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### A Concise Workbook for College Algebra 2nd Edition

Fei Ye

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# A Concise Workbook for College Algebra

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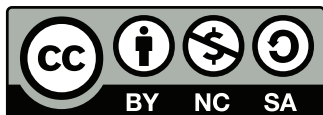
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*A Concise Workbook for College Algebra*

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## Preface to the First Edition

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This workbook mainly grew out of the author's worksheets for the college algebra course (MAD-119) at QCC of CUNY. It is intended to give a concise introduction to College Algebra.

In teaching College Algebra, I feel strongly that instructors should emphasize more the depth of reasoning and understanding rather than multitudes of approaches to similar types of questions. If you think carefully, whenever you choose an approach to solve a problem, there is always a reason. Students should learn why an approach works instead of simply following the procedure which in nowadays can be done by computers.

Motivated by the above thought, in this concise workbook, we try to expose only key concepts and ideas, which will save time for practicing and reinforcing critical thinking skills which include observing patterns, identifying and analyzing problems, making logic connections, determining problem-solving strategies, and solving problems systematically.

For example, only the method of undetermined coefficient was introduced for factoring trinomials in this book. To factor the trinomial  $Ax^2 + Bx + C$ , where  $A$ ,  $B$  and  $C$  are integers, we may use trial-and-error method to find integers  $m$ ,  $n$ ,  $p$  and  $q$  such that  $mn = A$ ,  $pq = C$  and  $mq + np = B$ . In practice, we first factor  $A$  and  $C$ , and then use the following diagram to check if  $mq + np = B$  holds.

$$\begin{array}{ccc} A = mn & & C = pq \\ m & \begin{array}{c} \diagdown \\ \diagup \end{array} & p \\ n & \begin{array}{c} \diagup \\ \diagdown \end{array} & q \\ \hline np & + & mq & \stackrel{?}{=} & B \end{array}$$

This method is based on the observation that  $Ax^2 + Bx + C$  can be factored into  $(mx + p)(nx + q)$ . Indeed, observing and making logic connections are very effective in problem-solving.

Topics are contained in 25 lessons. Each lesson corresponds to roughly one class meeting. A lesson starts with a page on concepts, formulas and examples, and ends with exercises that students are expected to complete in class. We would like to thank our colleagues and students for their feedback and support during the development of this project. In particular, we would like to thank Joseph Bertorelli, Beata Ewa Carvajal, Kwai Chiu, Lixu Li, Wenjian Liu, Nam Jong Moh, Tian Ren, Kostas Stroumbakis, Evelyn Tam, and Haishen Yao for their encouragement, support, and feedback.

Fei Ye  
Fall 2018

## Preface to the Second Edition

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Effective practice is the key for success. Focusing on key ideas, frequently used problem solving strategies and necessary knowledge is the best way to practice effectively.

In this second edition, more examples and exercises are provided. Some frequently used problem solving strategies and ideas are summarized as tips.

In algebra, problem solving is meaningful, for each step you take, there is always a reason. I hope that you will agree with me and enjoy thinking and understanding the powerful ideas hidden in algebraic problem solving.

Fei Ye  
Fall 2019



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## Lesson 1 Linear Inequalities

**Properties of Inequalities** An inequality defines a relationship between two expressions. The following properties show when the inequality relationship is preserved or reversed.

Property	Example
<p><b>The additive property</b>                      If <math>a &lt; b</math>, then <math>a + c &lt; b + c</math>, for any real number <math>c</math>.                      If <math>a &lt; b</math>, then <math>a - c &lt; b - c</math>, for any real number <math>c</math>.</p>	<p>If <math>x + 3 &lt; 5</math>, then <math>x + 3 - 3 &lt; 5 - 3</math>. Simplifying both sides, we get <math>x &lt; 2</math>.</p>
<p><b>The positive multiplication property</b>                      If <math>a &lt; b</math> and <math>c</math> is positive, then <math>ac &lt; bc</math>.                      If <math>a &lt; b</math> and <math>c</math> is positive, then <math>\frac{a}{c} &lt; \frac{b}{c}</math>.</p>	<p>If <math>2x &lt; 4</math>, then <math>\frac{2x}{2} &lt; \frac{4}{2}</math>. Simplifying both sides, we get <math>x &lt; 2</math>.</p>
<p><b>The negative multiplication property</b>                      If <math>a &lt; b</math> and <math>c</math> is negative, then <math>ac &gt; bc</math>.                      If <math>a &lt; b</math> and <math>c</math> is negative, then <math>\frac{a}{c} &gt; \frac{b}{c}</math>.</p>	<p>If <math>1 &lt; 2</math>, then <math>-2 = 1 \cdot (-2) &gt; 2 \cdot (-2) = -4</math>.                      If <math>-2x &lt; 4</math>, then <math>\frac{-2x}{-2} &gt; \frac{4}{-2}</math>. Simplifying both sides, we get <math>x &gt; 2</math>.</p>

**Note** These properties also apply to  $a \leq b$ ,  $a > b$  and  $a \geq b$ .

**Tips** It's always better to view  $a - c$  as  $a + (-c)$ . Because addition has the commutative property.

**Compound Inequalities** A *compound inequality* is formed by two inequalities with the word “and” or the word “or”. For examples, the following are three commonly seen type compound inequalities:

$$\begin{aligned}
 x - 1 > 2 \quad \text{and} \quad 2x + 1 < 3, \\
 3x - 5 < 4 \quad \text{or} \quad 3x - 2 > 10, \\
 -3 \leq \frac{2x - 4}{3} < 2.
 \end{aligned}$$

The third compound inequality is simplified expression for the compound inequality  $-3 \leq \frac{2x-4}{3}$  and  $\frac{2x-4}{3} < 2$ .

**Interval Notations** Solutions to an inequality normally form an interval which has boundaries and should reflect inequality signs. Depending on the form of an inequality, we may a single interval and a union of intervals. For example, suppose  $a < b$ , we have the following equivalent representations of inequalities.

$x < a$	$x \geq b$	$a \leq x < b$	$x \leq a$ or $x > b$
$(-\infty, a)$	$[b, \infty)$	$[a, b)$	$(-\infty, a] \cup (b, \infty)$

**Tips** **Think backward.** To solve a problem, knowing what to expect helps you narrow down the gap step by step by comparing the goal and your achievement.

An inequality (equation) is solved if the unknown variable is isolated. That's what to be expected. To isolate the unknown variable, you use comparisons to determine what mathematical operations should be applied. When an operation is applied to one side, the same operation should also be applied to the other side. For inequalities, we also need to determine whether the inequality sign should be preserved or reversed according to the operation.



**Example 1.1** Solve the linear inequality

$$2x + 4 > 0.$$

**Solution**

$$\begin{array}{rcl} & 2x + 4 & > 0 \\ \text{add } -4 & & 2x > -4 \\ \text{divide by } 2 & & x > -2 \end{array}$$

The solution set is  $(-2, \infty)$ .**Example 1.3** Solve the compound linear inequality

$$x + 2 < 3 \quad \text{and} \quad -2x - 3 < 1.$$

**Solution**

$$\begin{array}{rcl} x + 2 < 3 & \text{and} & -2x - 3 < 1 \\ x < 1 & & -2x < 4 \\ x < 1 & \text{and} & x > -2 \end{array}$$

That is  $-2 < x < 1$ . The solution set is  $(-2, 1)$ .**Example 1.5** Solve the compound linear inequality

$$-4 \leq \frac{2x - 4}{3} < 2.$$

**Solution**

$$\begin{array}{rcl} -4 \leq \frac{2x - 4}{3} < 2 \\ -12 \leq 2x - 4 < 6 \\ -8 \leq 2x < 10 \\ -4 \leq x < 5 \end{array}$$

The solution set is  $[-4, 5)$ .**Example 1.7** Suppose that  $-1 \leq x < 2$ . Find the range of  $5 - 3x$ . Write your answer in interval notation.**Solution** To get  $5 - 3x$  from  $x$ , we need first multiply  $x$  by  $-3$  and then add 5.

$$\begin{array}{rcl} -1 \leq x < 2 \\ 3 \geq -3x > -6 \\ 8 \geq 5 - 3x > -1 \end{array}$$

The range of  $5 - 3x$  is  $(-1, 8]$ .**Example 1.2** Solve the linear inequality

$$-3x - 4 < 2.$$

**Solution**

$$\begin{array}{rcl} & -3x - 4 & < 2 \\ \text{add } 4 & & -3x < 6 \\ \text{divide by } -3 \text{ and switch} & & x > -2 \end{array}$$

The solution set is  $(-2, \infty)$ .**Example 1.4** Solve the compound linear inequality

$$-x + 4 > 2 \quad \text{or} \quad 2x - 5 \geq -3.$$

**Solution**

$$\begin{array}{rcl} -x + 4 > 2 & \text{or} & 2x - 5 \geq -3 \\ -x > -2 & & 2x \geq 2 \\ x < 2 & \text{or} & x \geq 1 \end{array}$$


That is  $x \geq 1$  or  $x < 2$ . The solution set is  $(-\infty, +\infty)$ .**Example 1.6** Solve the compound linear inequality

$$-1 \leq \frac{-3x + 4}{2} < 3.$$

**Solution**

$$\begin{array}{rcl} -1 \leq \frac{-3x + 4}{2} < 3 \\ -2 \leq -3x + 4 < 6 \\ -6 \leq -3x < 2 \\ 2 \geq x > -\frac{2}{3} \end{array}$$

The solution set is  $(-\frac{2}{3}, 2]$ .

 **Exercise 1.1** Solve the linear inequality. **Write your answer in interval notation.**

(1)  $3x + 7 \leq 1$

(2)  $2x - 3 > 1$

( $-\infty, 2$ ) (2) 1.1

$[2, \infty)$  (1) 1.1


 **Exercise 1.2** Solve the linear inequality. **Write your answer in interval notation.**

(1)  $4x + 7 > 2x - 3$

(2)  $3 - 2x \leq x - 6$

( $-\infty, 3$ ] (2) 1.2

( $-\infty, 5$ ) (1) 1.2

 **Exercise 1.3** Solve the compound linear inequality. **Write your answer in interval notation.**


(1)  $3x + 2 > -1$  and  $2x - 7 \leq 1$

(2)  $4x - 7 < 5$  and  $5x - 2 \geq 3$

$[1, 3)$  (2) 1.3

$[-4, 1)$  (1) 1.3




 **Exercise 1.4** Solve the compound linear inequality. **Write your answer in interval notation.**

(1)  $-4 \leq 3x + 5 < 11$

(2)  $7 \geq 2x - 3 \geq -7$

$[-2, 5]$  (2) 1.4

$[-3, 2]$  (1) 1.4


 **Exercise 1.5** Solve the compound linear inequality. **Write your answer in interval notation.**

(1)  $3x - 5 > -2$  or  $10 - 2x \leq 4$

(2)  $2x + 7 < 5$  or  $3x - 8 \geq x - 2$

$(-\infty, 3] \cap (1, -\infty)$  (2) 1.5

$(-\infty, 1)$  (1) 1.5

 **Exercise 1.6** Solve the compound linear inequality. **Write your answer in interval notation.**


(1)  $-2 \leq \frac{2x - 5}{3} < 3$

(2)  $-1 < \frac{3x + 7}{2} \leq 4$

$[\frac{3}{1}, 3)$  (2) 1.6


$[\frac{2}{1}, 7]$  (1) 1.6



 **Exercise 1.7** Solve the linear inequality. **Write your answer in interval notation.**


$$\frac{1}{3}x + 1 < \frac{1}{2}(2x - 3) - 1$$

$(-\infty, \frac{7}{12})$  8.1

 **Exercise 1.8** Solve the compound linear inequality. **Write your answer in interval notation.**


$$0 \leq \frac{2}{5} - \frac{x+1}{3} < 1$$


$(\frac{5}{1}, \frac{5}{11})$  8.1

 **Exercise 1.9** Suppose  $0 < x \leq 1$ . Find the range of  $-2x + 1$ . **Write your answer in interval notation.**

$(-1, 1]$  6.1



 **Exercise 1.10** Suppose that  $x + 2y = 1$  and  $1 \leq x < 3$ . Find the range of  $y$ . **Write your answer in interval notation.**

 **Exercise 1.11** A toy store has a promotion “Buy one get the second one half price” on a certain popular toy. The sale price of the toy is \$20 each. Suppose the store makes more profit when you buy two. What do you think the store’s purchasing price of the toy is?

## Lesson 2 Absolute Value Equations

**Properties of Absolute Values** The *absolute value* of a real number  $a$ , denoted by  $|a|$ , is the distance from 0 to  $a$  on the number line. In particular,  $|a|$  is always greater than or equal to 0, that is  $|a| \geq 0$ . Absolute values satisfy the following properties:

$$|-a| = |a|, \quad |ab| = |a||b| \quad \text{and} \quad \left| \frac{a}{b} \right| = \frac{|a|}{|b|}.$$

**Absolute Value Equation** An absolute value equation may be rewritten as  $|X| = c$ , where  $X$  represents an algebraic expression.

If  $c$  is **positive**, then the equation  $|X| = c$  is equivalent to  $X = c$  or  $X = -c$ .

If  $c$  is **negative**, then the solution set of  $|X| = c$  is **empty**. An *empty set* is denoted by  $\emptyset$ .

More generally,  $|X| = |Y|$  is equivalent to  $X = Y$  or  $X = -Y$ .

**Example 2.1** Solve the equation

$$|2x - 3| = 7.$$

**Solution** The equation is equivalent to

$$\begin{aligned} 2x - 3 = -7 \quad \text{or} \quad 2x - 3 = 7 \\ 2x = -4 \qquad \qquad \qquad 2x = 10 \\ x = -2 \quad \text{or} \quad x = 5 \end{aligned}$$

The solutions are  $x = -2$  or  $x = 5$ . In set-builder notation, the solution set is  $\{-2, 5\}$ .

**Example 2.3** Solve the equation

$$3|2x - 5| = 9.$$

**Solution** Rewrite the equation into  $|X| = c$  form.

$$|2x - 5| = 3$$

Solve the equation.

$$\begin{aligned} 2x - 5 = -3 \quad \text{or} \quad 2x - 5 = 3 \\ 2x = 2 \qquad \qquad \qquad 2x = 8 \\ x = 1 \quad \text{or} \quad x = 4 \end{aligned}$$

The solutions are  $x = 1$  or  $x = 4$ . In set-builder notation, the solution set is  $\{1, 4\}$ .

**Example 2.5** Solve the equation

$$|3x - 2| = |x + 2|.$$

**Solution** Note that two numbers have the same absolute value only if they are the same or opposite to each other. Then the equation is equivalent to

$$\begin{aligned} 3x - 2 = x + 2 \quad \text{or} \quad 3x - 2 = -(x + 2) \\ 2x = 4 \qquad \qquad \qquad 4x = 0 \\ x = 2 \quad \text{or} \quad x = 0 \end{aligned}$$

The solutions are  $x = 2$  and  $x = 0$ . In set-builder notation, the solution set is  $\{0, 2\}$ .

**Example 2.2** Solve the equation

$$|2x - 1| - 3 = 8.$$

**Solution** Rewrite the equation into  $|X| = c$  form.

$$|2x - 1| = 11$$

Solve the equation.

$$\begin{aligned} 2x - 1 = -11 \quad \text{or} \quad 2x - 1 = 11 \\ 2x = -10 \qquad \qquad \qquad 2x = 12 \\ x = -5 \quad \text{or} \quad x = 6 \end{aligned}$$

The solutions are  $x = -5$  or  $x = 6$ . In set-builder notation, the solution set is  $\{-5, 6\}$ .

**Example 2.4** Solve the equation

$$2|1 - 2x| - 3 = 7.$$

**Solution** Rewrite the equation into  $|X| = c$  form.

$$|2x - 1| = 5$$

Solve the equation.

$$\begin{aligned} 2x - 1 = -5 \quad \text{or} \quad 2x - 1 = 5 \\ 2x = -4 \qquad \qquad \qquad 2x = 6 \\ x = -2 \quad \text{or} \quad x = 3 \end{aligned}$$

The solutions are  $x = -2$  or  $x = 3$ . In set-builder notation, the solution set is  $\{-2, 3\}$ .


**Example 2.6** Solve the equation

$$2|1 - x| = |2x + 10|.$$

**Solution** Since 2 is positive,  $2|1 - x| = |2||1 - x| = |2 - 2x|$ . Moreover, because  $|-X| = |X|$ , the equation is equivalent to

$$\begin{aligned} |2x - 2| = |2x + 10| \\ 2x - 2 = 2x + 10 \quad \text{or} \quad 2x - 2 = -(2x + 10) \\ -2 = 10 \qquad \qquad \qquad \text{or} \quad 4x = -8 \\ x = -2 \end{aligned}$$

The original equation only has one solution  $x = -2$ . In set-builder notation, the solution set is  $\{-2\}$ .


 **Exercise 2.1** Find the solution set for the equation.

(1)  $|2x - 1| = 5$

(2)  $\left| \frac{3x - 9}{2} \right| = 3$

{5, 1} (2) 1, 2

{-2, 3} (1) 1, 2


 **Exercise 2.2** Find the solution set for the equation.

(1)  $|3x - 6| + 4 = 13$

(2)  $3|2x - 5| = 9$

{1, 4} (2) 2, 2

{-1, 5} (1) 1, 2

 **Exercise 2.3** Find the solution set for the equation.


(1)  $|5x - 10| + 6 = 6$

(2)  $-3|3x - 11| = 5$

∅ (2) 2, 3

{2} (1) 2, 3




 **Exercise 2.4** Find the solution set for the equation.

(1)  $3|5x - 2| - 4 = 8$

(2)  $-2|3x + 1| + 5 = -3$

$\{1, \frac{3}{5}\}$  (2)

$\{-\frac{5}{2}, \frac{5}{6}\}$  (1)


 **Exercise 2.5** Find the solution set for the equation.

(1)  $|5x - 12| = |3x - 4|$

(2)  $|x - 1| = -5|(2 - x) - 1|$

$\{1\}$  (2)

$\{2, 4\}$  (1)

 **Exercise 2.6** Find the solution set for the equation.

(1)  $|2x - 1| = 5 - x$

(2)  $-2x = |x + 3|$

$\{-1\}$  (2)

$\{-4, 2\}$  (1)





## Lesson 3 Properties of Integral Exponents

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**Properties of Exponents** For an integer  $n$ , and an expression  $x$ , the mathematical operation of the  $n$ -times repeated multiplication of  $x$  is called exponentiation, written as  $x^n$ , that is,

$$x^n = \underbrace{x \cdot x \cdots x}_{n \text{ factors of } x}.$$

In the notation  $x^n$ ,  $n$  is called *the exponent*,  $x$  is called *the base*, and  $x^n$  is called *the power*, read as “ $x$  raised to the  $n$ -th power”, “ $x$  to the  $n$ -th power”, “ $x$  to the  $n$ -th”, “ $x$  to the power of  $n$ ” or “ $x$  to the  $n$ ”.

