangle$ around a single cosmic string. In perturbation theory, after subtracting the divergence present in flat space, we have

$$\langle A_\mu A^\mu\rangle = \langle A_\mu A^\mu(-S_{\rm wedge} - S_{\rm tip})\rangle_{C,0} = \frac{4\lambda}{6\pi^2 r^2} - \frac{\lambda}{\pi^2 r^2}. \tag{26}$$

The first term comes from $S_{\rm wedge}$ and is four times the scalar field result (17). The second term comes from $S_{\rm tip}$ and reflects the nonminimal coupling to curvature. The same quantity can be computed exactly,

$$\langle A_\mu A^\mu\rangle = \int_0^\infty ds\, g_{\mu\nu} K^{\mu\nu}_{\rm vector}(s,x,x), \tag{27}$$

where the vector heat kernel is [16]

$$g_{\mu\nu}K^{\mu\nu}_{\rm vector} = 4K_{\rm scalar}(s,x,x) + \frac{2}{r}\partial_r s K_{\rm scalar}(s,x,x). \tag{28}$$

Expanding to first order in the deficit angle and integrating over s reproduces (26).[7]

IV. CONCLUSIONS

In this paper, we considered a cosmic string spacetime and argued that to first order in the deficit angle, there is an equivalence between a gauge field and a collection scalar field with specific nonminimal couplings to curvature. More generally, the equivalence holds on the product of any weakly curved Einstein manifold with flat space. We tested the equivalence by computing the interaction energy between two cosmic strings to first order in perturbation theory, showing that it indeed matched for the appropriate value of the nonminimal coupling parameter.

Throughout this paper we worked in Feynman gauge, which is adequate for studying gauge-invariant quantities. However, it would be interesting to study the relation between gauge and scalar fields in other choices of gauge. Also, it would be interesting to study the interaction between strings at higher orders in perturbation theory. Beyond leading order there is no reason to expect an equivalence between gauge and scalar fields, since the spin connection distinguishes between the two types of fields and can appear in the interaction energy at second order in the deficit angle.

Besides their direct application to cosmic strings, our results also have relevance to the thermodynamics of black holes. In a Euclidean formalism the entropy of a black hole measures the response of the partition function to an infinitesimal conical deficit angle inserted at the horizon [17,18]. This has been used to study the renormalization of black hole entropy due to matter fields, with the somewhat surprising conclusion that a gauge field can make a negative contribution to the entropy. In Ref. [16], it was shown that this is due to a contact term in the partition function for a gauge field, associated with particle paths that begin and end on the horizon. Here we've shown that, to first order in the deficit angle, a gauge field is equivalent to a collection of nonminimal scalars. So the contact interaction of Ref. [16] is visible at the level of the equations of motion, as the explicit nonminimal coupling to curvature seen in (6). This makes the negative renormalization of black hole entropy less mysterious, since it maps a gauge field to the well-studied problem of a nonminimally coupled scalar field in a black hole background [19]. Our results also show the physical relevance of these contact interactions: besides contributing to black hole entropy, they make a (finite, observable, gauge-invariant) contribution to the force between two cosmic strings.

We conclude with some additional evidence in support of the relation between gauge and scalar fields at first order in the background curvature. The partition function for a gauge field on a cone was evaluated in Ref. [16]. Including the ghosts, the result is

$$\beta F_{\rm gauge} = (d-2)\beta F^{\rm minimal}_{\rm scalar} + A_\perp(2\pi-\beta)\int_{\epsilon^2}^\infty \frac{ds}{(4\pi s)^{d/2}}. \tag{29}$$

Here d is the total number of spacetime dimensions, $A_\perp$ is the area of the $d-2$ transverse dimensions corresponding to the horizon, s is a Schwinger parameter, and ϵ is a UV cutoff. The partition function for a nonminimal scalar was evaluated to first order in the deficit angle in Ref. [19], with the result

$$\beta F^{\xi}_{\rm scalar} = \beta F^{\rm minimal}_{\rm scalar} + \xi A_\perp(2\pi-\beta)\int_{\epsilon^2}^\infty \frac{ds}{(4\pi s)^{d/2}}. \tag{30}$$

[6]Again, it is not manifest that perturbation theory in $S_{\rm int}$ enforces the proper conical boundary conditions on A_α, but this effect is controlled by the spin connection which can only enter at second order in the deficit angle.

[7]Note that the last term in (28), which in the black hole context captures the contact interaction of a gauge field with the horizon, corresponds at first order in perturbation theory to effects associated with $S_{\rm tip}$.

Comparing the partition functions again shows that a gauge field corresponds to two nonminimal scalars with $\xi = 1/2$, together with $d-2$ minimal scalars (two of which are canceled by the ghosts). The same relation can be seen in the one-loop renormalization of Newton's constant,

$$\frac{1}{4G_{N,\mathrm{ren}}} = \frac{1}{4G_N} + \frac{c_1}{(4\pi)^{\frac{d-2}{2}}(d-2)\epsilon^{d-2}}, \tag{31}$$

where the Seeley-de Witt coefficients are [20]

$$c_1 = \begin{cases} \frac{1}{6} - \xi & \text{nonminimal scalar} \\ \frac{d-2}{6} - 1 & \text{gauge field including ghosts} \end{cases} \tag{32}$$

On a d-dimensional Einstein manifold, the gauge field result corresponds to d nonminimal scalars with $\xi = 1/d$, plus two minimally coupled scalar ghosts.

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