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M. Riesz Theorem on Conjugate Harmonic Functions for Octonion-Valued Monogenic Functions

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Abstract. The classical theorem of M. Riesz about the conjugate harmonic functions is extended onto octonion-valued monogenic functions.

1. Introduction and Statement of the Main Result

Analytic (monogenic) functions of octonion variables, due to their potential, yet not completely fulfilled utility in physics [9], continue to attract the attention of researchers. John C. Baez in his very readable and informative survey formulates the development of an octonionic analogue of the theory of analytic functions as the first item in his list of potential octonion-related problems [3, p. 201]. In this note we continue the study of this topic, see, e.g., [5, 10, 11, 13, 14] and the references therein. For the reader's convenience, in this section we briefly review some basic definitions and then state the principal result, extending the celebrated theorem of M. Riesz on the conjugate harmonic functions onto the monogenic functions. A quaternionic version of this theorem was recently proved by Avetisyan [2]. Let $\sum_{j=0}^7 x_j e_j$ be a generic octonion, where $e_0 \equiv 1$, and e_1, \dots, e_7 are the basis octonion (imaginary) units; we identify it with a vector $\mathbf{x} = (x_0, \dots, x_7) \in \mathbf{R}^8$. In notation we follow [5]. Let

$$f(\mathbf{x}) = \sum_{j=0}^7 e_j f_j(\mathbf{x})$$

be an octonion-valued left-monogenic function in a domain $\Omega \subset \mathbf{R}^8$, where $f_0(\mathbf{x}), \dots, f_7(\mathbf{x})$ are real-valued \mathcal{C}^1 functions. That means

$$D[f] = 0 \tag{1.1}$$

where $D = \sum_{k=0}^7 e_k \frac{\partial}{\partial x_k}$ is the Dirac (or Cauchy-Riemann) operator. It is known that all the components f_0, \dots, f_7 of a left-monogenic function are harmonic functions, that is,



$\Delta f_0 = \dots = \Delta f_7 = 0$ in Ω . The equation $Df = 0$ is a system of eight first-order linear partial differential equations with constant coefficients. It can be written as a matrix equation

$$\begin{bmatrix} \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_4} & -\frac{\partial}{\partial x_5} & -\frac{\partial}{\partial x_6} & -\frac{\partial}{\partial x_7} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_7} & \frac{\partial}{\partial x_6} & -\frac{\partial}{\partial x_5} & \frac{\partial}{\partial x_4} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_6} & -\frac{\partial}{\partial x_7} & \frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_5} \\ \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} & \frac{\partial}{\partial x_5} & -\frac{\partial}{\partial x_4} & -\frac{\partial}{\partial x_7} & \frac{\partial}{\partial x_6} \\ \frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_7} & \frac{\partial}{\partial x_6} & -\frac{\partial}{\partial x_5} & \frac{\partial}{\partial x_0} & \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_5} & -\frac{\partial}{\partial x_6} & \frac{\partial}{\partial x_7} & \frac{\partial}{\partial x_4} & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0} & \frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_6} & \frac{\partial}{\partial x_5} & -\frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_7} & \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_7} & -\frac{\partial}{\partial x_4} & -\frac{\partial}{\partial x_5} & -\frac{\partial}{\partial x_6} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0} \end{bmatrix} F = 0,$$

where $F = [f_0, \dots, f_7]^T$ is the unknown column vector-function. It can also be rewritten as a matrix equation

$$\sum_{j=0}^7 A_j \frac{\partial F}{\partial x_j} = 0.$$

Here A_0 is the identity matrix of order 8 and the other seven 8×8 antisymmetric matrices $A_1 - A_7$ are given by

$$A_1 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned}
 A_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 A_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 A_6 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 A_7 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
 \end{aligned}$$

It should be mentioned that $\det(A_0) = \dots = \det(A_7) = 1$. The octonion multiplication table can be written in various ways, for example, the table in [3, p. 150] is different from one in [4]. These different tables result in different systems (1.1), though all these systems are clearly equivalent. Solutions of the system $[f]D = 0$ are called right-monogenic functions; functions, which are both left- and right-monogenic, are called monogenic. Hereafter, we always discuss the left-monogenic functions, but the proofs go word-by-word for the right- and monogenic functions.