**1. The product rule**

$$x^m \cdot x^n = x^{m+n}.$$

**Example:**

$$2x^2 \cdot (-3x^3) = -6x^5.$$

**2. The quotient rule** (for  $x \neq 0$ .)

$$\frac{x^m}{x^n} = \begin{cases} x^{m-n} & \text{if } m \text{ is greater than} \\ & \text{or equal to } n. \\ \frac{1}{x^{n-m}} & \text{if } m \text{ is less than } n. \end{cases}$$

**Example:**

$$\frac{15x^5}{5x^2} = 3x^3, \quad \frac{-3x^2}{6x^3} = -\frac{1}{2x}$$

**3. The zero exponent rule** (for  $x \neq 0$ .)

$$x^0 = 1.$$

**Example:**

$$(-2)^0 = 1, \quad -x^0 = -1$$

**4. The negative exponent rule** (for  $x \neq 0$ .)

$$x^{-n} = \frac{1}{x^n} \quad \text{and} \quad \frac{1}{x^{-n}} = x^n.$$

**Example:**

$$(-2)^{-3} = \frac{1}{(-2)^3} = -\frac{1}{8}, \quad \frac{x^{-2}}{x^{-3}} = \frac{x^3}{x^2} = x.$$

**5. The power to a power rule:**

$$(x^a)^b = x^{ab}.$$

**Example:**

$$(2^2)^3 = 2^6 = 64, \quad (x^2)^3 = x^6.$$

**6. The product raised to a power rule:**

$$(xy)^n = x^n y^n.$$

**Example:**

$$(-2x)^2 = (-2)^2 x^2 = 4x^2$$

**7. The quotient raised to a power rule** (for  $y \neq 0$ .)

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}.$$

**Example:**

$$\left(\frac{x}{-2}\right)^3 = \frac{x^3}{(-2)^3} = -\frac{x^3}{8}.$$

**Order of Basic Mathematical Operations** In mathematics, the order of operations reflects conventions about which procedure should be performed first. There are four levels (from the highest to the lowest):

**Parenthesis; Exponentiation; Multiplication and Division; Addition and Subtraction.**

Within the same level, the convention is to perform from the left to the right.

**Example 3.1** Simplify. Write with positive exponents.

$$\left(\frac{2y^{-2}z^{-5}}{4x^{-3}y^6}\right)^{-4}.$$

**Solution**

$$\left(\frac{2y^{-2}z^{-5}}{4x^{-3}y^6}\right)^{-4} = \left(\frac{x^3}{2z^5y^8}\right)^{-4} = \left(\frac{2z^5y^8}{x^3}\right)^4 = \frac{2^4(z^5)^4(y^8)^4}{(x^3)^4} = \frac{16y^{32}z^{20}}{x^{12}}.$$

**Tips** Simplify (at least partially) the problem first.

To avoid mistakes when working with negative exponents, it's better to apply the negative exponent rule to change negative exponents to positive exponents and simplify the base first.

 **Exercise 3.1** Simplify. Write with positive exponents.

(1)  $(3a^2b^3c^2)(4abc^2)(2b^2c^3)$

(2)  $\frac{4y^3z^0}{x^2y^2}$

(3)  $(-2)^{-3}$

 **Exercise 3.2** Simplify. Write with positive exponents.

(1)  $\frac{-u^0v^{15}}{v^{16}}$

(2)  $(-2a^3b^2c^0)^3$

(3)  $\frac{m^5n^2}{(mn)^3}$



 **Exercise 3.3** Simplify. Write with positive exponents.

(1)  $(-3a^2x^3)^{-2}$

(2)  $\left(\frac{-x^0y^3}{2wz^2}\right)^3$

(3)  $\frac{3^{-2}a^{-3}b^5}{x^{-3}y^{-4}}$

 **Exercise 3.4** Simplify. Write with positive exponents.

(1)  $(-x^{-1}(-y)^2)^3$

(2)  $\left(\frac{6x^{-2}y^5}{2y^{-3}z^{-11}}\right)^{-3}$


(3)  $\frac{(3x^2y^{-1})^{-3}(2x^{-3}y^2)^{-1}}{(x^6y^{-5})^{-2}}$

3.4 (3)  $\frac{x^6}{54y^9}$


3.4 (2)  $\frac{x^6}{27y^2z^33}$

3.4 (1)  $-\frac{x^3}{y^6}$



 **Exercise 3.5** A pizza shop offer 12-inches pizza and 8-inches pizza at the price \$12 and \$6 respectively. With \$12, would you like to order one 12-inches and two 8-inches. Please explain.

3.5 12-inches pizza is better. Because the total area is bigger.

 **Exercise 3.6** A store has large size and small size watermelons. A large one cost \$4 and a small one \$1. The circumference of a larger watermelon is approximately twice of the circumference of a smaller one. Which size of watermelons is cheaper if it was sold by weight?

3.6 The larger one. Because the volume is 8 times larger.



## Lesson 4 Introduction to Functions

**Functions as Relations** A *relation* is a set of ordered pairs. The set of all first components of the ordered pairs is called the *domain*. The set of all second components of the ordered pairs is called the *range*.

A *function* is a relation such that each element in the domain corresponds to *exactly one* element in the range.

**Functions as Equations and Function Notation** For a function, we usually use the variable  $x$  to represent an element from the domain and call it the *independent variable*. The variable  $y$  is used to represent the value corresponding to  $x$  and is called the *dependent variable*. We say  $y$  is a function of  $x$ . When we consider several functions together, to distinguish them we named functions by a letter such as  $f$ ,  $g$ , or  $F$ . The notation  $f(x)$ , read as “ $f$  of  $x$ ” or “ $f$  at  $x$ ”, represents the output of the function  $f$  when the input is  $x$ . To find the value of a function at a given number, we substitute the independent variable  $x$  by the given number and then evaluate the expression. We call the procedure evaluating a function.

**Example 4.1** Find the indicated function value.

(1)  $f(-2)$ ,  $f(x) = 2x + 1$       (2)  $g(2)$ ,  $g(x) = 3x^2 - 10$       (3)  $h(a - t)$ ,  $h(x) = 3x + 5$ .

**Solution**

(1)  $f(-2) = 2 \cdot (-2) + 1 = -4 + 1 = -3$ .

(2)  $g(2) = 3 \cdot (2^2) - 10 = 3 \cdot 4 - 10 = 12 - 10 = 2$ .

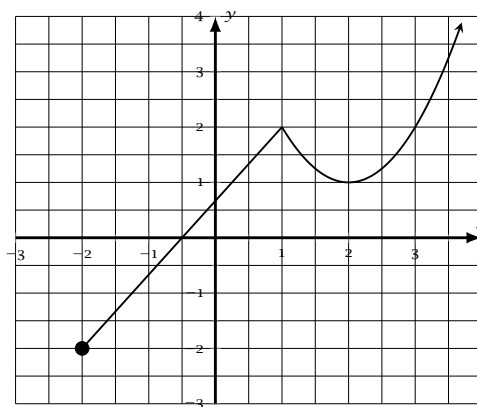
(3)  $h(a - t) = 3 \cdot (a - t) + 5 = 3a - 3t + 5$ .

**Functions as Graphs** The *graph of a function* is the graph of its ordered pairs. A graph of ordered pairs  $(x, y)$  in the rectangular system defines  $y$  as a function of  $x$  if any vertical line crosses the graph at most once. This test is called the *vertical line test*.

**Domain and Range** The domain of a graph is the set of all inputs ( $x$ -coordinates). The range of a graph is the set of all outputs ( $y$ -coordinates). To find the domain of a graph, we look for the left and the right endpoints. To find the range of a graph, we look for the highest and the lowest positioned points.


**Example 4.2** Use the graph on the right to answer the following questions.

- Determine whether the graph is a function and explain your answer.
- Find the domain (in interval notation) of the graph.
- Find the range (in interval notation) of the graph.
- Find the interval where the graph is above 2.
- Find the interval where the graph is decreasing.
- Find the extrema if they exist.
- Find the value of  $x$  such that  $(x, 0)$  is on the graph.




**Solution**

- The graph is a function. Because every vertical line crosses the graph at most once.
- The graph has the left endpoint at  $(-2, -2)$  and but no right endpoint. So the domain is  $[-2, +\infty)$ .
- The graph has a lowest positioned point  $(-2, -2)$  but no highest positioned point. So the range is  $[-2, +\infty)$ .
- The graph is above 2 over the interval  $(3, \infty)$ .
- The graph is decreasing over the interval  $(1, 2)$ .
- The graph has minima at  $(-2, -2)$  and  $(2, 1)$ .
- The point  $(x, 0)$  is on the  $x$ -axis. If  $x = -0.5$ , then it will be on the graph.

 **Exercise 4.1** Find the indicated function values for the functions  $f(x) = -x^2 + x - 1$  and  $g(x) = 2x - 1$ . Simplify your answer.


- (1)  $f(2)$                       (2)  $f(-x)$                       (3)  $g(-1)$                       (4)  $g(f(1))$

4.1 (1)  $f(2) = -2^2 + 2 - 1 = -4 + 2 - 1 = -3$       4.1 (2)  $f(-x) = -(-x)^2 + (-x) - 1 = -x^2 - x - 1$       4.1 (3)  $g(-1) = 2(-1) - 1 = -2 - 1 = -3$       4.1 (4)  $g(f(1)) = 2(-1) - 1 = -2 - 1 = -3$

 **Exercise 4.2** Suppose  $g(x) = -3x + 1$ .

- (1) Compute  $\frac{g(4) - g(1)}{4 - 1}$                       (2) Compute  $\frac{g(x + h) - g(x)}{h}$

4.2 (1)  $\frac{g(4) - g(1)}{4 - 1} = \frac{(-3(4) + 1) - (-3(1) + 1)}{4 - 1} = \frac{-12 + 1 - (-3 + 1)}{3} = \frac{-11 - (-2)}{3} = \frac{-9}{3} = -3$

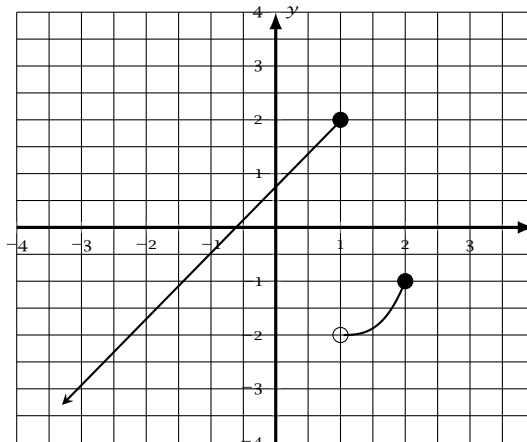
 **Exercise 4.3** Suppose the domain of the linear function  $l(x) = 1 - 2x$  is  $(0, 1)$ . Find the range of the function.

4.3  $(-1, 1)$ .



**Exercise 4.4** Use the graph shown on the right to answer the following questions.

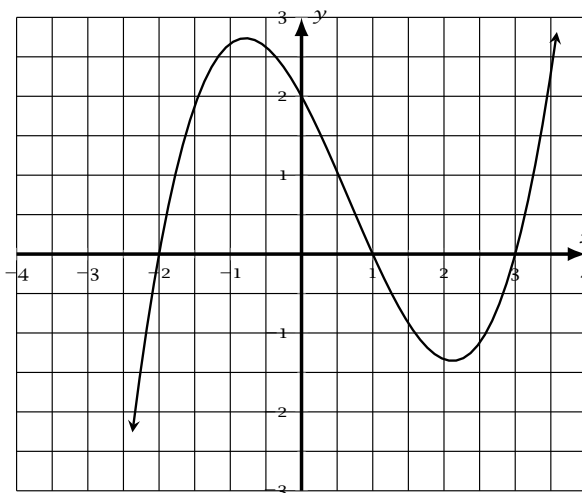
- (1) Determine whether the graph is a function and explain your answer.
- (2) Find the domain of the graph (write the domain in interval notation).
- (3) Find the range of the graph (write the range in interval notation).
- (4) Find the interval where the graph is above the  $x$ -axis.
- (5) Find all points where the graph reaches a maximum or a minimum.
- (6) Find the values of the  $x$ -coordinate of all points on the graph whose  $y$ -coordinate is 1.




4.4 (1) Yes 4.4 (2)  $(-\infty, 2]$  4.4 (3)  $(-\infty, 2)$  4.4 (4)  $(1, 1)$  4.4 (5)  $(1, 2)$  4.4 (6)  $x = 0$

**Exercise 4.5** Use the graph of a function  $f$  shown on the right to answer the following questions.

- (1) Find the  $y$ -intercept.
- (2) Find the value  $\frac{f(3) - f(0)}{3}$ .
- (3) Find the values  $x$  such that  $f(x) = 0$ .
- (4) Find the solution to the inequality  $f(x) > 0$ . Write in interval notation.



4.5 (0, 2) 4.5  $-\frac{3}{2}$  4.5  $\{-2, 1, 3\}$  4.5  $(-2, 1) \cup (3, \infty)$ .

 **Exercise 4.6** Today Matt drove from home to school in 30 minutes. He spent 6 minutes on local streets before driving on the highway and 4 minutes on local streets towards school after getting off the highway. On local streets, his average speed is 30 miles per hour. On the highway, his average speed is 60 miles per hours.

- (1) Write the distance  $d$  (in miles) he drove as a function of the time  $t$  (in minutes)?
- (2) After 15 minutes, where was he and how far did he drive?
- (3) How far did he drive from home to school?

$$d(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \leq t \leq 6 \\ (t - 6) + 3 & \text{if } 6 < t \leq 26 \\ \frac{1}{2}(t - 26) + 23 & \text{if } 26 < t \leq 30 \end{cases}$$

4.6 (3) 25 miles
4.6 (2) 12 miles

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## Lesson 5 Linear functions

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**The Slope-Intercept Form Equation** The slope of a line measures the steepness, in other words, “rise” over “run”, or rate of change of the line. Using the rectangular coordinate system, the *slope*  $m$  of a line is defined as

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in the output } y}{\text{change in the input } x},$$

where  $(x_1, y_1)$  and  $(x_2, y_2)$  are any two distinct points on the line. If the line intersects the  $y$ -axis at the point  $(0, b)$ , then a point  $(x, y)$  is on the line if and only if

$$y = mx + b.$$

This equation is called the *slope-intercept form* of the line.

**Point-Slope Form Equation of a Line** Suppose a line passing through the point  $(x_0, y_0)$  has the slope  $m$ . Solving from the slope formula, we see that any point  $(x, y)$  on the line satisfies the equation

$$y - y_0 = m(x - x_0)$$

which is called the *point-slope form* equation.

**Linear Function** A *linear function*  $f$  is a function whose graph is a line. An equation for  $f$  can be written as

$$f(x) = mx + b$$

where  $m$  is the slope and  $b = f(0)$ .

A function  $f$  is a linear function if the following equalities hold

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_3) - f(x_1)}{x_3 - x_1}$$

for any three distinct points  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  on the graph of  $f$ .

### How to Write an Equation for a Linear Function?

**Example 5.1** Find the slope-intercept form equation for the linear function  $f$  such that  $f(2) = 5$  and  $f(-1) = 2$ .

#### Solution

**Step 1.** Find the slope  $m$ :  $m = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{5 - 2}{2 - (-1)} = \frac{3}{3} = 1$ .

**Step 2.** Plug one of the points, say  $(2, 5)$  in the point-slope form equation, we get  $y - 5 = 1 \cdot (x - 2) + 5$

**Step 3.** Simplify the above equation, we get the slope-intercept form equation  $f(x) = x + 3$ .

### Graph a Linear Function by Plotting Points

**Example 5.2** Sketch the graph of the linear function  $f(x) = -\frac{1}{2}x + 1$ .

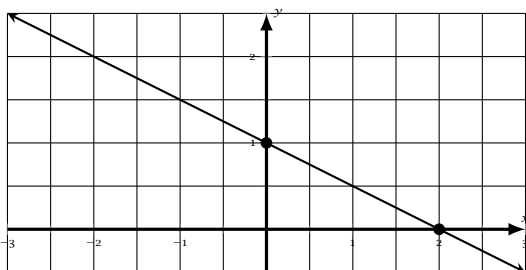
#### Solution

**Method 1: Get points by evaluating  $f(x)$ .**

**Step 1.** Choose two or more input values, e.g.  $x = 0$  and  $x = 2$ .

**Step 2.** Evaluate  $f(x)$ :  $f(0) = 1$  and  $f(2) = 0$ .

**Step 3.** Plot the points  $(0, 1)$  and  $(2, 0)$  and draw a line through them.

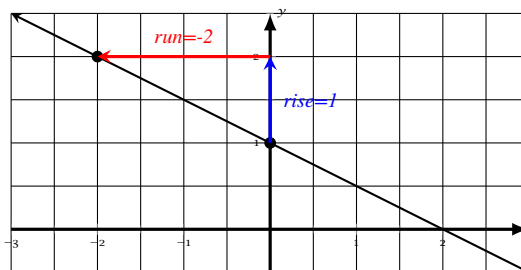



**Method 2: Get points by raise and run.**

**Step 1.** Plot the  $y$ -intercept  $(0, f(0)) = (0, 1)$ .


**Step 2.** Use  $\frac{\text{rise}}{\text{run}} = -\frac{1}{2}$  to get one or more points, e.g. we will get  $(-2, 2)$  by taking rise = 1 and run = -2, i.e. move up 1 unit, then move to the right 2 units.

**Step 3.** Plot the points  $(0, 1)$  and  $(-2, 2)$  and draw a line through them.




-  **Exercise 5.1** Find the slope of the line passing through
  - (3, 5) and (-1, 1)
  - (-2, 4) and (5, -2).

$$\frac{z}{9} = m(z) \quad \text{and} \quad 1 = m(1) \quad \text{and} \quad 1 = m(1) \quad \text{and} \quad 1 = m(1)$$


-  **Exercise 5.2** Find the point-slope form equation of the line with slope 5 that passes through (-2, 1).


$$1 + (z + x)5 = 1 \quad \text{and} \quad 1 + (z + x)5 = 1$$


-  **Exercise 5.3** Find the point-slope form equation of the line passing through (3, -2) and (1, 4).

$$4 + (1 - x)3 = 1 \quad \text{and} \quad 4 + (1 - x)3 = 1$$



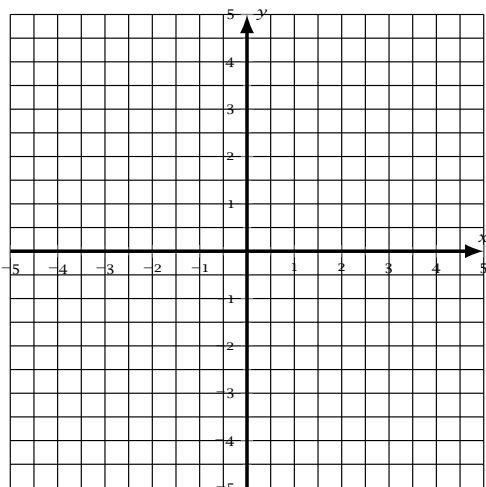
 **Exercise 5.4** Find the slope-intercept form equation of the line passing through  $(6, 3)$  and  $(2, 5)$ .

 **Exercise 5.5** Determine whether the linear functions  $f(x)$  and  $h(x)$  with the following values  $f(-2) = -4$ ,  $f(0) = h(0) = 2$  and  $h(2) = 8$  define the same function. Explain your answer.

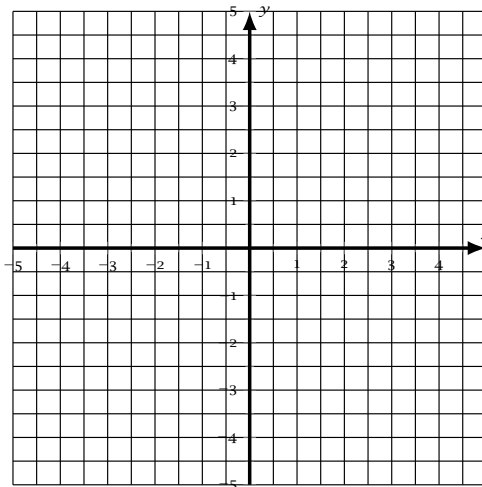
 **Exercise 5.6** Suppose the points  $(5, -1)$  and  $(2, 5)$  are on the graph of a linear function  $f$ . Find  $f(-3)$ .


 **Exercise 5.7** Graph the function.

(1)  $f(x) = -x + 1$



(2)  $f(x) = \frac{1}{2}x - 1$



 **Exercise 5.8** A storage rental company charges a base fee of \$15 and \$ $x$  per day for a small cube. Suppose the cost is \$20 dollars for 10 days.

- (1) Write the cost  $y$  (in dollars) as a linear function of the number of days  $x$ .
- (2) How much would it cost to rent a small cube for a whole summer (June, July and August)?



