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# A Hybrid Piecewise-Linear/Newton Approach To Solve Multi-Period Production And Distribution Planning Problems In Large Water Supply Networks

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# COMBINED PIECEWISE-LINEAR APPROXIMATIONS AND NEWTON'S METHOD TO SOLVE DAILY DRINKING WATER PRODUCTION AND DISTRIBUTION PLANNING PROBLEMS

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We consider the problem of optimizing daily production and distribution of drinking water in a large network. As already shown in a previous research work [8] the model is nonlinear due to pressure loss restrictions. This paper discusses an approach that integrates the previously developed piecewise linearization method with Newton's method to determine close to optimal feasible solutions. We first review the mixed integer program for the problem, then we describe the proposed algorithm for its solution and show its results on some real-world water networks.

## INTRODUCTION

This paper tackles the problem of efficient production and distribution in an urban drinking water supply network. The problem has been previously modeled in [7] as an MINLP (Mixed integer nonlinear program) and then reformulated as a mixed integer linear program (MILP) in [8] via piecewise linear approximation.

A common way to plan daily production and distribution in modern water supply companies is the use of simulation software such as EPANET [3]. These software solutions seek to iteratively update the values of flows and pressures in the network by using some gradient based method. An example of such methods is the Todini-Pilati implementation of Newton's iterative method [6], which is described in detail in part . Recently, this method has been improved [4]. The authors take into account the Reynold's number for the friction factor which restores quadratic convergence for the Darcy-Weisbach headloss formula (used in our model).

Unlike simulation, optimization approaches for water supply production and distribution usually assist drinking water companies to wisely use their water resources, both economically and ecologically. Undeniably, optimal operating solutions for water supply networks yield lower production costs and more efficient usage of network pumps. Unfortunately, not much research has been carried out in this field yet. The earliest papers proposed, for this complicated MINLP problem, some LP formulations, which oversimplify the actual situation and are therefore not of much use in practice [2, 5]. Later, more time-consuming nonlinear models were proposed, however without guaranteeing optimality of the solution. An example of a successful model is described in [1], which was implemented to manage and control Berlin's water supply network.

In this paper, we combine the method proposed in [8] with Newton's method to find good feasible solutions for the optimization problem. The proposed algorithm is detailed in the sections below.

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## REVIEW OF THE MATHEMATICAL PROGRAMMING MODEL

In this model, a division in wider intervals is made based on a typical hourly demand pattern and the different day and night tariffs for electricity. Furthermore, these intervals are denoted by $t \in [1, T]$, where $T$ represents the number of periods composing the day. We denote the length of period $t$ by $\tau_t$ (hours).

### Model sets, parameters and variables

To describe the water network, we use $\mathcal{J}$ to denote the junctions, $\mathcal{D}$ for delivery nodes, $\mathcal{B}$ for buffers, $\mathcal{S}$ for raw water sources, and the set $\mathcal{N} = \mathcal{J} \cup \mathcal{D} \cup \mathcal{B} \cup \mathcal{S}$ representing the nodes of the network. The set of arcs ($\mathcal{A} = \mathcal{P}i \cup \mathcal{P}u \cup \mathcal{R}$) is composed of $\mathcal{P}i$ denoting the set of pipes, $\mathcal{P}u$ (pure water pumps), and $\mathcal{R}$ (raw water pumps).