Systems (1.1), where each component f_j , $0 \leq j \leq 7$, is harmonic, are called the generalized Cauchy-Riemann systems (GCR) - see Stein and Weiss [15, pp. 260-262]. Systems $\sum_{j=0}^7 A_j \frac{\partial F}{\partial x_j} + BF = 0$, where B is also a constant matrix, were considered by Evgrafov [8]. Stein and Weiss have proved that for any GCR system there exists a non-negative index $p_0 < 1$ such that $|F|^p$ is a subharmonic function for all $p \geq p_0$. It is known (ibid, p. 262) that for the M. Riesz system in \mathbf{R}^n

$$\frac{\partial u_1}{\partial x_1} + \dots + \frac{\partial u_n}{\partial x_n} = 0,$$

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}, \quad i, j = 1, \dots, n, \quad i \neq j,$$

the exact value of p_0 is $(n - 2)/(n - 1)$. Earlier we proved in [10] that the same is valid for system (1.1) in \mathbf{R}^8 , that is for system (1.1)

$$p_0 = \left. \frac{n - 2}{n - 1} \right|_{n=8} = 6/7.$$

In this note we use the subharmonicity of powers of octonion-valued monogenic functions to prove for those functions an analog of M. Riesz theorem on the boundedness of the conjugation operator in L^p , $p > 1$. It should be noted that due to the positivity of the subharmonic function $|f(x)|^p$, $p \geq 6/7$, if f is integrable over the boundary of the domain, then f and all its harmonic components f_i , $i = 0, 1, \dots, 7$, have non-tangential boundary values almost everywhere on the boundary. We let B denote the unit ball and $S = \partial B$, and $d\sigma$ denote surface measure on S . We will let $f_v(x) = \sum_{i=1}^7 f_i(x)e_i$ denote the imaginary or vector part of f .

Theorem 1 *If f is monogenic in the unit ball B and $S = \partial B$, then for $0 \leq r < 1$*

$$\int_S |f(r\zeta)|^p d\sigma \leq C_p \int_S |f_v(r\zeta)|^p d\sigma \tag{1.2}$$

where

$$C_p = \left(\frac{8}{p-1}\right)^{1/p} \text{ for } 1 < p \leq 2$$

$$C_p = \frac{A^{p/2} - (1-\lambda)^p}{|\lambda|^p} \text{ for } 2 < p < \infty$$
(1.3)

$$A \geq 8p(p-1) \text{ and } 0 < |\lambda| \leq \min\left\{1, \frac{1}{32^{(p-2)}}\right\}$$

2. Octonionic version of M. Riesz theorem; the proof of Theorem 1.

Let $f(x) = \sum_{j=0}^7 u_j(x)e_j$ be a monogenic octonion-valued function, $f_v(x) = f(x) - u_0(x)e_0$ its imaginary part, and $f_\lambda(x) = \lambda u_0(x) + f_v(x)$

Lemma 2.1 *The function $g = A|f_v|^p - |f|^p$ is subharmonic for $1 < p \leq 2$ and $A \geq \frac{8}{p-1}$.*

Proof: We first assume $f_v(x) \neq 0$ and we show that $\Delta g \geq 0$. Letting $J(f)$ denote the Jacobian of $f(x)$, the following follows from (1.1):

$$\begin{aligned} |\nabla(u_0)|^2 &\leq 7|J(f_v)|^2 \\ 8|\nabla(u_0)|^2 &\leq 7|J(f)|^2 \\ |J(f)|^2 &\leq 8|J(f_v)|^2 \end{aligned} \tag{2.1}$$