## Lesson 6 Perpendicular and Parallel Lines

**Horizontal and Vertical Lines** A *horizontal line* is defined by an equation  $y = b$ . The slope of a horizontal line is simply zero. A *vertical line* is defined by an equation  $x = a$ . The slope of a vertical line is **undefined**.

A vertical line gives an example that a graph is not a function of  $x$ . Indeed, the vertical line test fails for a vertical line.

**Explicit Function** When studying functions, we prefer a clearly expressed function rule. For example, in  $f(x) = -\frac{2}{3}x + 1$ , the expression  $-\frac{2}{3}x + 1$  clearly tells us how to produce outputs. For a function  $f$  defined by an equation, for instance,  $2x + 3y = 3$ , to find the function rule (that is an expression), we simply solve the given equation for  $y$ .

$$\begin{aligned}2x + 3y &= 3 \\3y &= -2x + 3 \\y &= -\frac{2}{3}x + 1.\end{aligned}$$

Now, we get  $f(x) = -\frac{2}{3}x + 1$ .

**Perpendicular and Parallel Lines** Any two vertical lines are parallel. Two non-vertical lines are *parallel* if and only if they **have the same slope**.

A line that is parallel to the line  $y = mx + a$  has an equation  $y = mx + b$ , where  $a \neq b$ .

Horizontal lines are perpendicular to vertical lines. Two non-vertical lines are *perpendicular* if and only if **the product of their slopes is  $-1$** .

A line that is perpendicular to the line  $y = mx + a$  has an equation  $y = -\frac{1}{m}x + b$ .

### Finding Equations for Perpendicular or Parallel Lines

**Example 6.1** Find an equation of the line that is parallel to the line  $4x + 2y = 1$  and passes through the point  $(-3, 1)$ .

#### Solution

**Step 1.** Find the slope  $m$  of the original line from the slope-intercept form equation by solving for  $y$ .  $y = -2x + \frac{1}{2}$ . So  $m = -2$ .

**Step 2.** Find the slope  $m_{\parallel}$  of the parallel line.

$$m_{\parallel} = m = -2.$$

**Step 3.** Use the point-slope form.

$$\begin{aligned}y - 1 &= -2(x + 3) \\y &= -2x - 5.\end{aligned}$$

**Example 6.2** Find an equation of the line that is perpendicular to the line  $4x - 2y = 1$  and passes through the point  $(-2, 3)$ .

#### Solution


**Step 1.** Find the slope  $m$  of the original line from the slope-intercept form equation by solving for  $y$ .  $y = 2x - \frac{1}{2}$ . So  $m = 2$ .

**Step 2.** Find the slope  $m_{\perp}$  of the perpendicular line.


$$m_{\perp} = -\frac{1}{m} = -\frac{1}{2}.$$

**Step 3.** Use the point-slope form.


$$\begin{aligned}y - 3 &= -\frac{1}{2}(x + 2) \\y &= -\frac{1}{2}x + 2.\end{aligned}$$

 **Exercise 6.1** Find an equation for each of the following two lines which pass through the same point  $(-1, 2)$ .  
 (1) The vertical line. (2) The horizontal line.


$$z = \mathcal{C}(\mathcal{Z}) \text{ 1'9} \qquad \text{1-} = x \text{ (1) 1'9}$$

 **Exercise 6.2** Line  $L$  is defined by the equation  $2x - 5y = -3$ . What is the slope  $m_{\parallel}$  of the line that is parallel to the line  $L$ ? What is the slope  $m_{\perp}$  of the line that is perpendicular to the line  $L$ .


$$\frac{z}{5} = \text{T}u \qquad \frac{5}{z} = \|u \text{ 2'9}$$

 **Exercise 6.3** Line  $L_1$  is defined by  $3y + 5x = 7$ . Line  $L_2$  passes through  $(-1, -3)$  and  $(4, -8)$ . Determine whether  $L_1$  and  $L_2$  are parallel, perpendicular or neither.




 **Exercise 6.4** Find the point-slope form and then the slope-intercept form equations of the line parallel to  $3x - y = 4$  and passing through the point  $(2, -3)$ .


$$6 - x\varepsilon = \mathcal{L} \quad \varepsilon - (z - x)\varepsilon = \mathcal{L} \quad 4.9$$

 **Exercise 6.5** Find the slope-intercept form equation of the line that is perpendicular to  $4y - 2x + 3 = 0$  and passing through the point  $(2, -5)$

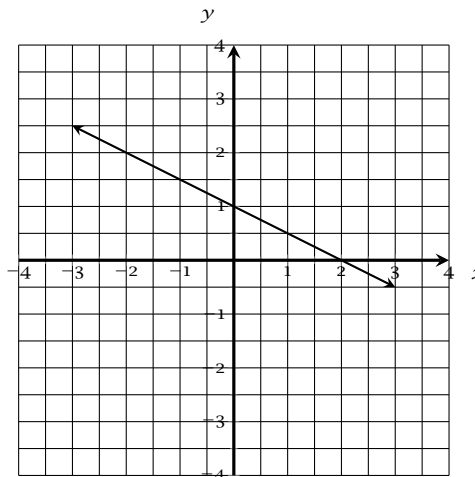
$$1 - xz - = \mathcal{L} \quad 5.9$$

 **Exercise 6.6** The line  $L_1$  is defined  $Ax + By = 3$ . The line  $L_2$  is defined by the equation  $Ax + By = 2$ . The line  $L_3$  is defined by  $Bx - Ay = 1$ . Determine whether  $L_1$ ,  $L_2$  and  $L_3$  are parallel or perpendicular to each other.



 **Exercise 6.7** Use the graph of the line  $L$  to answer the following questions


- (1) Find an equation for the line  $L$ .
- (2) Find an equation for the line  $L_{\perp}$  perpendicular to  $L$  and passing through  $(1, 1)$ .
- (3) Find an equation for the line  $L_{\parallel}$  parallel to  $L$  and passing through  $(-2, -1)$ .



$$z - x \frac{z}{1} = \mathcal{L} \text{ (3) } L'9$$

$$1 - xz = \mathcal{L} \text{ (2) } L'9$$

$$1 + x \frac{z}{1} = \mathcal{L} \text{ (1) } L'9$$

 **Exercise 6.8** Determine whether the points  $(-3, 1)$ ,  $(-2, 6)$ ,  $(3, 5)$  and  $(2, 0)$  form a square. Please explain your conclusion.

6.8 The points form a square. Because the rise and the run for each side up to a sign and a switch are the same.



## Lesson 7 Systems of Linear Equations

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**Methods to solve a linear systems** A *system of linear equations* of two variables consists of two equations. A *solution of a system* of linear equations of two variables is an ordered pair that satisfies both equations.

Two methods to solve a linear system are the substitution method and the elimination method. The idea of the substitution method is to view each equation as an implicitly defined function. It's less efficient than the elimination method in general. The idea of elimination is to reduce the number of variables using properties of equations.

### Substitution Method

**Example 7.1** Solve the system of linear equations using the substitution method.

$$x + y = 3 \quad (7.1)$$

$$2x + y = 4 \quad (7.2)$$

### Solution

**Step 1.** Solve one variable from one equation. For example, one may solve  $y$  from equation (7.1).

$$y = 3 - x.$$

**Step 2.** Plug  $y = 3 - x$  into the second equation and solve for  $x$ .

$$2x + (3 - x) = 4$$

$$x + 3 = 4$$

$$x = 1$$

**Step 3.** Plug the solution  $x = 1$  into the equation in Step 1 to solve for  $y$ .

$$y = 3 - x = 3 - 1 = 2$$

**Step 4.** Check the proposed solution. Plug  $(1, 2)$  into the first equation:

$$1 + 2 = 3.$$

Plug  $(1, 2)$  into the second equation:

$$2 \cdot 1 + 2 = 4.$$

### Elimination Method

**Example 7.2** Solve the system of linear equations using the addition method.

$$5x + 2y = 7 \quad (7.1)$$

$$3x - y = 13 \quad (7.2)$$

### Solution

**Step 1.** Eliminate one variable and solve for the other. For example, one may choose to eliminate  $y$ . In order to eliminate  $y$ , we add the opposite. We multiply both sides of the second equation by 2 to get the opposite  $-2y$ .

$$2(3x) - 2y = 2(13)$$

$$6x - 2y = 26 \quad (7.3)$$

Adding equations (7.1) and (7.3) will eliminate  $y$ .

$$5x + 2y = 7$$

$$+ \quad 6x - 2y = 26$$

---


$$11x + 0 = 33$$

$$x = 3$$

**Step 2.** Find the missing variable by plugging  $x = 3$  into the first equation and solve for  $y$ .

$$5 \cdot 3 + 2y = 7$$

$$15 + 2y = 7$$

$$2y = -8$$

$$y = -4$$

**Step 3.** Check the proposed solution. Plug  $(3, -4)$  into the first equation:

$$5 \cdot 3 + 2 \cdot (-4) = 15 - 8 = 7.$$

Plug  $(3, -4)$  into the second equation:

$$3 \cdot 3 - (-4) = 9 + 4 = 13.$$

 **Note** A linear system may have **one solution**, **no solution** or **infinitely many solutions**.

If the lines defined by equations in the linear system have the same slope but different  $y$ -intercepts or the elimination method ends up with  $0 = c$ , where  $c$  is a nonzero constant, then the system has no solution.

If the lines defined by equations in the linear system have the same slope and the same  $y$ -intercept or the elimination method ends up with  $0 = 0$ , then the system has infinitely many solutions. In this case, we say that the system is **dependent** and a solution can be expressed in the form  $(x, f(x)) = (x, mx + b)$ .

 **Exercise 7.1** Solve.

$$\begin{aligned} 2x - y &= 8 \\ -3x - 5y &= 1 \end{aligned}$$

 **Exercise 7.2** Solve.

$$\begin{aligned} x + 4y &= 10 \\ 3x - 2y &= -12 \end{aligned}$$

 **Exercise 7.3** Solve.

$$\begin{aligned} -x - 5y &= 29 \\ 7x + 3y &= -43 \end{aligned}$$

7.2 (-2, 3)

 **Exercise 7.4** Solve.

$$\begin{aligned} 2x + 15y &= 9 \\ x - 18y &= -21 \end{aligned}$$

7.1 (3, -2)

7.4 (-3, 1)

7.3 (-4, -5)

 **Exercise 7.5** Solve.

$$2x + 7y = 5$$

$$3x + 2y = 16$$

 **Exercise 7.6** Solve.

$$4x + 3y = -10$$

$$-2x + 5y = 18$$

 **Exercise 7.7** Solve.

$$3x + 2y = 6$$

$$6x + 4y = 16$$

(-4, 2) 7.6

 **Exercise 7.8** Solve.

$$2x - 3y = -6$$


$$-4x + 6y = 12$$

(1, 9) 7.5


(1 +  $\frac{3}{2}x$ ,  $\frac{3}{2}x + 1$ ) 7.8

no solution 7.7



 **Exercise 7.9** Last week, Mike got 5 apples and 4 oranges for \$7. The week the prices are still the same and he got 3 apples and 6 oranges for \$6. What's the price for 1 apple and 1 orange?

7.9 apple: \$1/each; orange: \$0.5/each.

 **Exercise 7.10** The sum of the digits of a certain two-digit number is 7. Reversing its digits increases the number by 27. What is the number?



## Lesson 8 Factoring Review

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**Factor by Removing the GCF** *The greatest common factor (GCF)* of two terms is a polynomial with the **greatest coefficient** and of the **highest possible degree** that divides each term.

To *factor a polynomial* is to **express the polynomial as a product** of polynomials of lower degrees. The first and the easiest step is to factor out the GCF of all terms.

**Example 8.1** Factor  $4x^3y - 8x^2y^2 + 12x^3y^3$ .

**Solution**

**Step 1.** Find the GCF of all terms. The GCF of  $4x^3y$ ,  $-8x^2y^2$  and  $12x^3y^3$  is  $4x^2y$ .

**Step 2.** Write each term as the product of the GCF and the remaining factor.

$$4x^3y = (4x^2y) \cdot x, \quad -8x^2y^2 = (4x^2y) \cdot (-2y), \quad \text{and} \quad 12x^3y^3 = (4x^2y)(3xy^2).$$

**Step 3.** Factor out the GCF from each term.

$$4x^3y - 8x^2y^2 + 12x^3y^3 = 4x^2y \cdot (x - 2y + 3xy^2).$$

**Factor by Grouping** For a four-term polynomial, in general, we will group them into two groups and factor out the GCF for each group and then factor further.

**Example 8.2** Factor  $2x^2 - 6xy + xz - 3yz$ .

**Solution**

**Step 1.** Group the first two terms and the last two terms.

$$\begin{aligned} &2x^2 - 6xy + xz - 3yz \\ &= (2x^2 - 6xy) + (xz - 3yz) \end{aligned}$$

**Step 2.** Factor out the GCF from each group.

$$= 2x(x - 3y) + z(x - 3y)$$

**Step 3.** Factor out the binomial GCF.

$$= (x - 3y)(2x + z).$$

**Example 8.3** Factor  $ax + 4b - 2a - 2bx$ .

**Solution**

**Step 1.** Group the first term with the third term and group the second term with the last term.


$$\begin{aligned} &ax + 4b - 2a - 2bx \\ &= (ax - 2a) + (-2bx + 4b) \end{aligned}$$

**Step 2.** Factor out the GCF from each group.

$$= a(x - 2) + (-2b)(x - 2)$$

**Step 3.** Factor out the binomial GCF.

$$= (x - 2)(a - 2b).$$

 **Tips** **Guess and check.** Once you factored one group, you may expect that the other group has the same binomial factor so that factoring may be continued.

### Factor Binomials in Special Forms

Difference of squares

$$a^2 - b^2 = (a - b)(a + b)$$

**Example 8.4** Factor  $25x^2 - 16$ .

**Solution**

**Step 1.** Recognize the binomial as a difference of squares.

$$\begin{aligned} &25x^2 - 16 \\ &= (5x)^2 - 4^2 \end{aligned}$$

**Step 2.** Apply the formula.

$$= (5x - 4)(5x + 4).$$

Sum of Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

**Example 8.5** Factor  $x^3 + 27$ .

**Solution**

**Step 1.** Recognize the binomial as a sum of cubes.

$$\begin{aligned} &x^3 + 27 \\ &= x^3 + 3^3 \end{aligned}$$

**Step 2.** Apply the formula.

$$\begin{aligned} &= (x + 3)(x^2 - 3x + 3^2) \\ &= (x + 3)(x^2 - 3x + 9). \end{aligned}$$

Difference of cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

**Example 8.6** Factor  $125x^3 - 8$ .

**Solution**

**Step 1.** Recognize the binomial as a difference of cubes.

$$\begin{aligned} &125x^3 - 8 \\ &= (5x)^3 - 2^3 \end{aligned}$$

**Step 2.** Apply the formula.

$$\begin{aligned} &= (5x - 2)((5x)^2 + 2(5x) + 2^2) \\ &= (5x - 2)(25x^2 + 10x + 4). \end{aligned}$$

**Example 8.7** Factor  $16x^3y + 2y^4$  completely.

**Solution**

$$16x^3y + 2y^4 = 2y(8x^3 + y^3) = 2y((2x)^3 + y^3) = 2y(2x + y)(4x^2 - 2xy + y^2).$$

 **Exercise 8.1** Factor out the GCF.

- (1)  $18x^2y^2 - 12xy^3 - 6x^3y^4$       (2)  $5x(x - 7) + 3y(x - 7)$       (3)  $-2a^2(x + y) + 3a(x + y)$

$(x + y)(x - y)$  (1) 1'8       $(x^2 + y^2)(x - y)$  (2) 1'8       $(x^2 - y^2)(x + y)$  (1) 1'8

 **Exercise 8.2** Factor by grouping.

- (1)  $12xy - 10y + 18x - 15$       (2)  $12ac - 18bc - 10ad + 15bd$       (3)  $5ax - 4bx - 5ay + 4by$

$(x^2 - y^2)(x - y)$  (3) 2'8       $(x^2 - y^2)(x + y)$  (2) 2'8       $(x + y)(x - y)$  (1) 2'8



 **Exercise 8.3** Factor completely.

(1)  $25x^2 - 4$

(2)  $8x^3 - 27$

(3)  $125y^3 + 1$

(1)  $(x+2)(x-2)$  (2)  $(x+1)(x-1)$  (3)  $(x+1)(x-1)$  (4)  $(x+2)(x-2)$  (5)  $(x+1)(x-1)$  (6)  $(x+2)(x-2)$  (7)  $(x+1)(x-1)$  (8)  $(x+2)(x-2)$  (9)  $(x+1)(x-1)$  (10)  $(x+2)(x-2)$

 **Exercise 8.4** Factor completely.

(1)  $27x^4 + xy^3$

(2)  $16xy^3 - 2x^4$

(3)  $x^4 + 3x^3 - 4x^2 - 12x$

(1)  $(x+2)(x-2)$  (2)  $(x+1)(x-1)$  (3)  $(x+2)(x-2)$  (4)  $(x+1)(x-1)$  (5)  $(x+2)(x-2)$  (6)  $(x+1)(x-1)$  (7)  $(x+2)(x-2)$  (8)  $(x+1)(x-1)$  (9)  $(x+2)(x-2)$  (10)  $(x+1)(x-1)$



## Lesson 9 Solve Quadratic Equations by Factoring

**Factor Trinomials** If a trinomial  $ax^2 + bx + c$ ,  $A \neq 0$ , can be factored, then it can be expressed as a product of two binomials:

$$ax^2 + bx + c = (mx + n)(px + q).$$

By simplify the product using the FOIL method and comparing coefficients, we observe that

$$a = \underbrace{mn}_F \quad b = \underbrace{mq}_O + \underbrace{np}_I \quad c = \underbrace{nq}_F$$

A trinomial  $ax^2 + bx + c$  is also called a **quadratic polynomial**. The function defined by  $f(x) = ax^2 + bx + c$  is called a **quadratic function**.

**Tips Trial and error.** The observation suggests to use trial and error to find the undetermined coefficients  $m$ ,  $n$ ,  $p$ , and  $q$  from factors of  $a$  and  $c$  such that the sum of cross products  $mq + np$  is  $b$ . A diagram as shown in the following examples can help check a trial.

**Example 9.1** Factor  $x^2 + 6x + 8$ .

**Solution**

**Step 1.** Factor  $a = 1$ :

$$1 = 1 \cdot 1.$$

**Step 2.** Factor  $c = 8$ :

$$8 = 1 \cdot 8 = 2 \cdot 4.$$

**Step 3.** Choose a proper combination of pairs of factors and check if the sum of cross product equals  $b = 6$ :

$$1 \cdot 4 + 1 \cdot 2 = 6.$$

This step can be checked easily using the following diagram.

$$\begin{array}{ccc}
 a = 1 = 1 \cdot 1 & c = 8 = 2 \cdot 4 & \\
 \begin{array}{cc}
 1 & 2 \\
 1 & 4
 \end{array} & & \\
 \hline
 1 \cdot 2 & + & 1 \cdot 4 = 6 = b
 \end{array}$$

**Step 4.** Factor the trinomial

$$x^2 + 6x + 8 = (x + 2)(x + 4).$$

**Example 9.2** Factor  $2x^2 + 5x - 3$ .

**Solution**

**Step 1.** Factor  $a = 2$ :

$$2 = 1 \cdot 2.$$

**Step 2.** Factor  $c = -3$ :

$$-3 = 1 \cdot (-3) = (-1) \cdot 3.$$

**Step 3.** Choose a proper combination of pairs of factors and if the sum of cross products equals  $b = 5$ :

$$2 \cdot 3 + 1 \cdot (-1) = 5.$$

This step can be checked easily using the following diagram.

$$\begin{array}{ccc}
 a = 2 = 1 \cdot 2 & c = -3 = 3 \cdot (-1) & \\
 \begin{array}{cc}
 1 & 3 \\
 2 & -1
 \end{array} & & \\
 \hline
 2 \cdot 3 & + & 1 \cdot (-1) = 5 = b
 \end{array}$$

**Step 4.** Factor the trinomial

$$2x^2 + 5x - 3 = (x + 3)(2x - 1).$$

**Tips Use Auxiliary Problem.** Some higher degree polynomials may be rewrite as a trinomial after a substitution. Factoring the trinomial helps factor the polynomial.