The additional parameters of the problem (with their respective units) are: $d_{it}$, the demand at node $i \in N$ (in m$^3$/h); $h_i$, the geographic height (in m) at node $i \in N$; $l_{it}^{min}$ and $l_{it}^{max}$, the minimum and maximum delivery in delivery nodes (in m$^3$/day); $h_{it}^{min}$ and $h_{it}^{max}$, the minimal and maximal piezometric pressure in delivery nodes (in m); $p_i$, the price of water in delivery nodes (€/m$^3$); $A_i$, the cross-sectional area of tank $i \in \mathcal{B}$ (in m$^2$); $l_i^{min}$ and $l_i^{max}$, minimum and maximum level in tank $i \in \mathcal{B}$ (in m); $h_i^{in}$, the level of inflow in tank $i \in \mathcal{B}$ (in m); $h_i^{fl}$, level of tank floor $i \in \mathcal{B}$ (in m); $q_{ij}^{cap}$, production capacity in a WPC (m$^3$/h); $q_{ij}^{lim}$, daily extraction limit in a WPC (m$^3$/h); $f_{ij}$, max fluctuation of production (m$^3$/h); $c_{ij}(p)$, production cost in a water production center (WPC) (€/m$^3$); $c_t(e)$ is the electricity cost in period $t$ (€/kWh); $k_{ij}$, roughness coefficient of pipe $(i, j)$ (in m); $\delta_{ij}$, pipe diameter (m); $l_{ij}$, pipe length (m); $v_{max}^{ij}$, maximum speed in a pipe (m/s); $h_{ij}^1, h_{ij}^2, h_{ij}^3$, head coefficients (varies); $e_{ij}^1, e_{ij}^2, e_{ij}^3$, efficiency coefficients (varies); $p_{ij}^1, p_{ij}^2$, power coefficients (varies); $q_{ij}^{min}$, pump minimum flow (m$^3$/h); $g$, gravity constant (m/s$^2$). Note that with the exception of the number of periods and the tank's cross-sectional area ($A$), all parameters are denoted by lower-case letters.

The variables of the model are $Q_{ijt}$, the flow in pipe $(i, j)$ in period $t$ (m$^3$/h); $H_{it}$, the piezometric head (m); $I_{it}^+$, the inflow at the entrance of buffers (m$^3$/h); $I_{it}^-$, the outflow at the entrance of buffers (m$^3$/h); $O_{it}$, the outflow at the exit of buffers (m$^3$/h); $V_{it}$, the volume in the tank at the end of period t (m$^3$); $L_{it}$, the level of the tank at the end of period t (m); $H_{it}^M$, Mean piezometric level of tank in period t (m); $X_{it}$, the binary variable for inflow at a buffer (-); $Y_{it}$, the binary variable for outflow at a buffer(-); $\Delta H_{ijt}$, the pump head increase (m); $Z_{ijt}$, the binary activity status of a pump (-). The piezometric head is given as the sum of the geometric height $h$ and the manometric water pressure $p/\gamma$.

The most important decision variables are the amount of water to be produced in the water production centers ($Q_{ijt}$), the activity status of a pump $Z_{ijt}$ (on/off) and the head to be delivered by each pump ($\Delta H_{ijt}$). Knowledge of the value of these variables will allow operators to optimally control the network. All other variables are thus dependent on these two decision variables. Note that variables are symbolized with a capital letter.

**Model formulation**

$$0 \leq (H_{it} - h_i) \leq 100 \quad \forall i \in \mathcal{N} \tag{1}$$

$$\sum_{k:(k,i)\in\mathcal{A}} Q_{kit} - \sum_{j:(i,j)\in\mathcal{A}} Q_{ijt} = d_{it}, \quad \forall i \in \mathcal{J} \tag{2}$$

$$l_{it}^{min} \leq \sum_{k:(k,i)\in\mathcal{A}} Q_{kit} - \sum_{j:(i,j)\in\mathcal{A}} Q_{ijt} \leq l_{it}^{max}, \quad \forall i \in \mathcal{D} \tag{3}$$

$$h_{it}^{min} \leq H_{it} \leq h_{it}^{max}, \quad \forall i \in \mathcal{D} \tag{4}$$

$$I_{it}^{+} \leq A_i\, l_i^{max} X_{it}, \quad \forall i \in \mathcal{B} \tag{5}$$

$$I_{it}^{-} \leq A_i\, l_i^{max} Y_{it}, \quad \forall i \in \mathcal{B} \tag{6}$$

$$H_{it} - h_i^{in} \geq (h_i - h_i^{in})\,(1 - X_{it}), \quad \forall i \in \mathcal{B} \tag{7}$$

$$H_{it}^{M} - H_{it} \geq (h_i^{fl} - 100 - h_i)\,(1 - Y_{it}), \quad \forall i \in \mathcal{B} \tag{8}$$