We will use a well known formula to calculate the laplacian, the following:

$$\Delta|f|^p = p|f|^{p-4}[|f|^2|J(f)|^2 + (p-2)\sum_{j=0}^7 (f \cdot f_{x_j})^2.] \tag{2.2}$$

Since $p \leq 2$, from equation (2.2) we immediately have

$$\Delta|f|^p \leq p|f|^{p-2}|J(f)|^2$$

Applying the Schwartz inequality to (2.2) gives

$$\Delta|f_v|^p \geq p(p-1)|f_v|^{p-2}|J(f_v)|^2$$

Using these inequalities with (2.1), $p-2 \leq 0$, and $|f|^{p-2} \leq |f_v|^{p-2}$ gives

$$\begin{aligned} \Delta g &\geq Ap(p-1)|f_v|^p|J(f_v)|^2 - p|f|^{p-2}|J(f)|^2 \\ &\geq p|f|^{p-2}[A(p-1)|J(f_v)|^2 - |J(f)|^2] \\ &\geq p|f|^{p-2}|J(f)|^2 \left[\frac{A}{8}(p-1) - 1 \right] \\ &\geq 0 \end{aligned}$$

The set of zero points of a harmonic function is a polar set, which completes the proof for all x .

In the proof of the next lemma we will need two identities.

$$\begin{aligned} \Delta(hg) &= fh\Delta g + g\Delta h + 2(\nabla g \cdot \nabla h) \\ \nabla|f|^p &= p|f|^{p-2} \langle f \cdot f_{x_0}, f \cdot f_{x_1}, \dots, f \cdot f_{x_7} \rangle = p|f|^{p-2}(f \cdot J(f)). \end{aligned} \tag{2.3}$$

Lemma 2.2 *Let $h = A|f_v|^2|f_\lambda|^{p-2} - |f_\lambda|^p$. Then h is subharmonic if*

$$\begin{aligned} p &\geq 2 \\ A &\geq 8p(p-1) \\ |\lambda| &\leq \min \left\{ 1, \frac{1}{32(p-2)} \right\} \end{aligned} \tag{2.4}$$

Proof: We assume that $f \neq 0$ and $f_v \neq 0$, and use (2.2) to calculate Δh .

$$\begin{aligned} \Delta h &= 2A|f_\lambda|^{p-2}|J(f_v)|^2 \\ &\quad + A(p-2)|f_v|^2|f_\lambda|^{p-6}(p-4) \sum_{j=0}^7 (f_\lambda \cdot f_{\lambda x_j})^2 \\ &\quad + A(p-2)|f_v|^2|f_\lambda|^{p-4}|J(f_\lambda)|^2 \\ &\quad + 4A(p-2)|f_\lambda|^{p-4} \sum_{j=0}^7 (f_v \cdot f_{v x_j})(f_\lambda \cdot f_{\lambda x_j}) \\ &\quad - \Delta|f_\lambda|^p \end{aligned} \tag{2.5}$$

Multiplying through by $|f_\lambda|^{2-p}$, and using $|f_v| \leq |f_\lambda| \leq |f|$ and using $|\lambda||f_{\lambda x_j}| \leq |f_{\lambda x_j}| \leq |f_{x_j}|$, we obtain

$$\begin{aligned} |f_\lambda|^{2-p}\Delta h &\geq 2A|J(f_v)|^2 \\ &\quad - 4A(p-2)|f_\lambda|^{-2} \sum_{j=0}^7 (f_v \cdot f_{v x_j})^2 \\ &\quad + 4A(p-2)|f_\lambda|^{-2} \sum_{j=0}^7 (f_v \cdot f_{v x_j})(f_\lambda \cdot f_{\lambda x_j}) \\ &\quad - |f_\lambda|^{2-p}\Delta|f_\lambda|^p \\ &\geq 2A|J(f_v)|^2 \\ &\quad + 4A(p-2)|f_\lambda|^{-2} \sum_{j=0}^7 (f_\lambda \cdot f_{\lambda x_j})[(f_v \cdot f_{v x_j}) - (f_\lambda \cdot f_{\lambda x_j})] \\ &\quad - |f_\lambda|^{2-p}\Delta|f_\lambda|^p. \end{aligned} \tag{2.6}$$