**Example 9.3** Factor the trinomial completely.

$$4x^4 - x^2 - 3$$

**Solution**

**Step 1.** Let  $x^2 = y$ . Then  $4x^4 - x^2 - 3 = 4y^2 - y - 3$ .

**Step 2.** Factor the trinomial in  $y$ :  $4y^2 - y - 3 = (4y + 3)(y - 1)$ .

**Step 3.** Replace  $y$  by  $x^2$  and factor further.

$$\begin{aligned}
 4x^4 - x^2 - 3 &= 4y^2 - y - 3 \\
 &= (4y + 3)(y - 1) \\
 &= (4x^2 + 3)(x^2 - 1) \\
 &= (4x^2 + 3)(x - 1)(x + 1).
 \end{aligned}$$



**Solving a Quadratic Equation by Factoring and Applications** A *quadratic equation* is a polynomial equation of degree 2, for example,  $2x^2 + 5x - 3 = 0$ . The *standard form* of a quadratic equation is

$$ax^2 + bx + c = 0,$$

where  $a$ ,  $b$  and  $c$  are numbers, and  $a \neq 0$ .

To solve a quadratic equation, we may first factor the polynomial and then apply the *zero product property*:

$$A \cdot B = 0 \quad \text{if and only if} \quad A = 0 \quad \text{or} \quad B = 0.$$

**Example 9.4** Solve the equation

$$2x^2 + 5x = 3.$$

**Solution**

**Step 1.** Rewrite the equation into “*Expression=0*” form and factor.

$$\begin{aligned} 2x^2 + 5x &= 3 \\ 2x^2 + 5x - 3 &= 0 \\ (2x - 1)(x + 3) &= 0 \end{aligned}$$

**Step 2.** Apply the zero product property.

$$2x - 1 = 0 \quad \text{or} \quad x + 3 = 0.$$

**Step 3.** Solve each equation.

$$\begin{aligned} 2x - 1 &= 0 & \text{or} & & x + 3 &= 0 \\ 2x &= 1 & & & x &= -3 \\ x &= \frac{1}{2} & \text{or} & & x &= -3 \end{aligned}$$

**Step 4.** The solution set is  $\{-3, \frac{1}{2}\}$ .

**Example 9.6** A rectangular garden is surrounded by a path of uniform width. If the dimension of the garden is 10 meters by 16 meters and the total area is 216 square meters, determine the width of the path.

**Solution**

**Step 1.** Suppose that the width of the frame is  $x$  meters. Translate given information into expressions in  $x$  and build an equation.

$$\text{Total Width: } 2x+10 \quad \text{Total Length: } 2x+16 \quad \text{Width} \times \text{Length} = \text{Total Area: } (2x + 10)(2x + 16) = 216.$$

**Step 2.** Solve the equation.

$$\begin{aligned} (2x + 10)(2x + 16) &= 216 \\ 4x^2 + 52x + 160 &= 216 \\ 4x^2 + 52x - 56 &= 0 \\ x^2 + 13x - 14 &= 0 \\ (x + 14)(x - 1) &= 0 \\ x = -14 & \quad \text{or} \quad x = 1 \end{aligned}$$

**Step 3.** So the width of the path is 1 meter.

**Tips Understand the Problem.** When solving a word problem, you may first outline what’s known and what’s unknown, and restate the problem using algebraic expressions. Once you reformulated the problem algebraically, you may solve it using your mathematical knowledge.

**Example 9.5** Solve the equation

$$(x - 2)(x + 3) = -4.$$

**Solution**

**Step 1.** Rewrite the equation into “*Expression=0*” form and factor.

$$\begin{aligned} (x - 2)(x + 3) &= -4 \\ x^2 + x - 6 &= -4 \\ x^2 + x - 2 &= 0 \\ (x - 1)(x + 2) &= 0 \end{aligned}$$

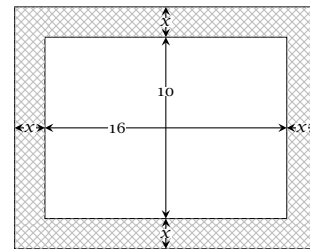
**Step 2.** Apply the zero product property.

$$x - 1 = 0 \quad \text{or} \quad x + 2 = 0.$$

**Step 3.** Solve each equation.

$$\begin{aligned} x - 1 &= 0 & \text{or} & & x + 2 &= 0 \\ x &= 1 & \text{or} & & x &= -2 \end{aligned}$$

**Step 4.** The solution set is  $\{-2, 1\}$ .



 **Exercise 9.1** Factor the trinomial.

(1)  $x^2 + 4x + 3$

(2)  $x^2 + 6x - 7$

(3)  $x^2 - 3x - 10$

(4)  $x^2 - 5x + 6$

(8 - x)(2 - x) (4) 1'6 (2 + x)(5 - x) (3) 1'6 (2 + x)(1 - x) (2) 1'6 (3 + x)(1 + x) (1) 1'6

 **Exercise 9.2** Factor the trinomial.

(1)  $5x^2 + 7x + 2$


(2)  $2x^2 + 5x - 12$

(3)  $3x^2 - 10x - 8$

(4)  $4x^2 - 12x + 5$


(5 - x)(2x - 1) (2) (4) 9.2 (2) (3 + x)(4 - x) (3) 9.2 (3) (2x - 4)(2x + 2) (2) 9.2 (2) (x + 4)(2x + 2) (2) 9.2 (1) (x + 1)(5x + 2) (1) 9.2



 **Exercise 9.3** Solve the equation by factoring.

(1)  $x^2 - 3x + 2 = 0$     (2)  $2x^2 - 3x = 5$     (3)  $(x - 1)(x + 3) = 5$     (4)  $\frac{1}{3}(2 - x)(x + 5) = 4$

9.3 (1)  $\{1, 2\}$     9.3 (2)  $\{-1, \frac{2}{5}\}$     9.3 (3)  $\{-4, 2\}$     9.3 (4)  $\{-2, -1\}$

 **Exercise 9.4** Find all real solutions of the equation by factoring.

(1)  $4(x - 2)^2 - 9 = 0$     (2)  $2x^3 - 18x = 0$     (3)  $3x^4 - 2x^2 = 1$     (4)  $x^3 - 3x^2 - 4x + 12 = 0$

9.4 (1)  $\{\frac{1}{2}, \frac{2}{2}\}$     9.4 (2)  $\{-3, 0, 3\}$     9.4 (3)  $\{-1, 1\}$     9.4 (4)  $\{-2, 2, 3\}$


 **Exercise 9.5** Find the  $x$ -intercepts for each of the the following functions.

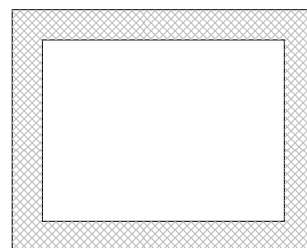
(1)  $f(x) = 2x^2 - x - 21$

(2)  $g(x) = (x + 1)(x - 2) - 4$


(0, 3) and (0, -2) (2) 5, 6

(0, 2) and (0, -3) (1) 5, 6

 **Exercise 9.6** A paint measuring 3 inches by 4 inches is surrounded by a frame of uniform width. If the combined area of the paint and the frame is 30 square inches, determine the width of the frame.




 **Exercise 9.7** A rectangle whose length is 2 meters longer than its width has an area 8 square meters. Find the width and the length of the rectangle.


 **Exercise 9.8** The product of two **consecutive negative odd** numbers is 35. Find the numbers.

9.8 The numbers are  $-7$  and  $-5$ .




 **Exercise 9.9** In a right triangle, the long leg is 2 inches more than double of the short leg. The hypotenuse of the triangle is 1 inch longer than the long leg. Find the length of the shortest side.

9.9 The shortest is the short leg which is 5 inches.

 **Exercise 9.10** A ball is thrown upwards from a rooftop. It will reach a maximum vertical height and then fall back to the ground. The height  $h(t)$  of the ball from the ground after time  $t$  seconds is  $h(t) = -16t^2 + 48t + 160$  feet. How long it will take the ball to hit the ground?

9.10 5 seconds.



 **Exercise 9.11** Each of trinomial below has a factor in the table. Match the letter on the left of a factor with a the number on the left a trinomial to decipher the following quotation.

"  $\overline{13}$   $\overline{10\ 2\ 9\ 15}$ ,  $\overline{9\ 5\ 14}$   $\overline{13}$   $\overline{4\ 3\ 15\ 7\ 2\ 1}$ ;

$\overline{13}$   $\overline{11\ 2\ 2}$ ,  $\overline{9\ 5\ 14}$   $\overline{13}$   $\overline{8\ 5\ 3\ 6}$ ;

$\overline{13}$   $\overline{14\ 3}$ ,  $\overline{9\ 5\ 14}$   $\overline{13}$   $\overline{12\ 5\ 14\ 2\ 15\ 11\ 1\ 9\ 5\ 14}$ ."

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<b>A:</b> $3x - 2$	<b>B:</b> $2x + 1$	<b>C:</b> $x + 6$	<b>D:</b> $x + 7$	<b>E:</b> $2x - 1$	<b>F:</b> $3x - 1$	<b>G:</b> $x + 10$
<b>H:</b> $x - 8$	<b>I:</b> $2x + 9$	<b>J:</b> $x - 1$	<b>K:</b> $x + 3$	<b>L:</b> $2x - 5$	<b>M:</b> $x + 5$	<b>N:</b> $x - 7$
<b>O:</b> $x - 13$	<b>P:</b> $5x - 3$	<b>Q:</b> $4x - 11$	<b>R:</b> $x - 9$	<b>S:</b> $2x + 3$	<b>T:</b> $x + 4$	<b>U:</b> $7x + 1$
<b>V:</b> $3x + 5$	<b>W:</b> $3x + 4$	<b>X:</b> $8x + 3$	<b>Y:</b> $x - 14$	<b>Z:</b> $5x - 6$		

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1.  $x^2 - 2x - 24$

2.  $6x^2 + x - 2$

3.  $x^2 - 16x + 39$

4.  $6x^2 + 13x - 5$

5.  $x^2 - 5x - 14$

6.  $3x^2 - 5x - 12$

7.  $x^2 - x - 110$

8.  $x^2 - 9$

9.  $-3x^2 + 11x - 6$

10.  $x^2 - 10x + 16$

11.  $-2x^2 + 5x + 12$

12.  $42x^2 - x - 1$

13.  $-2x^2 - 3x + 27$

14.  $x^2 + 14x + 49$

15.  $x^2 - 81$



## Lesson 10 Multiply or Divide Rational Expressions

**Rational Expressions** Let  $p$  and  $q$  be polynomial functions of  $x$  and  $p$  is not a constant function. We call the function  $r(x) = \frac{p(x)}{q(x)}$  a **rational function**. The domain of  $r$  is  $\{x \mid Q(x) \neq 0\}$ . The expression  $\frac{p(x)}{q(x)}$  is called a **rational expression**, the polynomial  $p(x)$  **the numerator**, and the polynomial  $q(x)$  **the denominator**. A rational expression is **simplified** if the numerator and the denominator have no common factor other than 1.

Let  $p(x)$ ,  $q(x)$  be polynomials with  $q(x) \neq 0$  and  $c(x)$  be a nonzero expression. Then

$$\frac{p(x) \cdot c(x)}{q(x) \cdot c(x)} = \frac{p(x)}{q(x)}.$$

**Example 10.1** Simplify  $\frac{x^2 + 4x + 3}{x^2 + 3x + 2}$ .

**Solution**

**Step 1.** Factor both the top and the bottom.

$$\frac{x^2 + 4x + 3}{x^2 + 3x + 2} = \frac{(x + 1)(x + 3)}{(x + 1)(x + 2)}.$$

**Step 2.** Divide out common factors.

$$\frac{(x + 1)(x + 3)}{(x + 1)(x + 2)} = \frac{x + 3}{x + 2}.$$

**Multiplying Rational Expressions** If  $p$ ,  $q$ ,  $s$ ,  $t$  are polynomials with  $q \neq 0$  and  $t \neq 0$ , then

$$\frac{p}{q} \cdot \frac{s}{t} = \frac{ps}{qt}.$$

**Example 10.3** Multiply and then simplify.

$$\frac{3x^2}{x^2 + x - 6} \cdot \frac{x^2 - 4}{6x}.$$

**Solution**

**Step 1.** Factor numerators and denominators.

$$\frac{3x^2}{x^2 + x - 6} \cdot \frac{x^2 - 4}{6x} = \frac{3 \cdot x \cdot x}{(x - 2)(x + 3)} \cdot \frac{(x - 2)(x + 2)}{2 \cdot 3 \cdot x}$$

**Step 2.** Multiply and simplify.

$$\frac{\cancel{3} \cdot x \cdot x \cdot \cancel{(x - 2)}(x + 2)}{\cancel{(x - 2)}(x + 3) \cdot 2 \cdot \cancel{3} \cdot x} = \frac{x(x + 2)}{2(x + 3)}$$

**Example 10.4** Multiply and then simplify.

$$\frac{3x^2 - 8x - 3}{x^2 + 8x + 16} \cdot \frac{x^2 - 16}{5x^2 - 14x - 3}.$$

**Solution**

$$\frac{3x^2 - 8x - 3}{x^2 + 8x + 16} \cdot \frac{x^2 - 16}{5x^2 - 14x - 3} = \frac{(3x + 1)\cancel{(x - 3)}\cancel{(x + 4)}(x - 4)}{\cancel{(x + 4)}(x + 4)(5x + 1)\cancel{(x - 3)}} = \frac{(3x + 1)(x - 4)}{(x + 4)(5x + 1)}$$

**Dividing Rational Expressions** If  $p$ ,  $q$ ,  $s$ ,  $t$  are polynomials where  $q \neq 0$ ,  $s \neq 0$  and  $t \neq 0$ , then

$$\frac{p}{q} \div \frac{s}{t} = \frac{p}{q} \cdot \frac{t}{s} = \frac{pt}{qs}.$$

**Example 10.5** Divide and then simplify.

$$\frac{2x + 6}{x^2 - 6x - 7} \div \frac{6x - 2}{2x^2 - x - 3}.$$

**Solution**

**Step 1.** Rewrite the division as a multiplication.

$$\frac{2x + 6}{x^2 - 6x - 7} \div \frac{6x - 2}{2x^2 - x - 3} = \frac{2x + 6}{x^2 - 6x - 7} \cdot \frac{2x^2 - x - 3}{6x - 2}$$

**Step 2.** Factor and simplify.

$$\frac{2x + 6}{x^2 - 6x - 7} \cdot \frac{2x^2 - x - 3}{6x - 2} = \frac{\cancel{2}(x + 3)\cancel{(x + 1)}(2x - 3)}{\cancel{2}\cancel{(x + 1)}(x - 7)(3x - 1)} = \frac{(x + 3)(2x - 3)}{(x - 7)(3x - 1)}$$



 **Exercise 10.1** Simplify.

(1)  $\frac{3x^2 - x - 4}{x + 1}$

(2)  $\frac{2x^2 - x - 3}{2x^2 + 3x + 1}$

(3)  $\frac{x^2 - 9}{3x^2 - 8x - 3}$

 **Exercise 10.2** Multiply and simplify.

(1)  $\frac{x + 5}{x + 4} \cdot \frac{x^2 + 3x - 4}{x^2 - 25}$

(2)  $\frac{3x^2 - 2x}{x + 2} \cdot \frac{3x^2 - 4x - 4}{9x^2 - 4}$

(3)  $\frac{6y - 2}{3 - y} \cdot \frac{y^2 - 6y + 9}{3y^2 - y}$

(1)  $\frac{x}{x - 3} \cdot \frac{x + 2}{x - 2}$

(2)  $\frac{x + 2}{x - 2} \cdot \frac{x - 2}{x - 2}$

(3)  $\frac{x - 5}{x - 1} \cdot \frac{x - 1}{x - 1}$



 **Exercise 10.3** Divide and simplify.

(1)  $\frac{9x^2 - 49}{6} \div \frac{3x^2 - x - 14}{2x + 4}$

(2)  $\frac{x^2 + 3x - 10}{2x - 2} \div \frac{x^2 - 5x + 6}{x^2 - 4x + 3}$

(3)  $\frac{y - x}{xy} \div \frac{x^2 - y^2}{y^2}$

$\frac{(x + x)x}{x}$  (3) 10.3

$\frac{z}{5 + x}$  (2) 10.3

$\frac{3}{z + x}$  (1) 10.3

 **Exercise 10.4** Simplify.

$\frac{-x^2 + 11x - 18}{x^2 - 4x + 4} \div \frac{x^2 - 5x - 36}{x^2 - 7x + 12} \cdot \frac{2x^2 + 7x - 4}{x^2 + 2x - 15}$

$\frac{(5 + x)(z - x)}{(x - 4)(2x - 1)}$  10.4



## Lesson 11 Add or Subtract Rational Expressions

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**Adding or Subtracting Rational Expressions with the Same Denominator** If  $P$ ,  $Q$  and  $R$  are polynomials with  $R \neq 0$ , then

$$\frac{P}{R} + \frac{Q}{R} = \frac{P + Q}{R} \quad \text{and} \quad \frac{P}{R} - \frac{Q}{R} = \frac{P - Q}{R}.$$

**Example 11.1** Add and simplify

$$\frac{x^2 + 4}{x^2 + 3x + 2} + \frac{5x}{x^2 + 3x + 2}.$$

**Solution**

**Step 1.** Determine if the rational expressions have the same denominator. If so, the new numerator will be the sum/difference of the numerators.

$$\frac{x^2 + 4}{x^2 + 3x + 2} + \frac{5x}{x^2 + 3x + 2} = \frac{x^2 + 5x + 4}{x^2 + 3x + 2}.$$

**Step 2.** Simplify the resulting rational expression.

$$\frac{x^2 + 5x + 4}{x^2 + 3x + 2} = \frac{(x + 1)(x + 4)}{(x + 1)(x + 2)} = \frac{x + 4}{x + 2}.$$

**Adding or Subtracting Rational Expressions with Different Denominators** To add or subtract rational expressions with different denominators, we need to rewrite the rational expressions to equivalent rational expressions with the same denominator, say the LCD.

**Tips Equivalent Reduction.** What if all denominators are the same? How to make denominators the same? Reducing the problem to an easier one using equivalent operations helps solve the problem.

**Example 11.2** Find the LCD of  $\frac{3}{x^2 - x - 6}$  and  $\frac{6}{x^2 - 4}$ .

**Solution**

**Step 1.** Factor each denominator.

$$x^2 - x - 6 = (x + 2)(x - 3) \quad x^2 - 4 = (x - 2)(x + 2)$$

**Step 2.** List the factors of the first denominator and add unlisted factors of the second factor to get the final list.

First list	$(x + 2)$	$(x - 3)$	
Second list	$(x + 2)$		$(x - 2)$
Final list	$(x + 2)$	$(x - 3)$	$(x - 2)$

**Step 3.** The LCD is the product of factors in the final list.

$$(x + 2)(x - 3)(x - 2).$$

**Example 11.3** Subtract and simplify

$$\frac{x - 3}{x^2 - 2x - 8} - \frac{1}{x^2 - 4}$$

**Solution**

**Step 1.** Find the LCD.

$$x^2 - 2x - 8 = (x + 2)(x - 4) \quad \begin{array}{l} \text{First list} \\ \text{Second list} \\ \text{Final list} \end{array} \begin{array}{|c|c|c|} \hline (x + 2) & & (x - 4) \\ \hline (x + 2) & (x - 2) & \\ \hline (x + 2) & (x - 2) & (x - 4) \\ \hline \end{array}$$

The LCD is  $(x + 2)(x - 2)(x - 4)$ .