$$X_{it} + Y_{it} \leq 1 \tag{9}$$

$$\sum_{k:(k,i)\in\mathcal{A}} Q_{kit} - \sum_{j:(i,j)\in\mathcal{A}\setminus\mathcal{P}u} Q_{ijt} = I_{it}^{+} - I_{it}^{-} + d_{it}, \quad \forall i \in \mathcal{B} \tag{10}$$

$$\sum_{j:(i,j)\in\mathcal{P}u} Q_{ijt} = O_{it}^{-}, \quad \forall i \in \mathcal{B} \tag{11}$$

$$V_{it} = V_{i,t-1} + (I_{it}^{+} - I_{it}^{-} - O_{it})\tau_t, \quad \forall i \in \mathcal{B} \tag{12}$$

$$V_{i0} \leq V_{iT}, \quad \forall i \in \mathcal{B} \tag{13}$$

$$H_{it}^{M} = h_i^{fl} + \frac{L_{it} + L_{i,t-1}}{2}, \quad \forall i \in \mathcal{B} \tag{14}$$

where $L_{it} = \dfrac{V_{it}}{A_i}$ is the water level in the tank.

$$l_i^{min} \leq L_{it} \leq l_i^{max}, \quad \forall i \in \mathcal{B} \tag{15}$$

$$-3600\frac{\pi}{4} v_{ij}^{max} (d_{ij})^2 \leq Q_{ijt} \leq 3600\frac{\pi}{4} v_{ij}^{max} (d_{ij})^2 (i,j), \quad \forall (i,j) \in \mathcal{P}i \tag{16}$$

$$H_{it} - H_{jt} = \kappa_{ij}\, Q_{ijt} |Q_{ijt}|, \quad \forall (i,j) \in \mathcal{P}i \tag{17}$$

$$Z_{ijt}\, q_{ij}(min) \leq Q_{ijt} \leq Z_{ijt}\, 3000(i,j), \quad \forall (i,j) \in \mathcal{P}u \tag{18}$$

$$H_{it} - H_{jt} - \kappa_{ij}\,(Q_{ijt})^2 + \Delta H_{ijt} = 0 \quad \forall (i,j) \in \mathcal{P}u : i \in \mathcal{N}\setminus\mathcal{B} \tag{19}$$

$$H_{it}^{M} - H_{jt} - \kappa_{ij}\,(Q_{ijt})^2 + \Delta H_{ijt} = 0 \quad \forall (i,j) \in \mathcal{P}u : i \in \mathcal{B} \tag{20}$$

$$\Delta H_{ijt} = h_{ij}^{1}\,(Q_{ijt})^2 + h_{ij}^{2}\, Q_{ijt} + h_{ij}^{3}\, Z_{ijt}, \quad \forall (i,j) \in \mathcal{P}u \tag{21}$$

$$P_{ijt} = p^1_{ij} Q_{ijt} + p^2_{ij} Z_{ijt}, \quad \forall (i,j) \in \mathcal{P}u \tag{22}$$

$$0 \leq Q_{ijt} \leq q^{cap}_{ij}(i,j), \quad \forall (i,j) \in \mathcal{R} \tag{23}$$

$$-f_{ij}\, q^{cap}_{ij} \leq Q_{ijt} - Q_{ij,t-1} \leq f_{ij}\, q^{cap}_{ij}, \quad \forall (i,j) \in \mathcal{R} \tag{24}$$

$$-2\, f_{ij}\, q^{cap}_{ij} \leq Q_{ijT} - Q_{ij,0} \leq 2\, f_{ij}\, q^{cap}_{ij}, \quad \forall (i,j) \in \mathcal{R} \tag{25}$$

$$\sum_{t=1}^{T} Q_{ijt} \leq q^{lim}_{ij}, \quad \forall (i,j) \in \mathcal{R} \tag{26}$$

