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On Groups of Homological Dimension One

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Summary: It has been conjectured that the groups of homological dimension one are precisely the nontrivial locally free groups. Some algebraic, geometric and analytic properties of any potential counter example to the conjecture are discussed.

1. Introduction

Classical results of Stallings and Swan state that the groups of cohomological dimension one are precisely the nontrivial free groups. There is no corresponding classification for the groups of homological one, but the only known examples are the nontrivial locally free groups, and it has been conjectured that these are the only examples. The purpose of this paper is to consider the evidence for this conjecture.

In Section 2, the aforementioned dimensions are briefly reviewed, along with related definitions and results from the theory of (co)homology of groups necessary for the exposition.

In Section 3, some standard observations about the conjecture are made, including why it suffices to consider only finitely generated groups. Then some properties of any group which is potential counterexample are discussed, and these roughly split into two categories. Firstly, the properties which make the group very difficult to construct, in particular that it would also be a counterexample to other well known conjectures, and secondly, some algebraic properties it must necessarily share with a free group.

2. Definitions and Background

The reader is referred to Bieri's lecture notes [1] and Brown's book [2] for further details concerning the terminology, definitions and background results described in this paper.

Let G be a nontrivial discrete group, let $\mathbf{Z}$ denote the ring of integers, let $\mathbf{Z}G$ be the group ring, and let M be a left $\mathbf{Z}G$ module. A projective resolution $P_* \to \mathbf{Z}$ of the trivial $\mathbf{Z}G$ module is an exact sequence of right $\mathbf{Z}G$ -modules

$$\to P_2 \to P_1 \to P_0 \to \mathbf{Z} \to 0$$

where each P_i is a projective module.

The cohomological dimension of G, denoted $\mathrm{cd}\, G$, is defined to be the smallest integer n such there exists a projective resolution for which $P_i = 0$ for all $i \geq n+1$. If no such n exists then G has infinite cohomological dimension. Recall that G is a free group if it contains a set S such that every element of G can be written in exactly one way as a product of elements of S and their inverses. The groups of cohomological dimension 1 have been classified by Stallings and Swan in the following theorem (see [2, Chapter VIII, Example 2].)

Theorem $\mathrm{cd}\, G = 1$ if and only if G is a free group.

A $K(G,1)$-complex is a connected CW-complex for which the fundamental group isomorphic to G and the higher homotopy groups are trivial. The geometric dimension of G, denoted $\mathrm{gd}\, G$, is defined to be the minimal dimension of a $K(G,1)$-complex.
The cellular chain complex of the universal cover of $K(G,1)$-complex yields a free resolution of the trivial $\mathbf{Z}G$-module. Since free modules are projective, it follows that $\mathrm{cd}\, G \leq \mathrm{gd}\, G$. Eilenberg and Ganea (see [2, Chapter VIII, Corollary 7.2]) showed that if $\mathrm{cd}\, G \geq 3$ then $\mathrm{cd}\, G = \mathrm{gd}\, G$, and $\mathrm{gd}\, G = 1$ if and only if G is a free group, thus the only possibility for these dimensions not to be equal is $\mathrm{cd}\, G = 2$ and $\mathrm{gd}\, G = 3$. The Eilenberg-Ganea conjecture states that no such group exists.

The homology groups of G with coefficients in M, denoted $H_i(G,M)$, are defined to be the homology groups of the chain complex formed by tensoring the projective resolution with M

$$\to P_2 \otimes_G M \to P_1 \otimes_G M \to P_0 \otimes_G M \to 0.$$

Homology is independent, up to isomorphism, of the chosen projective resolution and in fact the modules P_i may even be chosen to be flat modules. The homological dimension of G, denoted $\mathrm{hd}\, G$, is thus defined to be the smallest integer n such there exists a *flat* resolution for which $P_i = 0$ for all $i \geq n+1$. Since projective modules are necessarily flat, it follows that $\mathrm{hd}\, G \leq \mathrm{cd}\, G$. Homological dimension is equivalently be defined as the smallest integer n such that $H_i(G,M) = 0$ for $i \geq n+1$ and for every module M.

Remarks The following hold for each definition of dimension described in this section.