**Step 2.** Rewrite each rational expression into an equivalent expression with the LCD as the new denominator.

$$\frac{x - 3}{x^2 - 2x - 8} - \frac{1}{x^2 - 4} = \frac{(x - 3)(x - 2)}{(x + 2)(x - 2)(x - 4)} - \frac{(x - 4)}{(x + 2)(x - 2)(x - 4)}$$

**Step 3.** Subtract and simplify.

$$\frac{(x - 3)(x - 2) - (x - 4)}{(x + 2)(x - 2)(x - 4)} = \frac{(x^2 - 5x + 6) - (x - 4)}{(x + 2)(x - 2)(x - 4)} = \frac{x^2 - 6x + 10}{(x + 2)(x - 2)(x - 4)}$$

 **Exercise 11.1** Add/subtract and simplify.

(1)  $\frac{x^2 + 2x - 2}{x^2 + 2x - 15} + \frac{5x + 12}{x^2 + 2x - 15}$     (2)  $\frac{3x - 10}{x^2 - 25} - \frac{2x - 15}{x^2 - 25}$     (3)  $\frac{4}{(x - 3)(x + 2)} + \frac{3x - 2}{x^2 - x - 6}$

$\frac{(z + x)(8 - x)}{z + x8}$  (3) 1.11

$\frac{5 - x}{1}$  (2) 1.11

$\frac{8 - x}{z + x}$  (1) 1.11

 **Exercise 11.2** Find the LCD of rational expressions.

(1)  $\frac{2x}{2x^2 - 5x - 3}$  and  $\frac{x - 1}{x^2 - x - 6}$     (2)  $\frac{9}{7x^2 - 28x}$  and  $\frac{2}{x^2 - 8x + 16}$



 **Exercise 11.3** Add and simplify.

(1)  $\frac{x}{x+1} + \frac{x-1}{x+2}$

(2)  $\frac{x+2}{2x^2-x-3} + \frac{1}{x^2+3x+2}$

(3)  $\frac{4}{x-3} + \frac{3x-2}{x^2-x-6}$

$\frac{(z+x)(3-x)}{9+xz}$  (1) 11.3

$\frac{(3-xz)(z+x)(1+x)}{1+x9+zx}$  (2) 11.3

$\frac{(z+x)(1+x)}{1-1-2xz+zxz}$  (1) 11.3

 **Exercise 11.4** Subtract and simplify.

(1)  $\frac{3x+5}{x^2-7x+12} - \frac{3}{x-3}$

(2)  $\frac{y}{y^2-5y-6} - \frac{7}{y^2-4y-5}$

(3)  $\frac{2x-3}{x^2+3x-10} - \frac{x+2}{x^2+2x-8}$

$\frac{(x+x)(4+x)(2-x)}{x^2-2x-22}$  (3) 11.4

$\frac{(y^2-12y+42)(y-6)(y-5)(y+1)}{y^2-12y+42}$  (2) 11.4

$\frac{(x-4)(x-3)}{17}$  (1) 11.4



 **Exercise 11.5** Simplify.

$$\frac{x + 11}{7x^2 - 2x - 5} + \frac{x - 2}{x - 1} - \frac{x}{7x + 5}$$

 **Exercise 11.6** Subtract and simplify.

$$\frac{x - 1}{x^2 - 3x} + \frac{4}{x^2 - 2x - 3} - \frac{1}{x(x + 1)}$$

$$\frac{5 + x^2}{1 - x^9} \quad 11.5$$

$$\frac{(x - 3)x}{x + 2} \quad 11.6$$




## Lesson 12 Complex Rational Expressions

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**Simplifying Complex Rational Expressions** A *complex rational expression* is a rational expression whose denominator or numerator contains a rational expression.

A complex rational expression is equivalent to the quotient of its numerator by its denominator. That suggests the following strategy to simplify a complex rational expression.

 **Tips Simplify and Change the Viewpoint.** A complex rational expression is a quotient of two rational expressions. You may rewrite it as an multiplication by flipping the denominator. However, it's better to simply the numerator and denominator or you won't see a good looking new expression.

**Example 12.1** Simplify

$$\frac{\frac{2x-1}{x^2-1} + \frac{x-1}{x+1}}{\frac{x+1}{x-1} - \frac{1}{x^2-1}}$$

**Step 1.** Simplify the numerator and the denominator.


$$\begin{aligned} \frac{\frac{2x-1}{x^2-1} + \frac{x-1}{x+1}}{\frac{x+1}{x-1} - \frac{1}{x^2-1}} &= \frac{\frac{2x-1}{(x-1)(x+1)} + \frac{(x-1)(x-1)}{(x-1)(x+1)}}{\frac{(x+1)(x+1)}{(x-1)(x+1)} - \frac{1}{(x-1)(x+1)}} \\ &= \frac{\frac{(2x-1) + (x-1)(x-1)}{(x-1)(x+1)}}{\frac{(x+1)(x+1) - 1}{(x-1)(x+1)}} \\ &= \frac{(2x-1) + (x^2 - 2x + 1)}{\frac{(x-1)(x+1)}{(x^2 + 2x + 1) - 1}} \\ &= \frac{x^2}{\frac{(x-1)(x+1)}{x^2 + 2x}} \\ &= \frac{x^2}{(x-1)(x+1)} \end{aligned}$$

**Step 2.** Rewrite as a product.

$$\frac{\frac{x^2}{(x-1)(x+1)}}{\frac{x^2 + 2x}{(x-1)(x+1)}} = \frac{x^2}{(x-1)(x+1)} \cdot \frac{(x-1)(x+1)}{x^2 + 2x}$$

**Step 3.** Multiply and simplify.

$$\begin{aligned} \frac{x^2}{(x-1)(x+1)} \cdot \frac{(x-1)(x+1)}{x^2 + 2x} &= \frac{x \cdot x}{(x-1)(x+1)} \cdot \frac{(x-1)(x+1)}{x(x+2)} \\ &= \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{(x-1)} \cdot \cancel{(x+1)}}{x(x+2) \cdot \cancel{(x-1)} \cdot \cancel{(x+1)}} \\ &= \frac{x}{x+2} \end{aligned}$$

 **Note** Another way to simplify a complex rational expression is to multiply the LCD to both the denominator and numerator and then simplify.

 **Exercise 12.1** Simplify.

(1)  $\frac{1 + \frac{2}{x}}{1 - \frac{2}{x}}$

(2)  $\frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} - \frac{1}{x}}$

12.1 (2)  $x + 1$

12.1 (1)  $\frac{x - 2}{x + 2}$

 **Exercise 12.2** Simplify.

(1)  $\frac{\frac{x^2 - y^2}{y^2}}{\frac{1}{x} - \frac{1}{y}}$

(2)  $\frac{\frac{2}{(x + 1)^2} - \frac{1}{x + 1}}{1 - \frac{4}{(x + 1)^2}}$

12.2 (2)  $\frac{x + 3}{1}$

12.2 (1)  $\frac{x}{x(x + y)}$






 **Exercise 12.3** Simplify.

$$(1) \frac{\frac{5x}{x^2 - x - 6}}{\frac{x+2}{x-3} + \frac{3}{x+2}}$$

$$(2) \frac{\frac{x+1}{x+1} + \frac{x-1}{x-1}}{\frac{x-1}{x-1} - \frac{x+1}{x+1}}$$

$$\frac{xz}{1+z^2} \quad (2) \quad 12.3$$

12.3 (1) 1


 **Exercise 12.4** Tim and Jim refill their cars at the same gas station twice last month. Each time Tim got \$20 gas and Jim got 8 gallon. Suppose they refill their cars on same days. The price was \$2.5 per gallon the first time. The price on the second time changed. Can you find out who had the better average price?

12.4 Tim had the better average price.



## Lesson 13 Rational Equations

**Solving Rational Equations by Clearing Denominators** A *rational equation* is an equation that contains a rational expression.

 **Tips Reduction with Auxiliary Conditions** Assume that denominators are not zero. One idea to solve a rational equation is to reduce the equation to a polynomial equation by clearing denominators, that is multiplying the LCD to both sides of the equation. Once the polynomial equations is solve, remember to check if there is an *extraneous solution* which is a solution making a denominator zero.

**Example 13.1** Solve

$$\frac{5}{x^2 - 9} = \frac{3}{x - 3} - \frac{2}{x + 3}.$$

**Step 1.** Find the LCD.

Since  $x^2 - 9 = (x + 3)(x - 3)$ , the LCD is  $(x + 3)(x - 3)$ .

**Step 2.** Clear denominators.

Multiply each rational expression in both sides by  $(x + 3)(x - 3)$  and simplify:

$$\begin{aligned}(x + 3)(x - 3) \cdot \frac{5}{x^2 - 9} &= (x + 3)(x - 3) \cdot \frac{3}{x - 3} - (x + 3)(x - 3) \cdot \frac{2}{x + 3} \\ 5 &= 3(x + 3) - 2(x - 3)\end{aligned}$$

**Step 3.** Solve the resulting equation.

$$\begin{aligned}5 &= 3(x + 3) - 2(x - 3) \\ 5 &= x + 15 \\ -10 &= x\end{aligned}$$

**Step 4.** Check for any extraneous solution by plugging the solution into the LCD to see if it is zero. If it is zero, then the solution is extraneous.

$$(-10 + 3)(-10 - 3) \neq 0$$

So  $x = -10$  is a valid solution of the original equation.

**Example 13.2** Solve for  $x$  from the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$

**Step 1.** The LCD is  $xyz$ .


**Step 2.** Clear denominators.

$$\begin{aligned}xyz \cdot \frac{1}{x} + xyz \cdot \frac{1}{y} &= xyz \cdot \frac{1}{z} \\ yz + xz &= xy\end{aligned}$$

**Step 3.** Solve the resulting equation.

$$\begin{aligned}yz + xz &= xy \\ yz &= xy - xz \\ yz &= x(y - z) \\ \frac{yz}{y - z} &= x \quad \text{if } y \neq z\end{aligned}$$

**Step 4.** The solution is  $x = \frac{yz}{y - z}$  if  $y \neq z$ . If  $y = z$ , the equation has no solution.

 **Note** Another way to solve a rational equations is to rewrite the equation in the form  $\frac{p(x)}{q(x)} = 0$  using properties of rational expressions, then solve  $p(x) = 0$  and check.

 **Exercise 13.1** Solve.

(1)  $\frac{1}{x+1} + \frac{1}{x-1} = \frac{4}{x^2-1}$

(2)  $\frac{30}{x^2-25} = \frac{3}{x+5} + \frac{2}{x-5}$

13.1 (2)  $x = 7$

13.1 (1)  $x = 2$

 **Exercise 13.2** Solve.


(1)  $\frac{2x-1}{x^2+2x-8} = \frac{1}{x-2} - \frac{2}{x+4}$

(2)  $\frac{3x}{x-5} = \frac{2x}{x+1} - \frac{42}{x^2-4x-5}$

13.2 (2)  $x = -6$  or  $x = -7$

13.2 (1)  $x = 3$




 **Exercise 13.3** Solve a variable from a formula.

(1) Solve for  $f$  from  $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$ .

(2) Solve for  $x$  from  $A = \frac{f + cx}{x}$ .

13.3 (2)  $x = \frac{c - A}{f}$  if  $A \neq c$ .

13.3 (1)  $f = \frac{b + d}{bd}$  if  $b \neq d$ .

 **Exercise 13.4** Solve for  $x$  from the equation.

(1)  $2(x + 1)^{-1} + x^{-1} = 2$ .

(2)  $\frac{a^2x + 2a}{x^{-1}} = -1$ .

13.4 (2)  $x = -\frac{2}{a}$  if  $a \neq 0$ .

13.4 (1)  $x = -\frac{2}{1}$  or  $x = 1$



## Lesson 14 Radical Expressions: Concepts and Properties

**Radical Expressions** If  $b^2 = a$ , then we say that  $b$  is a **square root** of  $a$ . We denote the positive square root of  $a$  as  $\sqrt{a}$ , called the **principal square root**.

We call the function  $f(x) = \sqrt{x}$  a **square root function**. As a real-valued function, the domain of the square root function consists of all real numbers  $x$  such that  $x \geq 0$ , in interval notation, the domain is  $[0, +\infty)$ .

For any real number  $a$ , the expression  $\sqrt{a^2}$  can be simplified as

$$\sqrt{a^2} = |a|.$$

If  $b^3 = a$ , then we say that  $b$  is a **cube root** of  $a$ . The cube root of a real number  $a$  is denoted by  $\sqrt[3]{a}$ .

For any real number  $a$ , the expression  $\sqrt[3]{a^3}$  can be simplified as

$$\sqrt[3]{a^3} = a.$$

In general, if  $b^n = a$ , then we say that  $b$  is an  **$n$ -th root** of  $a$ . If  $n$  is **even**, the **positive  $n$ -th root** of  $a$ , called the **principal  $n$ -th root**, is denoted by  $\sqrt[n]{a}$ . If  $n$  is odd, the  $n$ -th root  $\sqrt[n]{a}$  of  $a$  has the same sign with  $a$ .

In  $\sqrt[n]{a}$ , the symbol  $\sqrt{\phantom{x}}$  is called the **radical sign**,  $a$  is called the **radicand**, and  $n$  is called the **index**.

If  $n$  is even, then the  $n$ -th root of a negative number is not a real number.

For any real number  $a$ , the expression  $\sqrt[n]{a^n}$  can be simplified as

1.  $\sqrt[n]{a^n} = |a|$  if  $n$  is even.
2.  $\sqrt[n]{a^n} = a$  if  $n$  is odd.

A radical is simplified if the radicand has no perfect power factors against the radical.

**Example 14.1** Simplify the radical expression using the definition.

$$(1) \sqrt{4(y-1)^2} \qquad (2) \sqrt[3]{-8x^3y^6}$$

**Solution** (1)  $\sqrt{4(y-1)^2} = \sqrt{[2(y-1)]^2} = 2|y-1|$ . (2)  $\sqrt[3]{-8x^3y^6} = \sqrt[3]{(-2xy^2)^2} = -2xy^2$ .

**Rational Exponents** If  $\sqrt[n]{a}$  is a real number, then we define  $a^{\frac{m}{n}}$  as

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

Rational exponents have the same properties as integral exponents:

1.  $a^m \cdot a^n = a^{m+n}$
2.  $\frac{a^m}{a^n} = a^{m-n}$
3.  $a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}$
4.  $(a^m)^n = a^{mn}$
5.  $(ab)^m = a^m \cdot b^m$
6.  $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

**Example 14.2** Simplify the radical expression or the expression with rational exponents. **Write in radical notation.**

$$(1) \sqrt{x} \sqrt[3]{x^2} \qquad (2) \sqrt[3]{\sqrt{x^3}} \qquad (3) \left(\frac{x^{\frac{1}{2}}}{x^{-\frac{5}{6}}}\right)^{\frac{1}{4}} \qquad (4) \sqrt{\frac{x^{-\frac{1}{2}}y^2}{x^{\frac{3}{2}}}}$$


**Solution**

$$(1) \sqrt{x} \sqrt[3]{x^2} = x^{\frac{1}{2}} x^{\frac{2}{3}} = x^{\frac{1}{2} + \frac{2}{3}} = x^{\frac{7}{6}} = x^{\frac{6}{6}} x^{\frac{1}{6}} = x \sqrt[6]{x}.$$

$$(2) \sqrt[3]{\sqrt{x^3}} = (\sqrt{x^3})^{\frac{1}{3}} = [(x^3)^{\frac{1}{2}}]^{\frac{1}{3}} = x^{3 \cdot \frac{1}{2} \cdot \frac{1}{3}} = x^{\frac{1}{2}} = \sqrt{x}.$$

$$(3) \left(\frac{x^{\frac{1}{2}}}{x^{-\frac{5}{6}}}\right)^{\frac{1}{4}} = (x^{\frac{1}{2}} x^{\frac{5}{6}})^{\frac{1}{4}} = (x^{\frac{1}{2} + \frac{5}{6}})^{\frac{1}{4}} = (x^{\frac{4}{3}})^{\frac{1}{4}} = x^{\frac{1}{3}} = \sqrt[3]{x}.$$

$$(4) \sqrt{\frac{x^{-\frac{1}{2}}y^2}{x^{\frac{3}{2}}}} = \sqrt{\frac{y^2}{x^2}} = \sqrt{\left(\frac{y}{x}\right)^2} = \left|\frac{y}{x}\right|.$$

 **Exercise 14.1** Evaluate the square root. If the square root is not a real number, state so.

(1)  $-\sqrt{\frac{4}{25}}$

(2)  $\sqrt{49} - \sqrt{9}$

(3)  $-\sqrt{-1}$

14.1 (3) not a real number

14.1 (2) 4

14.1 (1)  $-\frac{2}{5}$

 **Exercise 14.2** Simplify the radical expression.

(1)  $\sqrt{(-7x^2)^2}$


(2)  $\sqrt{(x+2)^2}$

(3)  $\sqrt{25x^2y^6}$

14.2 (3)  $|5xy^3|$

14.2 (2)  $|x+2|$

14.2 (1)  $7x^2$

 **Exercise 14.3** Simplify the radical expression.

(1)  $\sqrt[3]{-27x^3}$

(2)  $\sqrt[4]{16x^8}$


(3)  $\sqrt[5]{(2x-1)^5}$

14.3 (3)  $2x - 1$

14.3 (2)  $2x^2$

14.3 (1)  $-3x$



 **Exercise 14.4** Simplify the radical expression. Assume all variables are positive.

(1)  $\sqrt{50}$

(2)  $\sqrt[3]{-8x^2y^3}$

(3)  $\sqrt[5]{32x^{12}y^2z^8}$

14.4 (1)  $5\sqrt{2}$

14.4 (2)  $-\sqrt[3]{2xy}$

14.4 (3)  $2\sqrt[5]{2x^3y^2z^2}$

 **Exercise 14.5** Write the radical expression with rational exponents.

(1)  $\sqrt[3]{(2x)^5}$


(2)  $(\sqrt[5]{3xy})^7$

(3)  $\sqrt[4]{x^2 + 3^3}$

14.5 (1)  $(2x)^{\frac{5}{3}}$

14.5 (2)  $(3xy)^{\frac{7}{5}}$

14.5 (3)  $(x^2 + 3^3)^{\frac{1}{4}}$

 **Exercise 14.6** Write in radical notation and simplify.

(1)  $4^{\frac{3}{2}}$


(2)  $-81^{\frac{3}{4}}$

(3)  $\left(\frac{27}{8}\right)^{-\frac{2}{5}}$

14.6 (1)  $8\sqrt[4]{9}$

14.6 (2)  $-27$

14.6 (3)  $8$

 **Exercise 14.7** Simplify the expression. Write with radical notations. Assume all variables represent nonnegative numbers.