### Goal function

The function to be minimized is the total cost, consisting of energy delivered by the pumps, production/electricity at the water production centers and the delivery cost:

$$\text{Minimize} \sum_{t=1}^{T} \left[ \sum_{(i,j)\in\mathcal{P}u} (p^1_{ij} Q_{ijt} + p^2_{ij}) \frac{c_t(e)}{1000} + \sum_{(i,j)\in\mathcal{R}} Q_{ijt}\, c_{ij}(p) + \sum_{i\in\mathcal{D}} \left( \sum_{j:(i,j)\in\mathcal{A}} Q_{ijt} - \sum_{k:(k,i) in \mathcal{A}} Q_{kit} \right) p_i \right] \tau_t \tag{27}$$

## COMBINED PWL APPROXIMATION AND NEWTON'S METHOD

This section describes the proposed algorithm to determine a close to optimal feasible solution to the mathematical programming problem described in section 2. The algorithm consists of two inner procedures: a piecewise linearization step (described in detail in part ) and the gradient method (Newton's method, detailed in part ).

### Major steps of the general algorithm

In a first step, the quadratic constraints (17), (19)-(21) are iteratively linearized using $2^n$ segments, starting with $n = 1$. The resulting linear mixed-integer problem is solved using a commercial MILP code (Gurobi). With this approximate solution, we fix the values of certain variables, drop the sign restrictions and then start Newton's method to determine a distribution solution for the first period. If we succeed in finding a feasible solution for the first period, the Newton's method then applied to the next period. We repeat this process until we generate a feasible solution for the whole planning horizon. Finally, a reparation step is added to fulfill the production requirements between the different periods. If at any time it is impossible to generate a feasible solution, we set $n := n+1$ and we restart the procedure all over again. The steps of the approach are summarized in algorithm 1.

**Note:** In the second step we fix the integers because Newton's method is a linear algorithm. It is possible that during one or more iterations the values of some variables assume values outside their bounds. Even if they would eventually converge to feasible values, this causes the algorithm to stop. Therefore we drop the bound restrictions during this phase. However, if during any iteration we have reverse flow in a pump, we set the binary activity status of this pump, $B_P$, to 0. In the same fashion $B_P = 1$ if the flow is positive. This way we expect to accelerate the convergence process.

**Algorithm 1** (Combined piecewise-linear approximations and Newton's method)

**Step 0. (Initialization):**
Let $n = 1, t_0 = 1$

**Step 1. (Piecewise-Linear Approximation):**
Given $n$, linearize the quadratic constraints (17), (19)-(21) and solve the resulting linear mixed-integer problem (MILP) to obtain solution $(Q^n_{LP}, H^n_{LP})$ on the horizon $\{1,...,T\}$.

**Step 2. (Newton's Method):**
For period $t_0$, drop restrictions at each $WPC$ for available capacity, and for each buffer $i \in \mathcal{B}$ fix the volume $V_{it}$ and the inflow binary variables $X$ and $Y$. Furthermore, fix the binary activity status $Z_{it}$ of pumps and drop sign restrictions on flow rates. With this updated information, solve the water distribution problem for $t_0$ using Newton's procedure. If a feasible solution is obtained, then let $t_0 := t_0 + 1$ and repeat step 2, otherwise let $t_0 := 1, n := n+1$ and return to step 1. Here, a solution is considered feasible if (1), (4), (16), (18) and all sign restrictions on the variables hold.