1. The dimension is equal to zero if and only if G is the trivial group.
2. If the dimension is finite then G is necessarily torsion-free.
3. The dimension of a subgroup of G is less than or equal to the dimension of G.

Finally, before writing down the conjecture, the following observation and definition are required. Every group can be written as the directed union of its collection of finitely generated subgroups $\{G_i \mid i \in I\}$, and since homology 'commutes' with colimits over directed systems, it follows that $H_n(G,M) = \mathrm{colim}_{i \in I} H_n(G_i, M)$ [1, Proposition 4.8]. A group G is said to be locally free if every finitely generated subgroup of G is a free group. If G_i is a finitely generated free group, then $\mathrm{hd}\, G_i = 1$, because $0 < \mathrm{hd} G_i \leq \mathrm{cd} G_i = 1$ and so by the colimit result, if G is locally free group then $\mathrm{hd}\, G = 1$.

Conjecture: $\mathrm{hd}\, G = 1$ if, and only if, G is a locally free group.

Examples of non-free, locally free groups include the non-cyclic subgroups of the group of additive rational numbers. Non-abelian examples are generally constructed by forming ascending chains of finitely generated free groups in such a way that the union is not free.

3. Some Properties of a Potential Counterexample

It follows from the colimit result that if there is a counterexample to the conjecture, then there is a finitely generated counterexample. Therefore, from now on, it will be assumed that G is a finitely generated by the set S with s generators, that $\operatorname{hd} G = 1$ and that G is not a free group.

Lemma $\operatorname{cd} G = 2$ and G is *not* a finitely presentable group.
Proof A finitely generated group is necessarily countable, and so it follows from [1, Theorem 4.6b] that $\operatorname{cd} G = 2$, and thus there is a projective resolution of the form

$$0 \to P \to \mathbf{Z}G^s \to \mathbf{Z}G \to \mathbf{Z} \to 0$$

where $\mathbf{Z}G^s$ is a free module indexed by the generating set S [2, Chapter VIII, Theorem 7.1]. If P were a finitely generated module then [1, Theorem 4.6c] would imply that $1 = \operatorname{hd} G = \operatorname{cd} G$, contradicting the fact that G is not a free group, and so P must be an infinitely generated module. But if the resolution corresponds to a finite presentation of G then P is finitely generated.

If the resolution in the proof of the lemma is used to compute the homology groups of G with trivial coefficients $\mathbf{Z}$, then the map $P \otimes_G \mathbf{Z} \to \mathbf{Z}G^s \otimes_G \mathbf{Z} \cong \mathbf{Z}^s$ is injective because $H_2(G, \mathbf{Z}) = 0$. Consequently, P is not a free module, otherwise the map would be an embedding of a free abelian group of infinite rank into a free abelian group of finite rank. However, trivial integer (or rational) coefficients are probably not sufficient to detect finite generation of P, and different concepts of rank or dimension are needed.

In this direction, the strongest positive results about the conjecture have been obtained in [4] and [3] using Von Neumann dimension to find conditions under which P must necessarily be finitely generated. Their methods are beyond the scope of this paper, the interested reader is referred to [5] as well as the original articles for details, and only the results will be recorded here.

The *Atiyah conjecture* for a torsion-free group states that the Von Neumann dimension of a finitely presented $\mathbf{Z}G$-module is an integer.

Theorem [4, Theorem 2] If G is a group such that $\operatorname{hd} G = 1$ and for which the Atiyah conjecture holds, then G is locally free.

A group G is *left orderable* if there exists a total order on G which is left G- invariant. Such a group is necessarily torsion-free. Examples of such groups include free groups, nilpotent groups and more generally, residually torsion-free nilpotent groups. The group ring $\mathbf{Z}G$ of a left orderable group has no non-zero divisors, and this fact is used to prove the following.

Theorem [3, Theorem 6.11] If G is a group generated by 2 elements, $\operatorname{hd} G = 1$ and G is left orderable, then G is free.

The statements of these two theorems thus provide conditions which the counterexample G must fail. In particular, G must be a counterexample to the Atiyah conjecture. Turning next to geometric dimension, one can deduce the following about G.

Proposition Either

1. $\operatorname{gd} G = 3$, and hence G is a counterexample to the Eilenberg-Ganea conjecture, or
2. $\operatorname{gd} G = 2$, but every 2 dimensional $K(G,1)$-complex corresponds to a presentation of G with infinitely many generators despite G being finitely generated.