(1)  $\frac{12x^{\frac{1}{2}}}{4x^{\frac{2}{3}}}$


(2)  $(x^{-\frac{3}{5}}y^{\frac{1}{2}})^{\frac{1}{3}}$

(3)  $\left(\frac{x^{\frac{1}{2}}}{x^{-\frac{1}{3}}}\right)^4$

14.7 (3)  $\sqrt[3]{x^{10}}$

14.7 (2)  $\frac{\sqrt[6]{x}}{\sqrt[5]{y}}$

14.7 (1)  $\frac{\sqrt[6]{x}}{3}$

 **Exercise 14.8** Simplify the expression. Write in radical notation. Assume  $x$  is nonnegative.

(1)  $\frac{\sqrt{x}}{\sqrt[3]{x}}$

(2)  $\sqrt{\sqrt[3]{x}}$

(3)  $\sqrt{x^3 \sqrt{x}}$

14.8 (3)  $\sqrt[6]{x^5}$

14.8 (2)  $\sqrt[6]{x}$

14.8 (1)  $\sqrt[6]{x}$





 **Exercise 14.9** Simplify the expression. Write in radical notation. Assume  $x$  is nonnegative.

(1)  $\sqrt[5]{32x^{\frac{1}{3}}}$


(2)  $\left(\frac{\sqrt[4]{9x}}{3}\right)^{-2}$

(3)  $\sqrt{\frac{1}{\sqrt[3]{x^{-2}}}}$

14.9 (3)  $\sqrt[3]{x}$

14.9 (2)  $\frac{x^{\frac{2}{3}}}{3}$

14.9 (1)  $\sqrt[15]{x^2}$

 **Exercise 14.10** Simplify the expression. Write in radical notation. Assume all variables are nonnegative.

(1)  $\sqrt[3]{(-x)^{-2}} \sqrt[2]{x^3}$

(2)  $\left(\frac{8a^{-\frac{5}{2}}b}{a^{\frac{1}{2}}b^{-5}}\right)^{-\frac{2}{3}}$

(3)  $\left(\frac{y^{-\frac{1}{3}}}{\sqrt[3]{x^2}}\right)^{-3}$

14.10 (3)  $x^2 y$

14.10 (2)  $\frac{a^2}{4b^4}$

14.10 (1)  $\sqrt[6]{x^5}$



## Lesson 15 Algebra of Radicals

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**Product and Quotient Rules for Radicals** If  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$  are real numbers, then

$$\sqrt[n]{a} \sqrt[n]{b} = \sqrt[n]{ab}.$$

If  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$  are real numbers and  $b \neq 0$ , then

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

**Example 15.1** Simplify the expression.

(1)  $\sqrt[4]{8xy^4} \sqrt[4]{2x^7y}$ .

(2)  $\frac{\sqrt[5]{96x^9y^3}}{\sqrt[5]{3x^{-1}y}}$ .

**Solution**

(1)  $\sqrt[4]{8xy^4} \sqrt[4]{2x^7y} = \sqrt[4]{(8xy^4) \cdot (2x^7y)} = \sqrt[4]{16x^8y^5} = \sqrt[4]{2^4(x^2)^4y^4 \cdot y} = 2x^2y \sqrt[4]{y}$ .

(2)  $\frac{\sqrt[5]{96x^9y^3}}{\sqrt[5]{3x^{-1}y}} = \sqrt[5]{\frac{96x^9y^3}{3x^{-1}y}} = \sqrt[5]{32x^{10}y^2} = \sqrt[5]{(2x^2)^5 \cdot y^2} = 2x^2 \sqrt[5]{y^2}$ .

**Combining Like Radicals** Two radicals are called *like radicals* if they have the same index and the same radicand. We add or subtract like radicals by combining their coefficients.

**Example 15.2** Simplify the expression.

$$\sqrt{8x^3} - \sqrt{(-2)^2x^4} + \sqrt{2x^5}.$$

**Solution**

$$\sqrt{8x^3} - \sqrt{(-2)^2x^4} + \sqrt{2x^5} = 2x\sqrt{2x} - 2x^2 + x^2\sqrt{2x} = (x^2 + 2x)\sqrt{2x} - 2x^2.$$

**Multiplying Radicals** Multiplying radical expressions with many terms is similar to that multiplying polynomials with many terms.

**Example 15.3** Simplify the expression.

$$(\sqrt{2x} + 2\sqrt{x})(\sqrt{2x} - 3\sqrt{x}).$$

**Solution**

$$\begin{aligned} (\sqrt{2x} + 2\sqrt{x})(\sqrt{2x} - 3\sqrt{x}) &= \sqrt{2x} \cdot \sqrt{2x} - 3\sqrt{x} \cdot \sqrt{2x} + 2\sqrt{x} \cdot \sqrt{2x} - 6\sqrt{x} \cdot \sqrt{x} \\ &= 2x - 3x\sqrt{2} + 2x\sqrt{2} - 6x \\ &= -4x - x\sqrt{2} \\ &= -(4 + \sqrt{2})x. \end{aligned}$$

**Rationalizing Denominators** Rationalizing denominator means rewriting a radical expression into an equivalent expression in which the denominator no longer contains radicals.

**Example 15.4** Rationalize the denominator.

(1)  $\frac{1}{2\sqrt{x^3}}$

(2)  $\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$

**Solution**

(1) In this case, to get rid of the radical in the bottom, we multiply the expression by  $\frac{\sqrt{x}}{\sqrt{x}}$  so that the radicand in the bottom becomes a perfect power.

$$\frac{1}{2\sqrt{x^3}} = \frac{1}{2\sqrt{x^3}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}}{2\sqrt{x^2}\sqrt{x}} = \frac{\sqrt{x}}{2x^2}.$$

(2) In this case, we use the formula  $(a - b)(a + b) = a^2 + b^2$ . Multiply the expression by  $\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$ .

$$\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}} = \frac{(\sqrt{x} + \sqrt{y})^2}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} = \frac{x + y + 2\sqrt{xy}}{x - y}.$$

**Exercise 15.1** Multiply and simplify.

(1)  $\sqrt[3]{4} \sqrt[3]{5}$

(2)  $\sqrt{|x+7|} \sqrt{|x-7|}$

(3)  $\sqrt[3]{(x-y)^{\frac{5}{2}}} \sqrt[3]{(x-y)^{\frac{7}{2}}}$

**Exercise 15.2** Simplify the radical expression. Assume all variables are positive.

(1)  $\sqrt{20xy} \cdot \sqrt{4xy^2}$

(2)  $\sqrt[3]{16} \cdot 5 \sqrt[3]{2}$

(3)  $\sqrt[5]{8x^4y^3z^3} \cdot \sqrt[5]{8xy^4z^8}$

**Exercise 15.3** Divide. Assume all variables are positive. Answers must be simplified.

(1)  $\sqrt{\frac{9x^3}{y^8}}$

(2)  $\sqrt[3]{\frac{32x^4}{x}}$

(3)  $\frac{\sqrt{40x^5}}{\sqrt{2x}}$

(4)  $\frac{\sqrt[3]{24a^6b^4}}{\sqrt[3]{3b}}$


15.3 (4)  $\sqrt[3]{2a^2b}$

15.3 (3)  $\sqrt[3]{2x^2}$

15.3 (2)  $\sqrt[3]{4}$


15.3 (1)  $\sqrt[3]{\frac{8x}{y^4}}$



 **Exercise 15.4** Add or subtract, and simplify. Assume all variables are positive.


- (1)  $5\sqrt{6} + 3\sqrt{6}$                       (2)  $4\sqrt{20} - 2\sqrt{5}$                       (3)  $3\sqrt{32x^2} + 5x\sqrt{8}$

15.4 (1)  $8\sqrt{6}$                       15.4 (2)  $6\sqrt{5}$                       15.4 (3)  $22x\sqrt{2}$

 **Exercise 15.5** Add or subtract, and simplify. Assume all variables are positive

- (1)  $7\sqrt{4x^2} + 2\sqrt{25x} - \sqrt{16x}$                       (2)  $5\sqrt[3]{x^2y} + \sqrt[3]{27x^5y^4}$                       (3)  $3\sqrt{9y^3} - 3y\sqrt{16y} + \sqrt{25y^3}$


15.5 (1)  $14x + 6\sqrt{x}$                       15.5 (2)  $(5 + 3xy)\sqrt[3]{x^2y}$                       15.5 (3)  $2y\sqrt{y}$

 **Exercise 15.6** Multiply and simplify. Assume all variables are positive.

- (1)  $\sqrt{2}(3\sqrt{3} - 2\sqrt{2})$                       (2)  $(\sqrt{5} + \sqrt{7})(3\sqrt{5} - 2\sqrt{7})$                       (3)  $(\sqrt{3} + \sqrt{2})^2$

15.6 (1)  $3\sqrt{6} - 4$                       15.6 (2)  $1 + \sqrt{35}$                       15.6 (3)  $5 + 2\sqrt{6}$




 **Exercise 15.7** Multiply and simplify. Assume all variables are positive.

- (1)  $(\sqrt{6} - \sqrt{5})(\sqrt{6} + \sqrt{5})$       (2)  $(\sqrt{x+1} - 1)(\sqrt{x+1} + 1)$       (3)  $(2\sqrt[3]{x} + 6)(\sqrt[3]{x} + 1)$

 **Exercise 15.8** Simplify the radical expression and rationalize the denominator. Assume all variables are positive.

- (1)  $\sqrt[3]{\frac{2}{25}}$       (2)  $\sqrt{\frac{2x}{7y}}$       (3)  $\frac{\sqrt[3]{x}}{\sqrt[3]{3y^2}}$       (4)  $\frac{3x}{\sqrt[4]{x^3y}}$

 **Exercise 15.9** Simplify the radical expression and rationalize the denominator. Assume all variables are positive.

- (1)  $\frac{6\sqrt{3}}{\sqrt{3}-1}$       (2)  $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$       (3)  $\frac{3+\sqrt{2}}{2+\sqrt{3}}$       (4)  $\frac{2\sqrt{x}}{\sqrt{x}-\sqrt{y}}$



## Lesson 16 Solve Radical Equations

---

**Solve Radical Equations by Taking a Power** The idea to solve a radical equation  $\sqrt[n]{X} = a$  is to first take  $n$ -th power of both sides to get rid of the radical sign, that is  $X = a^n$  and then solve the resulting equation.

**Tips Solve by Reduction.** The goal to solve a single variable equation is to isolate the variable. When an equation involves radical expressions, you can not isolate the variable arithmetically without eliminating the radical sign. To remove a radical sign, you make take a power. However, you'd better to isolate it first. Because simply taking powers of both sides may create new radical expressions.

**Example 16.1** Solve the equation  $x - \sqrt{x+1} = 1$ .

### Solution

**Step 1.** Arrange terms so that one radical is isolated on one side of the equation.

$$x - 1 = \sqrt{x+1}$$

**Step 2.** Square both sides to eliminate the square root.

$$(x - 1)^2 = x + 1$$

**Step 3.** Solve the resulting equation.

$$x^2 - 2x + 1 = x + 1$$

$$x^2 - 3x = 0$$

$$x(x - 3) = 0$$

$$x = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = 0 \quad \text{or} \quad x = 3$$

**Step 4.** Check all proposed solutions. Plug  $x = 0$  into the original equation, we see that the left hand side is  $0 - \sqrt{0+1} = 0 - \sqrt{1} = 0 - 1 = -1$  which is not equal to the right hand side. So  $x = 0$  cannot be a solution. Plug  $x = 3$  into the original equation, we see that the left hand side is  $3 - \sqrt{3+1} = 3 - \sqrt{4} = 3 - 2 = 1$ . So  $x = 3$  is a solution.

**Example 16.2** Solve the equation  $\sqrt{x-1} - \sqrt{x-6} = 1$ .

### Solution

**Step 1.** Isolated one radical.

$$\sqrt{x-1} = \sqrt{x-6} + 1$$

**Step 2.** Square both sides to remove radical sign and then isolate the remaining radical.

$$x - 1 = (x - 6) + 2\sqrt{x-6} + 1$$

$$x - 1 = x - 5 + 2\sqrt{x-6}$$

$$4 = 2\sqrt{x-6}$$

$$2 = \sqrt{x-6}.$$

**Step 3.** Square both sides to remove the radical sign and then solve.

$$\sqrt{x-6} = 2$$

$$x - 6 = 4$$

$$x = 10.$$

Since  $10 - 1 > 0$  and  $10 - 6 > 0$ ,  $x = 10$  is a valid solution. Indeed,

$$\sqrt{10-1} - \sqrt{10-6} = \sqrt{9} - \sqrt{4} = 3 - 2 = 1.$$

**Example 16.3** Solve the equation  $-2\sqrt[3]{x-4} = 6$ .

### Solution

**Step 1.** Isolated the radical.

$$\sqrt[3]{x-4} = -3$$


**Step 2.** Cube both sides to eliminate the cube root and then solve the resulting equation.

$$x - 4 = (-3)^3$$

$$x - 4 = -27$$


$$x = -23$$

The solution is  $x = -23$ .

 **Exercise 16.1** Solve each radical equation.

(1)  $\sqrt{3x + 1} = 4$


(2)  $\sqrt{2x - 1} - 5 = 0$

 **Exercise 16.2** Solve each radical equation.

(1)  $\sqrt{5x + 1} = x + 1$

(2)  $x = \sqrt{3x + 7} - 3$



 **Exercise 16.3** Solve each radical equation.

(1)  $\sqrt{6x + 7} - x = 2$

(2)  $\sqrt{x + 2} + \sqrt{x - 1} = 3$

16.3 (2)  $x = 2$

16.3 (1)  $x = -1$  or  $x = 3$

 **Exercise 16.4** Solve each radical equation.

(1)  $\sqrt{x + 5} - \sqrt{x - 3} = 2$

(2)  $3\sqrt[3]{3x - 1} = 6$

16.4 (2)  $x = 3$

16.4 (1)  $x = 4$






 **Exercise 16.5** Solve each radical equation.

(1)  $(x + 3)^{\frac{1}{2}} = x + 1$

(2)  $2(x - 1)^{\frac{1}{2}} - (x - 1)^{-\frac{1}{2}} = 1$

 **Exercise 16.6** Solve each radical equation.

(1)  $(x - 1)^{\frac{3}{2}} = 8$

(2)  $(x + 1)^{\frac{2}{3}} = 4$



## Lesson 17 Complex Numbers

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**Complex Numbers** The imaginary unit  $\mathbf{i}$  is defined as  $\mathbf{i} = \sqrt{-1}$ . Hence  $\mathbf{i}^2 = -1$ .

If  $b$  is a positive number, then  $\sqrt{-b} = \mathbf{i}\sqrt{b}$ .

Let  $a$  and  $b$  are two real numbers. We define a complex number by the expression  $a + b\mathbf{i}$ . The number  $a$  is called the real part and the number  $b$  is called the imaginary part. If  $b = 0$ , then the complex number  $a + b\mathbf{i} = a$  is just the real number. If  $b \neq 0$ , then we call the complex number  $a + b\mathbf{i}$  an imaginary number. If  $a = 0$  and  $b \neq 0$ , then the complex number  $a + b\mathbf{i} = b\mathbf{i}$  is called a purely imaginary number.

Adding, subtracting, multiplying, dividing or simplifying complex numbers are similar to those for radical expressions. In particular, adding and subtracting become similar to combining like terms.

**Example 17.1** Simplify and write your answer in the form  $a + b\mathbf{i}$ , where  $a$  and  $b$  are real numbers and  $\mathbf{i}$  is the imaginary unit.

(1)  $\sqrt{-3}\sqrt{-4}$       (2)  $(4\mathbf{i} - 3)(-2 + \mathbf{i})$       (3)  $\frac{-2+5\mathbf{i}}{\mathbf{i}}$       (4)  $\frac{1}{1-2\mathbf{i}}$       (5)  $\mathbf{i}^{2018}$

**Solution**

(1)

$$\sqrt{-3}\sqrt{-4} = \mathbf{i}\sqrt{3} \cdot \mathbf{i}\sqrt{4} = \mathbf{i}^2 \cdot \sqrt{3} \cdot 2 = -2\sqrt{3}.$$

(2)

$$\begin{aligned} (4\mathbf{i} - 3)(-2 + \mathbf{i}) &= 4\mathbf{i} \cdot (-2) + 4\mathbf{i} \cdot \mathbf{i} + (-3) \cdot (-2) + (-3) \cdot \mathbf{i} \\ &= -8\mathbf{i} + (-4) + 6 + (-3\mathbf{i}) = 2 - 11\mathbf{i}. \end{aligned}$$

(3)

$$\begin{aligned} \frac{-2 + 5\mathbf{i}}{\mathbf{i}} &= \frac{(-2 + 5\mathbf{i})\mathbf{i}}{\mathbf{i} \cdot \mathbf{i}} = \frac{-2\mathbf{i} + 5\mathbf{i}^2}{\mathbf{i}^2} \\ &= \frac{-2\mathbf{i} - 5}{-1} = 5 + 2\mathbf{i}. \end{aligned}$$

(4)

$$\begin{aligned} \frac{1}{1 - 2\mathbf{i}} &= \frac{(1 + 2\mathbf{i})}{(1 - 2\mathbf{i})(1 + 2\mathbf{i})} = \frac{1 + 2\mathbf{i}}{1 - (2\mathbf{i})^2} \\ &= \frac{1 + 2\mathbf{i}}{5} = \frac{1}{5} + \frac{2}{5}\mathbf{i}. \end{aligned}$$

(5)

$$\mathbf{i}^{2018} = \mathbf{i}^{4 \cdot 504 + 2} = (\mathbf{i}^4)^{504} \cdot \mathbf{i}^2 = -1.$$

**Example 17.2** Evaluate  $f(1 + \mathbf{i})$  for the function  $f(z) = z^2 + \frac{z-1}{z+1}$ . Write your answer in the form  $a + b\mathbf{i}$ .

**Solution**

$$\begin{aligned} f(1 + \mathbf{i}) &= (1 + \mathbf{i})^2 + \frac{\mathbf{i}}{2 + \mathbf{i}} \\ &= 1 + 2\mathbf{i} + \mathbf{i}^2 + \frac{\mathbf{i}(2 - \mathbf{i})}{4 - \mathbf{i}^2} \\ &= 2\mathbf{i} + \frac{1 + 2\mathbf{i}}{5} \\ &= \frac{1}{5} + \frac{12}{5}\mathbf{i}. \end{aligned}$$

 **Exercise 17.1** Add, subtract, multiply complex numbers and write your answer in the form  $a + bi$ .


- (1)  $\sqrt{-2} \cdot \sqrt{-3}$       (2)  $\sqrt{2} \cdot \sqrt{-8}$       (3)  $(5 - 2i) + (3 + 3i)$       (4)  $(2 + 6i) - (12 - 4i)$

17.1 (1)  $\sqrt{6}$       17.1 (2)  $4i$       17.1 (3)  $i + 8$       17.1 (4)  $-10 + 10i$

 **Exercise 17.2** Add, subtract, multiply complex numbers and write your answer in the form  $a + bi$ .

- (1)  $(3 + i)(4 + 5i)$       (2)  $(7 - 2i)(-3 + 6i)$       (3)  $(3 - x\sqrt{-1})(3 + x\sqrt{-1})$       (4)  $(2 + 3i)^2$

17.2 (1)  $7 + 19i$       17.2 (2)  $-9 + 48i$       17.2 (3)  $9 + x^2$       17.2 (4)  $-5 + 12i$

 **Exercise 17.3** Divide the complex number and write your answer in the form  $a + bi$ .

- (1)  $\frac{2i}{1+i}$       (2)  $\frac{5-2i}{3+2i}$       (3)  $\frac{2+3i}{3-i}$       (4)  $\frac{4+7i}{-3i}$

17.3 (1)  $1 + i$       17.3 (2)  $\frac{11}{16} - \frac{13}{16}i$       17.3 (3)  $\frac{10}{3} + \frac{10}{11}i$       17.3 (4)  $-\frac{3}{2} + \frac{3}{4}i$




 **Exercise 17.4** Simplify the expression.


(1)  $(-i)^8$

(2)  $i^{15}$

(3)  $i^{2017}$

(4)  $\frac{1}{i^{2018}}$

 **Exercise 17.5** Evaluate the function  $p(x) = 2x^2 - 3x + 5$  at  $x = 1 - i$ . Write your answer in the form  $a + bi$ .

 **Exercise 17.6** Evaluate the function  $g(x) = ix^2 - x + \frac{2}{x-1}$  at  $x = i - 1$ . Write your answer in the form  $a + bi$ .



## Lesson 18 Complete the Square

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**The Square Root Property** Suppose that  $X^2 = d$ . Then  $X = \sqrt{d}$  or  $X = -\sqrt{d}$ , or simply  $X = \pm\sqrt{d}$ .

**Complete the Square** The square root property provides another ideal to solve a quadratic equation, which is by completing the square. This method is based on the following observations:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2,$$

and more generally, let  $f(x) = ax^2 + bx + c$ , and  $h = -\frac{b}{2a}$ , then

$$ax^2 + bx + c = a(x - h)^2 + f(h) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}.$$

The procedure to rewrite a trinomial as the sum of a perfect square and a constant is called **completing the square**.