**Step 3. (Reparation step):**
Since we relaxed the transient conditions (23 - 26), we should now check periods where the production capacity was exceeded for each $(i, j) \in \mathcal{R}$, starting from $t_1 = T$ .
Iteration:
- if $(Q_{ijt_1} > q^{cap}_{ij})$ then (capacity exceeded)
let $V_{j,t_1-1} := V_{j,t_1-1} + \tau_{t_1} * (Q_{ijt_1} - q^{cap}_{ij})$; (increase volume at end of previous period)
let $Q_{ij,t_1-1} := Q_{ijt_1} + \tau_{t_1} * (Q_{ijt_1} - q^{cap}_{ij})/\tau_{t_1-1}$;(increase production in previous period to fill buffer)
let $Q_{ijt_1} := q^{cap}_{ij}$; (set production equal to capacity)
- if $(Q_{ijt_1} > Q_{ij,t_1+1} + k\, f_{ij}\, q^{cap}_{ij})$ then (quality issues)
let $V_{j,t_1-1} := V_{j,t_1-1} + \tau_{t_1} * (Q_{ijt_1} - Q_{ij,t_1+1} - k\, f_{ij}\, q^{cap}_{ij})$; (increase volume at end of previous period)
let $Q_{ij,t_1-1} := Q_{ij,t_1-1} + \tau_{t_1} * (Q_{ijt_1} - Q_{ij,t_1+1} - k\, f_{ij}\, q^{cap}_{ij})/\tau_{t_1-1}$; (increase production in previous period to fill buffer)
let $Q_{ijt_1} := Q_{ijt_1+1} + k\, f_{ij}\, q^{cap}_{ij}$;(set production equal to limit with regard to next period)
- if $(Q_{ijt_1} > Q_{ij,t_1-1} + k\, f_{ij}\, q^{cap}_{ij})$ then (quality issues)
let $V_{j,t_1-1} := V_{j,t_1-1} + (\tau_{t_1} + \tau_{t_1-1}) * (Q_{ijt_1} - Q_{ij,t_1-1} - k\, f_{ij}\, q^{cap}_{ij})/2$; (increase volume at end of previous period)
let $Q_{ij,t_1-1} := Q_{ij,t_1-1} + (Q_{ijt_1} - Q_{ij,t_1-1} - k\, f_{ij}\, q^{cap}_{ij})/2$; (increase production in previous period)
let $Q_{ijt_1} := Q_{ijt_1} - (Q_{ijt_1} - Q_{ij,t_1-1} - k\, f_{ij}\, q^{cap}_{ij})/2$; (decrease production in current period), where $k = 2$ if $t_1 \in \{0, T\}$, 1 otherwise.
Note that $T + 1$ and 1, resp. $T$ and 0 denote the same period. Also note that we always adjust flow and volume in previous period, working from T –> 1 in this step. It is possible that the initial (and thus final) value of volume in the buffer is adjusted in the process.
If $l^{min}_i \leq L_{it} \leq l^{max}_i, \quad \forall i \in \mathcal{B}$ let $t_1 := t_1 - 1$. Otherwise let $n := n+1$ and return to step 1.

**Step 4. (Stopping rule):**
If $t_0$ is the last period and $t_1$ the first one then stop, we have a feasible solution.

#### A piecewise linear approximation based solution approach

For a more detailed description of this approach, we refer the reader to [8]. The method in this paper approximates the equations for pressure losses in pipes and pumps (17, 19-21) by dividing the axis $Q$ in intervals and linearize the functions over $Q$. The pressure losses are simultaneously over- and underestimated by linear functions (see figure 1).

The formulation that is chosen here requires only a logarithmic amount of binary variables ([9, 10]). This encourages the use of $2^n$ intervals. For each arc, doubling the number of intervals reduces the approximation error by a factor 4 but increases the number of binary variables by just 1.

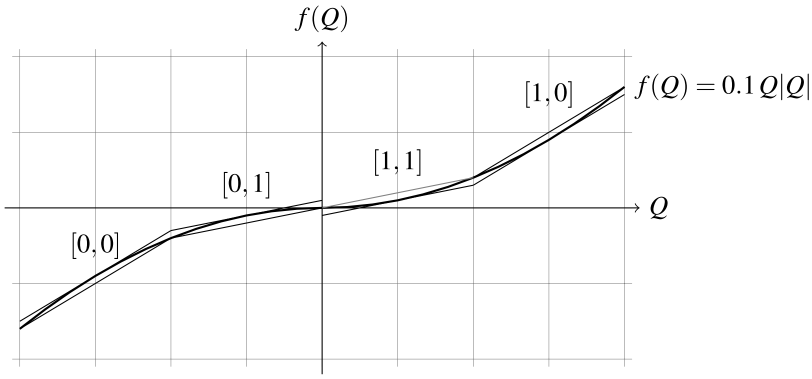


Figure 1: Example division in intervals with logarithmic number of binary variables

For this paper, the model PWL-2 (standard model with logarithmic number of binaries for intervals, see [8]) was used.