Proof The only possible values of $\operatorname{gd} G$ are 2 and 3, because $\operatorname{cd} G = 2$. If $\operatorname{gd} G = 2$ then, by definition, there exists a 2-dimensional $K(G,1)$-complex Y, and the cellular chain complex of the universal covering space of Y gives rise to a projective resolution of the form

$$0 \to F_2 \to F_1 \to \mathbf{Z}G \to \mathbf{Z} \to 0$$

corresponding to the presentation of G, where F_1 is a free module indexed by the generators of G, and F_2 is a free module indexed by the relations of G. As pointed out previously, it is not possible for F_1 to be finitely generated and F_2 to be infinitely generated if G is a counterexample to the conjecture but, on the other hand, G is finitely generated but not finitely presentable. This means that Y must correspond to a presentation of G with infinitely many generators, and thus that every presentation of G with finitely many generators yields a 3-dimensional $K(G,1)$-complex.

Remark While there is no obvious reason why a group described in 2. of the proposition could not exist, the construction of an example seems to be a difficult problem.

Finally, some algebraic properties of G are considered.

G is an accessible group, since it is a finitely generated torsion-free group [1, Lemma 7.2]. This essentially means that if G is split into a free product $G_1 * G_2$, and this is repeated for G_1 and G_2 and so on, then the process must stop after finitely many steps. Thus, by replacing G with an indecomposable factor, one could assume that G cannot be written as a free product, and attempt to show that G is an infinite cyclic group.

The abelianization of a free group is a free abelian group, and the center of a free group is the trivial group. These properties also hold for G.

Theorem $G/[G,G]$ is a finitely generated free abelian group.

Proof The abelianization $G/[G,G]$ is naturally isomorphic to the first homology group with trivial coefficients $H_1(G,\mathbf{Z})$ (see [2, Chapter II, Section 3]), and so it is a finitely generated abelian group since G is finitely generated. Now, by assumption $H_2(G,\mathbf{Z}/p\mathbf{Z}) = 0$ for every prime number p, but this homology group maps surjectively onto $\mathrm{Tor}_1(H_1(G,\mathbf{Z}),\mathbf{Z}/p\mathbf{Z})$ by the Universal Coefficient Theorem [2, p.60 ex3], and so $H_1(G,\mathbf{Z})$ has no torsion elements.

More generally, applying [1, Theorem 8.15], it can be deduced that every quotient of the derived series of G is torsion-free.

The Hirsch rank of a torsion-free solvable group is equal to its homological dimension [1, Theorem 7.10], and thus every solvable subgroup of G is in fact abelian, and isomorphic to a subgroup of the group of additive rational numbers. Much more can be said about the center of G.

Theorem If G is non-abelian, then the center of G is the trivial group.
Proof Let Z denote the center of G. According to [1, Corollary 8.9] Z is either the trivial group or an infinite cyclic group because $\mathrm{cd}\, G = 2$, and if Z is infinite cyclic then $[G,G]$ is a free group. Moreover, $[G,G]$ is not trivial unless G is abelian, and it is not infinite cyclic, otherwise G would be finitely presented, and thus it is a free group of rank at least 2. Now, the inclusion of Z into G induces a map from Z to $G/[G,G]$ which is either injective or the zero map since both are free abelian groups. If the map is injective, then Z and any non-trivial element of $x \in [G,G]$ generate a free abelian subgroup $< x > \times Z$ of rank 2, but G cannot have such a subgroup because $\mathrm{hd}\,(< x > \times Z) = 2$. If the map is the zero map then $[G,G]$ contains non-trivial central elements, contradicting the fact that it is free. Thus Z must be the trivial group.

4. Conclusions
If there exists a finitely generated non-free group G with $\mathrm{hd}\, G = 1$, then it must have some analytical and geometric properties very different to those of a free group, but it must also have some algebraic properties quite similar to those of a free group. The construction of such a group, with even a few of the necessary properties seems to be very difficult with currently known techniques. However, there are huge gaps in the known structure theory of groups of low dimension, and new methods are probably needed if there is to be any hope of proving the conjecture.

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