**Solving by Completing the Square using**  $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

**Example 18.1** Solve the equation  $x^2 + 2x - 1 = 0$ .

**Solution**

**Step 1.** Isolate the constant.

$$x^2 + 2x = 1$$

**Step 2.** With  $b = 2$ , add  $\left(\frac{2}{2}\right)^2$  to both sides to complete a square for the binomial  $x^2 + bx$ .

$$x^2 + 2x + \left(\frac{2}{2}\right)^2 = 1 + \left(\frac{2}{2}\right)^2$$

$$\left(x + \frac{2}{2}\right)^2 = 1 + 1$$

$$(x + 1)^2 = 2$$

**Step 3.** Solve the resulting equation using the square root property.

$$x + 1 = \sqrt{2} \quad \text{or} \quad x + 1 = -\sqrt{2}$$

$$x = -1 + \sqrt{2} \quad \text{or} \quad x = -1 - \sqrt{2}$$

**Example 18.2** Solve the equation  $-2x^2 + 8x - 9 = 0$ .

**Solution**

**Step 1.** Isolate the constant.

$$-2x^2 + 8x = 9$$

**Step 2.** Divide by  $-2$  to rewrite the equation in  $x^2 + Bx = C$  form

$$x^2 - 4x = -\frac{9}{2}$$

**Step 3.** With  $b = -4$ , add  $\left(\frac{-4}{2}\right)^2 = 4$  to both sides to complete the square for the binomial  $x^2 - 4x$ .


$$x^2 - 4x + 4 = -\frac{9}{2} + 4$$


$$(x - 2)^2 = -\frac{1}{2}$$

**Step 4.** Solve the resulting equation and simplify.

$$x - 2 = \frac{\mathbf{i}}{\sqrt{2}} \quad \text{or} \quad x - 2 = -\frac{\mathbf{i}}{\sqrt{2}}$$


$$x = 2 + \frac{\sqrt{2}}{2}\mathbf{i} \quad \text{or} \quad x = 2 - \frac{\sqrt{2}}{2}\mathbf{i}$$

 **Note** Another way to complete the square is to use the formula  $ax^2 + bx + c = a(x - h)^2 + f(h)$ .

 **Exercise 18.1** Solve the quadratic equation by the square root property.

(1)  $4x^2 = 20$


(2)  $2x^2 - 6 = 0$

 **Exercise 18.2** Solve the quadratic equation by the square root property.

(1)  $(x - 3)^2 = 10$

(2)  $4(x + 1)^2 + 25 = 0$



 **Exercise 18.3** Solve the quadratic equation *by completing the square*.

(1)  $x^2 - 6x + 25 = 0$


(2)  $x^2 + 4x - 3 = 0$

(3)  $x^2 - 3x - 5 = 0$

$\frac{2}{6x^2-8} = x$  (3) 18.3

$\frac{2}{x^2-2} = x$  (2) 18.3

$\frac{2}{x^2+3} = x$  (1) 18.3

 **Exercise 18.4** Solve the quadratic equation *by completing the square*.

(1)  $x^2 + x - 1 = 0$

(2)  $x^2 + 8x + 12 = 0$

(3)  $3x^2 + 6x - 1 = 0$

$\frac{3}{2x^2+1} = x$  (3) 18.4

$\frac{2}{x^2-6} = x$  (2) 18.4

$\frac{2}{x^2+5} = x$  (1) 18.4



## Lesson 19 Quadratic Formula

**The Quadratic Formula** The solutions of a quadratic equation in the *standard form*  $ax^2 + bx + c = 0$  with  $a \neq 0$  are given by *the quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity  $b^2 - 4ac$  is called the *discriminant* of the quadratic equation.

- If  $b^2 - 4ac > 0$ , the equation has two real solutions.
- If  $b^2 - 4ac = 0$ , the equation has one real solution.
- If  $b^2 - 4ac < 0$ , the equation has two imaginary solutions (no real solutions).

**Example 19.1** Determine the type and the number of solutions of the equation  $(x - 1)(x + 2) = -3$ .

### Solution

**Step 1.** Rewrite the equation in the form  $ax^2 + bx + c = 0$ .

$$\begin{aligned}(x - 1)(x + 2) &= -3 \\ x^2 + x + 1 &= 0\end{aligned}$$

**Step 2.** Find the values of  $a$ ,  $b$  and  $c$ .

$$a = 1, b = 1 \text{ and } c = 1.$$

**Step 3.** Find the discriminant  $b^2 - 4ac$ .

$$b^2 - 4ac = 1^2 - 4 \cdot 1 \cdot 1 = -3.$$

The equation has two imaginary solutions.

**Example 19.2** Solve the equation  $2x^2 - 4x + 7 = 0$ .

### Solution

**Step 1.** Find the values of  $a$ ,  $b$  and  $c$ .

$$a = 2, b = -4 \text{ and } c = 7.$$

**Step 2.** Find the discriminant  $b^2 - 4ac$ .

$$b^2 - 4ac = (-4)^2 - 4 \cdot 2 \cdot 7 = 16 - 56 = -40.$$

**Step 3.** Apply the quadratic formula and simplify.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{-40}}{2 \cdot 2} = \frac{4 \pm 2\sqrt{10}i}{4} = 1 \pm \frac{\sqrt{10}}{2}i.$$

**Example 19.3** Find the base and the height of a **triangle** whose base is three inches more than twice its height and whose area is 5 square inches. Round your answer to the nearest tenth of an inch.

### Solution

**Step 1.** We may suppose the height is  $x$  inches. The base can be expressed as  $2x + 3$  inches.

**Step 2.** By the area formula for a triangle, we have an equation.

$$\frac{1}{2}x(2x + 3) = 5.$$

**Step 3.** Rewrite the equation in  $ax^2 + bx + c = 0$  form.

$$\begin{aligned}x(2x + 3) &= 10 \\ 2x^2 + 3x - 10 &= 0.\end{aligned}$$


**Step 4.** By the quadratic formula, we have

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot (-10)}}{2 \cdot 2} = \frac{-3 \pm \sqrt{89}}{4}.$$

Since  $x$  can not be negative,  $x = \frac{-3 + \sqrt{89}}{4} \approx 1.6$  and  $2x + 3 \approx 6.2$ .

The height and base of the triangle are approximately 1.6 inches and 6.2 inches respectively.




 **Exercise 19.1** Determine the number and the type of solutions of the given equation.

(1)  $x^2 + 8x + 3 = 0$

(2)  $3x^2 - 2x + 4 = 0$

(3)  $2x^2 - 4x + 2 = 0$

19.1 (1) Two real solutions      19.1 (2) Two imaginary solutions      19.1 (3) One real solution

 **Exercise 19.2** Solve using the quadratic formula.

(1)  $x^2 + 3x - 7 = 0$

(2)  $2x^2 = -4x + 5$


(3)  $2x^2 = x - 3$

19.2 (3)  $x = \frac{1 \pm \sqrt{23}}{4}$

19.2 (2)  $x = \frac{-2 \pm \sqrt{14}}{2}$

19.2 (1)  $x = \frac{-3 \pm \sqrt{37}}{2}$



 **Exercise 19.3** Solve using the quadratic formula.

(1)  $(x - 1)(x + 2) = 3$


(2)  $2x^2 - x = (x + 2)(x - 2)$

(3)  $\frac{1}{2}x^2 + x = \frac{1}{3}$

$\frac{3}{51 \pm 3} = x$  (3) 3.61


$\frac{2}{51 \pm 1} = x$  (2) 3.61

$\frac{2}{12 \pm 1} = x$  (1) 3.61


 **Exercise 19.4** A **triangle** whose area is 7.5 square meters has a base that is one meter less than triple the height. Find the length of its base and height. Round to the nearest hundredth of a meter.

19.4 The height is  $\frac{6}{1 + \sqrt{181}} \approx 2.41$  meters and the base is  $\frac{2}{-1 + \sqrt{181}} \approx 6.23$  meters.



-  **Exercise 19.5** A **rectangular** garden whose length is 2 feet longer than its width has an area 66 square feet. Find the dimensions of the garden, rounded to the nearest hundredth of a foot.

19.5 The width is  $\sqrt{67} - 1 \approx 7.19$  feet and the length is  $\sqrt{67} + 1 \approx 9.19$  feet.

-  **Exercise 19.6** A 2 hour river cruise goes in a constant speed 16 miles upstream and then back again. The river has a current of 3 miles an hour. What is the boat's speed, rounded to the nearest hundredth of a mile/hour? How long is the journey upstream, rounded to the nearest tenth of a hour?

19.6 The speed is  $8 + \sqrt{73} \approx 16.54$  miles/hour and the upstream journey is  $16 / (5 + \sqrt{73}) \approx 1.2$  hours.

## Lesson 20 Quadratic Functions

**The Graph of a Quadratic Function** The graph of a quadratic function  $f(x) = ax^2 + bx + c$ ,  $a \neq 0$ , is called a parabola.

A quadratic function  $f(x) = ax^2 + bx + c$  can be written in the form  $f(x) = a(x - h)^2 + k$ , where  $h = -\frac{b}{2a}$  and  $k = f(h) = f\left(-\frac{b}{2a}\right)$ .

- The line  $x = h = -\frac{b}{2a}$  is called the axis of symmetry of the parabola.
- The point  $(h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$  is called the vertex of the parabola.

**The Minimum or Maximum of a Quadratic Function** Consider the quadratic function  $f(x) = ax^2 + bx + c$ ,  $a \neq 0$ .

- If  $a > 0$ , then the parabola opens upward and  $f$  has a minimum  $f\left(-\frac{b}{2a}\right)$  at the vertex.
- If  $a < 0$ , then the parabola opens downward and  $f$  has a maximum  $f\left(-\frac{b}{2a}\right)$  at the vertex.

**Intercepts of a Quadratic Function** Consider the quadratic function  $f(x) = ax^2 + bx + c$ ,  $a \neq 0$ .

- The  $y$ -intercept is  $(0, f(0)) = (0, c)$ .
- The  $x$ -intercepts, if exist, are the solutions of the equation  $ax^2 + bx + c = 0$ .

**Example 20.1** Does the function  $f(x) = 2x^2 - 4x - 6$  have a maximum or minimum? Find it.

**Solution**

**Step 1.** Since  $a > 2$ , the function opens upward and has a minimum.

**Step 2.** Find the line of symmetry  $x = -\frac{b}{2a}$ :  $x = \frac{-(-4)}{2 \cdot 2} = 1$ .

**Step 3.** Find the minimum by plugging  $x = 1$  into the function  $f$ . The minimum is

$$f\left(-\frac{b}{2a}\right) = f(1) = 2 - 4 - 6 = -8.$$

**Example 20.2** Consider the function  $f(x) = -x^2 + 3x + 6$ . Find values of  $x$  such that  $f(x) = 2$ .

**Solution**

**Step 1.** Set up the equation for  $x$ .

$$-x^2 + 3x + 6 = 2$$

**Step 2.** Solve the equation  $-x^2 + 3x + 6 = 2$ .

We get  $x = -1$  or  $x = 4$ .

The values of  $x$  such that  $f(x) = 2$  are  $x = -1$  and  $x = 4$ .

**Example 20.3** A quadratic function  $f$  whose the vertex is  $(1, 2)$  has a  $y$ -intercept  $(0, -3)$ . Find the equation that defines the function.

**Solution**

**Step 1.** Write down the general form of  $f$  using only the vertex.


Quadratic functions with the vertex at  $(1, 2)$  are defined by  $y = a(x - 1)^2 + 2$ , where  $a$  is a nonzero real number.

**Step 2.** Determine the unknown  $a$  using the remaining information.

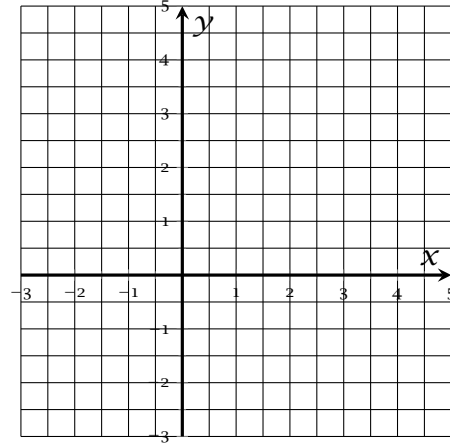
Since  $(0, -3)$  is on the graph of the function, the number  $a$  must satisfy the equation  $-3 = a(0 - 1)^2 + 2$ .

**Step 3.** Solving for  $a$  from the equation, we get  $a = -5$ .


The quadratic function  $f$  is given by  $f(x) = -5(x - 1)^2 + 2$ .

 **Exercise 20.1** Sketch the graph of the quadratic functions  $f(x) = -(x - 2)^2 + 4$  and find

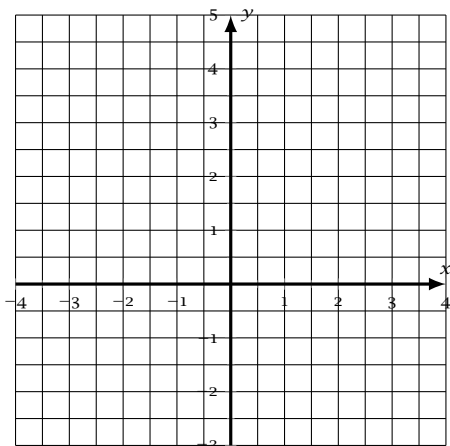
- (1) the coordinates of the  $x$ -intercepts,
- (2) the coordinates of the  $y$ -intercept,
- (3) the equation of the axis of symmetry,
- (4) the coordinates of the vertex.
- (5) the interval of  $x$  values such that  $f(x) \geq 0$ .



20.1 (1) (0, 0) and (4, 0), 20.1 (2) (0, 0), 20.1 (3)  $x = 2$ , 20.1 (4) (2, 4), 20.1 (5) [0, 4].


 **Exercise 20.2** Sketch the graph of the quadratic functions  $f(x) = x^2 + 2x - 3$  and find

- (1) the coordinates of the  $x$ -intercepts,
- (2) the coordinates of the  $y$ -intercept,
- (3) the equation of the axis of symmetry,
- (4) the coordinates of the vertex.
- (5) the interval of  $x$  values such that  $f(x) > 0$ .

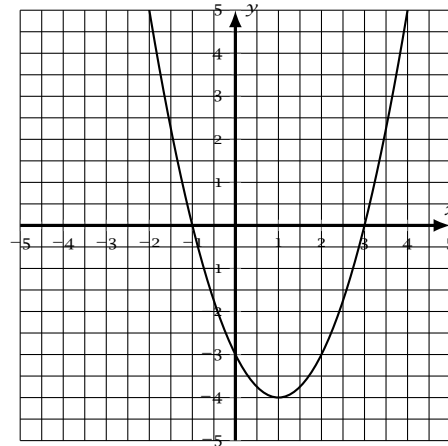


20.2 (1) (-3, 0) and (1, 0), 20.2 (2) (0, -3), 20.2 (3)  $x = -1$ , 20.2 (4) (-1, -4), 20.2 (5)  $(-\infty, -3) \cup (1, \infty)$ .



 **Exercise 20.3** Consider the parabola in the graph.

- (1) Determine the coordinates of the  $x$ -intercepts.
- (2) Determine the coordinates of the  $y$ -intercept.
- (3) Determine the coordinates of the vertex.
- (4) For what values of  $x$  is  $f(x) = -3$ .
- (5) Find an equation for the function.

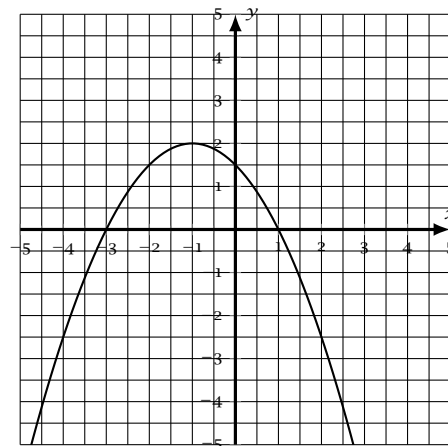


20.3 (5)  $f(x) = (x + 1)(x - 3)$

20.3 (1)  $(-1, 0)$  and  $(3, 0)$ ,  
20.3 (2)  $(0, -4)$ ,  
20.3 (3)  $(1, -4.5)$   
20.3 (4)  $x = 0$  or  $x = 2$

 **Exercise 20.4** Consider the parabola in the graph.


- (1) Determine the coordinates of the  $x$ -intercepts.
- (2) Determine the coordinates of the  $y$ -intercept.
- (3) Determine the coordinates of the vertex.
- (4) For what values of  $x$  is  $f(x) = \frac{3}{2}$ .
- (5) Find an equation for the function.




20.4 (5)  $f(x) = -\frac{1}{2}(x - 1)(x + 3)$

20.4 (1)  $(-3, 0)$  and  $(1, 0)$ ,  
20.4 (2)  $(0, 2)$ ,  
20.4 (3)  $(-1, 2.5)$   
20.4 (4)  $x = -2$  or  $x = 0$



-  **Exercise 20.5** A store owner estimates that by charging  $x$  dollars each for a certain cell phone case, he can sell  $d(x) = 40 - x$  phone cases each week. The revenue in dollars is  $R(x) = xd(x)$  when the selling price of a computer is  $x$ , Find the selling price that will maximize revenue, and then find the amount of the maximum revenue.

20.5 When the selling price is \$20/each, the revenue reaches the maximum \$400.

-  **Exercise 20.6** A ball is thrown upward from the ground with an initial velocity  $v_0$  ft/sec. The height  $h(t)$  of the ball after  $t$  seconds is  $h(t) = -16t^2 + v_0t$ . The ball hits the ground after 4 seconds. Find the maximum height and how long it will take the ball to reach the maximum height.

20.6 After 2 seconds the ball reaches its maximum height 64 feet.

## Lesson 21 Rational and Radical Functions

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**The Domain of a Rational Function** A rational function  $f$  is defined by an equation  $f(x) = \frac{p(x)}{q(x)}$ , where  $p(x)$  and  $q(x)$  are polynomials and the degree of  $q(x)$  is at least one. Since the denominator cannot be zero, the domain of  $f$  consists all real numbers except the numbers such that  $q(x) = 0$

**Example 21.1** Find the domain of the function  $f(x) = \frac{1}{x-1}$ .

**Solution** Solve the equation  $x - 1 = 0$ , we get  $x = 1$ . Then the domain is  $\{x \mid x \neq 1\}$ . In interval notation, the domain is

$$(-\infty, 1) \cup (1, \infty).$$


**The Domain of a Radical Function** A radical function  $f$  is defined by an equation  $f(x) = \sqrt[n]{r(x)}$ , where  $r(x)$  is an algebraic expression. For example  $f(x) = \sqrt{x+1}$ . When  $n$  is odd number,  $r(x)$  can be any real number. When  $n$  is even,  $r(x)$  has to be nonnegative, that is  $r(x) \geq 0$  so that  $f(x)$  is a real number.