#### Newton's method

This section describes the Todini-Pilati implementation of true Newton method [6]. Basically this is the original Newton's method adapted to pipe networks. It is also implemented in EPANET [3].

It is assumed that the demands in the network are known. Furthermore, sources have unlimited capacities.

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**Algorithm 2** (Newton's method)

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**Initialization:**

Initial configuration $q^0$(vectors of $Q$), Nonlinear equation $f(x) = 0$.

**Iteration** $m$**:**

Update $q^m$ through Newton's iterative method

$f'(x^{(m)})(x^{(m+1)} - x^{(m)}) = -f(x^{(m)}), \quad x^{(0)} prescribed, m = 0, 1, 2, \ldots$

**Until:**

$\phi(q^{(m+1)}) = \sum_{(i,j)\in\mathcal{A},t\in 1..T} |Q_{ijt}^{(m+1)} - Q_{ijt}^{(m)}| / \sum_{(i,j)\in\mathcal{A},t\in 1..T} |Q_{ijt}^{(m+1)}| \leq \delta_{\text{stop}}$

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The nonlinear functions in the system are the pressure losses and pump constraints. For pipes, we easily find $f'(x^{(m)}) = -2\kappa_{ij}\,|Q^{(m)}_{ijt}|$, such that the linearized constraint becomes:

$$-2\kappa_{ij}\,|Q^{(m)}_{ijt}|Q^{(m+1)}_{ijt} = -H_{it} + H_{jt} - \kappa_{ij}\,Q^{(m)}_{ijt}|Q^{(m)}_{ijt}|$$

For pumps, we can derive these equations in a similar fashion. Together with additional constraints, the system of equations can then be solved until convergence is achieved.

## PRELIMINARY RESULTS

Tests were conducted on the exact same test network that is described in [8], that consists of 9 buffers (5 of which are water towers (WT) and the other 4 pure water reservoirs (R)), three raw water sources (WS), two delivery nodes (D) and 13 junctions (J). There is a total of 30 arcs: 3 raw water pumps (one for each WPC), 5 regular speed pumps and 22 pipes.

When we run the algorithm for $n = 1$, we manage to get a solution in step 2 despite the very rough approximation. Unfortunately, pump R1-WPC1 produces only 18,85 $m^3/h$, which is lower than the minimum requirement of 25 $m^3/h$. Therefore, we set $n = 2$. In the second iteration of Newton's method, the algorithm unfortunately does not converge. During some periods, the flow in certain pumps fluctuates between positive and negative values. We are still investigating how we can prevent this from happening. For now, we set $n = 3$ (correpsonding to 8 intervals) and find a solution that fulfills all requirements in step 2 after 4 iterations of Newton's method. A reparation step is needed however, since the capacity in the third water production center is exceeded in periods 4 and 5, while there are some quality issues in between periods in other WPC's. In node R2, there is a minor increase in initial volume in the buffer from 16000 to 16007,6. The final objective value is 4203.773, which is the best optimal value found to date on this test problem. Table  shows a summary of the results. We find a solution after 43s. with 8 intervals. The time it takes to run Newton's method is negligible and therefore not mentioned.

Table 1: Comparison of number of variables, constraints and computation time for different formulations

| | | # Intervals | | |
|---|---|---|---|---|
| | | n = 2 | n = 4 | n = 8 |
| PWL-2 | # variables | 1268 | 1673 | 2348 |
| | # constraints | 2122 | 2392 | 2662 |
| | time | 1.14 | 26.47 | 15.15 |
| | goal | 3812.07 | 4067.93 | 4194.07 |
| | error | 26.42 | 6.60 | 1.65 |
| | PWL+Newton | 4199.75* | - | 4203.77 |

*infeasible

## CONCLUSION

Based on previous research [8] we extend a piecewise linear approximation algorithm with Newton's method to obtain good feasible solutions for the daily production and distribution planning in a large drinking water network. Preliminary results show that such solutions can be obtained in short computation time, although criteria for convergence need to be defined in future research.

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