**Example 21.2** Find the domain of the function  $f(x) = \sqrt{x+1}$ .

**Solution** Since the index is 2 which is even, the function has real outputs only if the radicand  $x + 1 \geq 0$ . Solve the inequality, we get  $x \geq -1$ . In interval notation, the domain is


$$[-1, \infty).$$



 **Exercise 21.1** Find the domain of each function. Write in interval notation.

- (1)  $f(x) = \frac{x^2}{x-2}$       (2)  $f(x) = \frac{x}{x^2-1}$       (3)  $f(x) = \sqrt{2x-3}$       (4)  $f(x) = \sqrt{x^2+1}$

(1)  $(-\infty, 2) \cup (2, \infty)$  (2)  $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$  (3)  $[\frac{3}{2}, \infty)$  (4)  $(-\infty, \infty)$

 **Exercise 21.2** Find the domain of each function. Write in interval notation.

- (1)  $f(x) = 1 - \frac{2x}{x+3}$       (2)  $f(x) = \frac{x-2}{x^2-4}$       (3)  $f(x) = \sqrt{1-x^2}$       (4)  $f(x) = -\sqrt{\frac{1}{x-5}}$

(1)  $(-\infty, -3) \cup (-3, \infty)$  (2)  $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$  (3)  $[-1, 1]$  (4)  $(5, \infty)$

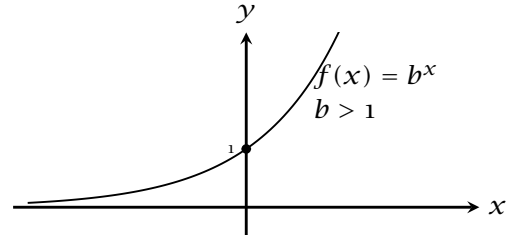
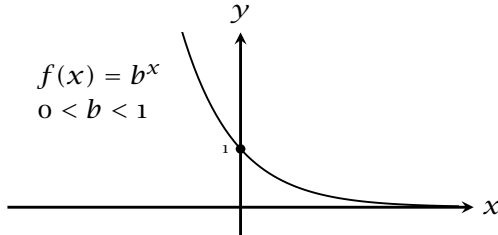


## Lesson 22 Exponential Functions

**Exponential Functions** Let  $b$  be a positive number other than 1 (i.e.  $b > 0$  and  $b \neq 1$ ). The exponential function  $f$  of  $x$  with the base  $b$  is defined as

$$f(x) = b^x \quad \text{or} \quad y = b^x.$$

Graphs of exponential functions:



**Note** The exponential function  $f(x) = b^x$  is an one-to-one function: any vertical line or any horizontal line crosses the graph at most once. Equivalently, the equation  $b^x = c$  has at most one solution for any real number  $c$ .

**The Natural Number  $e$**  The natural number  $e$  is the number to which the quantity  $\left(1 + \frac{1}{n}\right)^n$  approaches as  $n$  takes on increasingly large values. Approximately,  $e \approx 2.718281827$ .

**Compound Interests** After  $t$  years, the balance  $A$  in an account with a principal  $P$  and annual interest rate  $r$  is given by the following formulas:

1. For  $n$  compounding periods per year:  $A = P \left(1 + \frac{r}{n}\right)^{nt}$ .
2. For compounding continuously:  $A = Pe^{rt}$ .

**Example 22.1** A sum of \$10,000 is invested at an annual rate of 8%, Find the balance, to the nearest hundredth dollar, in the account after 5 years if the interest is compounded

- (1) monthly,                      (2) quarterly,                      (3) semiannually,                      (4) continuous.

**Solution**

**Step 1.** Find values of  $P$ ,  $r$ ,  $t$  and  $n$ . In this case,  $P = 10,000$ ,  $r = 8\% = 0.08$ ,  $t = 5$  and  $n$  depends on compounding.

**Step 2.** Plug the values in the formula and calculate.

- (1) “Monthly” means  $n = 12$ . Then

$$A = 10000 \left(1 + \frac{0.08}{12}\right)^{5 \cdot 12} \approx 14898.46.$$

- (2) “Quarterly” means  $n = 4$ . Then

$$A = 10000 \left(1 + \frac{0.08}{4}\right)^{5 \cdot 4} \approx 14859.47.$$

- (3) “semiannually” means  $n = 2$ . Then

$$A = 10000 \left(1 + \frac{0.08}{2}\right)^{5 \cdot 2} \approx 14802.44.$$


- (4) For continuously compounded interest, we have

$$A = 10000e^{0.08 \cdot 5} \approx 14918.25.$$


**Example 22.2** The population of a country was about 0.78 billion in the year 2015, with an annual growth rate of about 0.4%. The predicted population is  $P(t) = 0.78(1.004)^t$  billions after  $t$  years since 2015. To the nearest thousandth of a billion, what will the predicted population of the country be in 2030?

**Solution** The population is approximately

$$P(15) = 0.78(1.004)^{15} \approx 0.828 \text{ billions.}$$

 **Exercise 22.1** The value of a car is depreciating according to the formula:  $V = 25000(3.2)^{-0.05x}$ , where  $x$  is the age of the car in years. Find the value of the car, to the nearest dollar, when it is five years old.

22.1 \$18692

 **Exercise 22.2** A sum of \$20,000 is invested at an annual rate of 5.5%, Find the balance, to the nearest dollar, in the account after 5 years subject to

(1) monthly compounding, (2) continuously compounding.

22.2 (1) \$26314

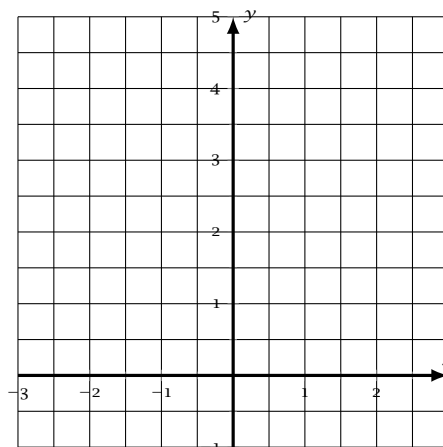
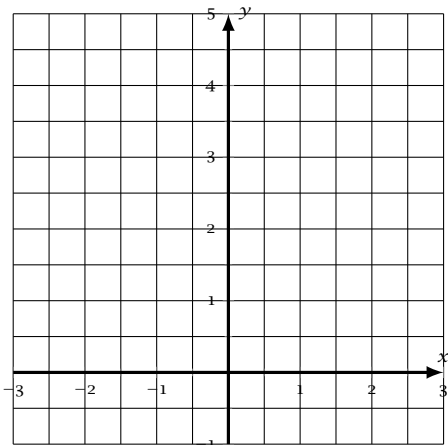
22.2 (2) \$26331




 **Exercise 22.3** Sketch the graph of the function and find its range.

(1)  $f(x) = 3^x$

(2)  $f(x) = \left(\frac{1}{3}\right)^x$




 **Exercise 22.4** Use the given function to compare the values of  $f(-1.05)$ ,  $f(0)$  and  $f(2.4)$  and determine which value is the largest and which value is the smallest. Explain your answer.

(1)  $f(x) = \left(\frac{5}{2}\right)^x$


(2)  $f(x) = \left(\frac{2}{3}\right)^x$



 **Exercise 23.1** Write each equation into equivalent exponential form.

- (1)  $\log_3 7 = y$                       (2)  $3 = \log_b 64$                       (3)  $\log x = y$                       (4)  $\ln(x - 1) = c$

23.1 (1)  $3^y = 7$                       23.1 (2)  $b^3 = 64$                       23.1 (3)  $x^y = x$                       23.1 (4)  $e^{x-1} = c$

 **Exercise 23.2** Write each equation into equivalent logarithmic form.

- (1)  $7^x = 10$                       (2)  $b^5 = 2$                       (3)  $e^{2y-1} = x$                       (4)  $10^x = c^2 + 1$

23.2 (1)  $x = \log_7 10$                       23.2 (2)  $\log_b 2 = 5$                       23.2 (3)  $2y - 1 = \ln x$                       23.2 (4)  $x = \log(c^2 + 1)$

 **Exercise 23.3** Evaluate.

- (1)  $\log_2 16$                       (2)  $\log_9 3$                       (3)  $\log 10$                       (4)  $\ln 1$

23.3 (1) 4                      23.3 (2)  $\frac{2}{1}$                       23.3 (3) 1                      23.3 (4) 0

**Exercise 23.4** Evaluate.

(1)  $e^{\ln 2}$

(2)  $\log 10^{\frac{1}{3}}$

(3)  $\ln(\sqrt{e})$

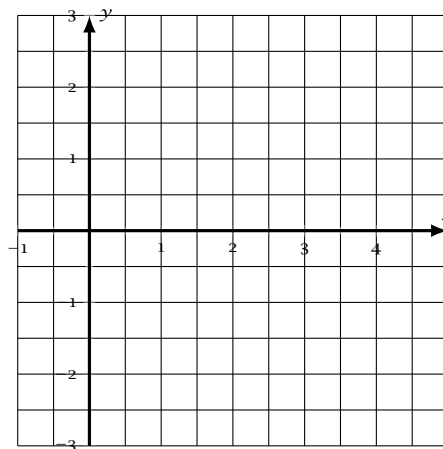
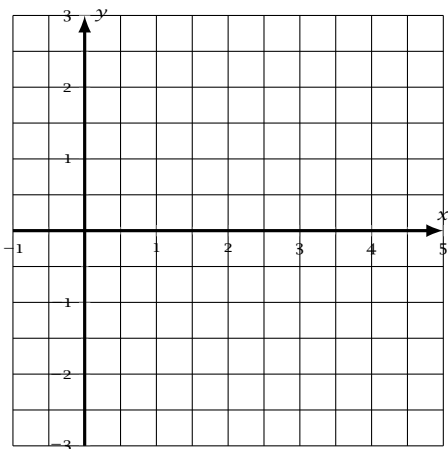
(4)  $\log_2(\frac{1}{2})$

**Exercise 23.5** Find the domain of the function  $f(x) = \log(x - 5)$ . Write in interval notation.

**Exercise 23.6** Sketch the graph of each function and find its range.

(1)  $f(x) = \log_2 x$

(2)  $f(x) = \log_{\frac{1}{2}} x$



## Lesson 24 Properties of Logarithms

**Properties of Logarithms** For  $M > 0$ ,  $N > 0$ ,  $b > 0$  and  $b \neq 1$ , we have

- (The product rule)  $\log_b(MN) = \log_b M + \log_b N$
- (The quotient rule)  $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$ .
- (The power rule)  $\log_b(M^p) = p \log_b M$ , where  $p$  is any real number.
- (The change-of-base property)  $\log_b M = \frac{\log_a M}{\log_a b}$ , where  $a > 0$  and  $a \neq 1$ . In particular,  
$$\log_b M = \frac{\log M}{\log b} \text{ and } \log_b M = \frac{\ln M}{\ln b}.$$

**Example 24.1** Expand and simplify the logarithm  $\log_2\left(\frac{8\sqrt{y}}{x^3}\right)$ .

**Solution**

$$\begin{aligned}\log_2\left(\frac{8\sqrt{y}}{x^3}\right) &= \log_2(8\sqrt{y}) - \log_2(x^3) \\ &= \log_2 8 + \log_2(y^{\frac{1}{2}}) - 3 \log_2 x \\ &= 3 + \frac{1}{2} \log_2 y - 3 \log_2 x.\end{aligned}$$

**Example 24.2** Write the expression  $2 \ln(x - 1) - \ln(x^2 + 1)$  as a single logarithm.

**Solution**

$$2 \ln(x - 1) - \ln(x^2 + 1) = \ln((x - 1)^2) - \ln(x^2 + 1) = \ln\left(\frac{(x - 1)^2}{x^2 + 1}\right).$$

**Example 24.3** Evaluate the logarithm  $\log_3 4$  and round it to the nearest tenth.

**Solution** On most scientific calculator, there are only the common logarithmic function **LOG** and the natural logarithmic function **LN**. To evaluate a logarithm based on a general number, we use the change-of-base property. In this case, the value of  $\log_3 4$  is

$$\log_3 4 = \frac{\log 4}{\log 3} \approx 1.3.$$


**Example 24.4** Simplify the logarithmic expression

$$\log_2(x^{\log 3}) \log_3 2.$$

**Solution**

$$\log_2(x^{\log 3}) \log_3 2 = (\ln 3 \log_2 x) \log_3 2 = \ln 3 \left(\frac{\ln x}{\ln 2}\right) \left(\frac{\ln 2}{\ln 3}\right) = \ln x.$$



 **Exercise 24.1** Expand the logarithm and simplify.


(1)  $\log(100x)$

(2)  $\ln\left(\frac{10}{e^2}\right)$

(3)  $\log_b(\sqrt[3]{x})$

(4)  $\log_7\left(\frac{x^2\sqrt{y}}{z}\right)$

24.1 (1)  $2 + \log x$  (2)  $\ln 10 - 2$  (3)  $\frac{3}{1} \log_b x$  (4)  $2 \log_7 x + \frac{2}{1} \log_7 y - \log_7 z$


 **Exercise 24.2** Expand the logarithm and simplify.

(1)  $\log_b \sqrt{\frac{x^2y}{5}}$

(2)  $\ln(\sqrt[3]{x^2+1}y^{-2})$

(3)  $\log(x\sqrt{10x} - \sqrt{10x})$

24.2 (1)  $\log_b x + \frac{2}{1} \log_b y - \frac{2}{1} \log_b 5$  (2)  $\frac{3}{1} \ln(x^2+1) - 2 \ln y$  (3)  $\frac{2}{1} \log(x) + \log(x)$

 **Exercise 24.3** Write as a single logarithm.


(1)  $\frac{1}{3} \log x + \log y$

(2)  $\frac{1}{2} \ln(x^2+1) - 2 \ln x$

(3)  $\frac{1}{3} \log_2 x - 3 \log_2(x+1) + 1$


24.3 (1)  $\log\left(\sqrt[3]{x}y\right)$  (2)  $\ln\left(\frac{x^2}{\sqrt{x^2+1}}\right)$  (3)  $\log_2\left(\frac{x^2}{\sqrt{x^2+1}}\right)$



 **Exercise 24.4** Write as a single logarithm.


(1)  $2 \log(2x + 1) - \frac{1}{2} \log x$     (2)  $3 \ln x - 5 \ln y + \frac{1}{2} \ln z$     (3)  $3 \log_3 x - 2 \log_3(1 - x) + \frac{1}{3} \log_3(x^2 + 1)$ .

24.4 (1)  $\log \left( \frac{x^{\sqrt{2x+1}}}{z} \right)$     (2)  $\ln \left( \frac{x^5}{z^{\sqrt{3x}}} \right)$     (3)  $\log_3 \left( \frac{z^{(1-x)}}{1+z^{\sqrt{3x}}}$

 **Exercise 24.5** Evaluate the logarithm and round it to the nearest hundredth.

(1)  $\log_2 10$     (2)  $\log_3 5$     (3)  $\frac{1}{\log_5 2}$     (4)  $\log_4 5 - \log_2 9$

24.5 (1) 3.32    (2) 1.46    (3) 2.32    (4) -2.01

 **Exercise 24.6** Simplify the logarithmic expression

$$\frac{\log_3(x^2) \log_y \sqrt{3}}{\log x}$$



## Lesson 25 Exponential and Logarithmic Equations

**Solving Exponential and Logarithmic Equations** To solve an exponential or logarithmic equation, the first step is to rewrite the equation with a single exponentiation or logarithm. Then we can use the equivalent relation between exponentiation and logarithm to rewrite the equation and solve the resulting equation.

**Example 25.1** Solve the equation  $10^{2x-1} - 5 = 0$ .

**Solution**

**Step 1.** Rewrite the equation in the form  $b^u = c$ :

$$10^{2x-1} = 5.$$

**Step 2.** Take logarithm of both sides and simplify:

$$2x - 1 = \log 5.$$

**Step 3.** Solve the resulting equation:

$$x = \frac{1}{2}(\log 5 + 1).$$

**Example 25.2** Solve the equation  $\log_2 x + \log_2(x - 2) = 3$ .

**Solution**

**Step 1.** Rewrite the equation in the form  $\log_b u = c$ :

$$\log_2(x(x - 2)) = 3$$

**Step 2.** Rewrite the equation in the exponential form (moving the base):

$$x(x - 2) = 2^3$$

**Step 3.** Solve the resulting equation  $x^2 - 2x - 8 = 0$ . The solutions are  $x = -2$  and  $x = 4$

**Step 4.** Check proposed solutions. Both  $x$  and  $x - 2$  has to be positive. So  $x = -2$  is not a solution of the original equation. When  $x = 4$ , we have  $\log_2 4 + \log_2 2 = 2 + 1 = 3$ . So  $x = 4$  is a solution.

### Solving Compound Interest Model

**Example 25.3** A check of \$5000 was deposited in a savings account with an annual interest rate 6% which is compounded monthly. How many years will it take for the money to raise by 20%?

**Solution** The question tells us the following information:  $P = 5000$ ,  $r = 0.06$ ,  $n = 12$ , and  $A = 5000 \cdot (1 + 0.2) = 6000$ . What we want to know is the number of years  $t$ . The compound interest model tells us that  $t$  satisfies the following equation:

$$6000 = 5000 \left(1 + \frac{0.06}{12}\right)^{12t}.$$

This is an exponential equation and can be solve using logarithms.

$$5000 \left(1 + \frac{0.06}{12}\right)^{12t} = 6000$$


$$\left(1 + \frac{0.06}{12}\right)^{12t} = 1.2$$


$$12t \cdot \log(1 + 0.06 \div 12) = 1.2$$

$$12t = \log(1.2) \div \log(1 + 0.06 \div 12)$$

$$t = \log(1.2) \div \log(1 + 0.06 \div 12) \div 12 \approx 3.$$

So it takes about 3 years for the savings to raise by 20%.

 **Note** When solving exponential and logarithmic equations, you may also use the one-to-one property if both sides are powers with the same base or logarithms with the same base.


 **Exercise 25.1** Solve the exponential equation.

(1)  $2^{x-1} = 4$

(2)  $7e^{2x} - 5 = 58$

25.1 (2)  $x = \ln 3$

25.1 (1)  $x = 3$

 **Exercise 25.2** Solve the exponential equation.


(1)  $3^{x^2-2x} = e^{-\ln 3}$

(2)  $2^{(x+1)} = 3^{(1-x)}$

25.2 (2)  $x = \frac{\ln 6}{\ln 3 - \ln 2}$

25.2 (1)  $x = 1$




 **Exercise 25.3** Solve the logarithmic equation.

(1)  $\log_5 x + \log_5(4x - 1) = 1$

(2)  $\ln \sqrt{x + 1} = 1$

25.3 (2)  $e^2 = x - 1$

25.3 (1)  $x = \frac{4}{5}$


 **Exercise 25.4** Solve the logarithmic equation.

(1)  $\log_2(x + 2) - \log_2(x - 5) = 3$

(2)  $\log_3(x - 5) = 2 - \log_3(x + 3)$

25.4 (2)  $x = 6$


25.4 (1)  $x = 6$

 **Exercise 25.5** For the given function, find values of  $x$  satisfying the given equation.

(1)  $f(x) = \log_4 x - 2\log_4(x + 1)$ ,  $f(x) = -1$       (2)  $g(x) = \log(2 - 5x) + \log(-x)$ ,  $g(x) = 1$

$$\frac{5}{15^{x-1}} = x \quad (2) \quad 5:57$$

$$1 = x \quad (1) \quad 5:57$$


 **Exercise 25.6** Find intersections of the given pairs of curves.

(1)  $f(x) = e^{x^2}$  and  $g(x) = e^x + 12$ .      (2)  $f(x) = \log_7\left(\frac{1}{2}(x + 2)\right)$  and  $g(x) = 1 - \log_7(x - 3)$


$$5 = x \quad (2) \quad 25.6$$

$$25.6 \quad (1) \quad x = 2 \ln 2$$



 **Exercise 25.7** Using the formula  $A = P(1 + \frac{r}{n})^{nt}$  to determine how many years, to the nearest hundredth, it will take to double an investment \$20,000 at the interest rate 5% compounded monthly.

25.7 14 years

 **Exercise 25.8** Newton's Law of Cooling states that the temperature  $T$  of an object at any time  $t$  satisfying the equation  $T = T_s + (T_0 - T_s)e^{-rt}$ , where  $T_s$  is the the temperature of the surrounding environment,  $T_0$  is the initial temperature of the object, and  $r$  is positive constant characteristic of the system, which is in units of  $\text{time}^{-1}$ . In a room with a temperature of  $22^\circ\text{C}$ , a cup of tea of  $97^\circ\text{C}$  was freshly brewed. Suppose that  $r = \ln 5/20 \text{ minute}^{-1}$ . In how many minutes, the temperature of the tea will be  $37^\circ\text{C}$ ?

25.8 20 minutes