Observing that

$$(f_v \cdot f_{v x_j}) - (f_\lambda \cdot f_{\lambda x_j}) = -\lambda^2 u_0 \left\{ \frac{\partial u_0}{\partial x_j} \right\}$$

we have

$$\begin{aligned} & |f_\lambda|^{-2} |(f_\lambda \cdot f_{\lambda x_j})| |(f_v \cdot f_{v x_j}) - (f_\lambda \cdot f_{\lambda x_j})| \\ & \leq \frac{|\lambda u_0|}{|f_\lambda|} |f_{\lambda x_j}| |\lambda| \left| \left\{ \frac{\partial u_0}{\partial x_j} \right\} \right| \\ & \leq |f_{\lambda x_j}|^2 |\lambda| \\ & \leq |\lambda| |f_{x_j}|^2 \end{aligned} \tag{2.7}$$

Hence,

$$\begin{aligned} |f_\lambda|^{2-p} \Delta h & \geq 2A |J(f_v)|^2 \\ & - |f_\lambda|^{2-p} \Delta |f_\lambda|^p - 4A |\lambda| (p-2) |J(f)|^2 \\ & = A [|J(f_v)|^2 - 4|\lambda| (p-2) |J(f)|^2] + [A |J(f_v)|^2 - |f_\lambda|^{2-p} \Delta |f_\lambda|^p]. \end{aligned}$$

We have

$$|J(f_v)|^2 - 4|\lambda| (p-2) |J(f)|^2 \geq [1 - 32|\lambda| (p-2)] |J(f_v)|^2 \geq 0$$

and using (2.7) with the Cauchy-Schartz inequality we have

$$\begin{aligned} & A |J(f_v)|^2 - |f_\lambda|^{2-p} \Delta |f_\lambda|^p \\ & \geq A |J(f_v)|^2 - p(p-2) |f_\lambda|^{-2} \sum_{j=0}^7 |f_\lambda|^2 \left| \left\{ \frac{\partial f_\lambda}{\partial x_j} \right\} \right|^2 - p |J(f_\lambda)|^2 \\ & = A |J(f_v)|^2 - p(p-2) |J(f_\lambda)|^2 - p |J(f_\lambda)|^2 \\ & \geq \left[\frac{A}{8} - p(p-1) \right] |J(f_\lambda)|^2 \geq 0 \end{aligned}$$

since $A \geq 8p(p-1)$. This completes the proof since once again the zero set of a non-constant harmonic function is a polar set.

We now prove Theorem 1.

Proof: We may assume $u_0(0) = 0$. For $1 < p \leq 2$, letting $A = \frac{8}{p-1}$ in Lemma 1, we have

$$g(0) = \frac{8}{p-1} |f_v(0)|^p - |f(0)|^p = \frac{9-p}{p-1} |f(0)|^p \geq 0$$

Hence, by subharmonicity

$$\int_S (g(r\zeta)) d\sigma(\zeta) \geq 0$$

so

$$\int_S |f(r\zeta)|^p d\sigma \leq \left(\frac{8}{p-1} \right)^{1/p} \int_S |f_v(r\zeta)|^p d\sigma, \quad 0 \leq r < 1$$

For $2 < p < \infty$ we again assume $u_0(0) = 0$ so $h(x)$ is subharmonic and A and $|\lambda|$ satisfying the conditions of Lemma 2. Since $h(0) \geq 0$, so it follows that

$$\int_S |f_\lambda(r\zeta)|^p d\sigma(\zeta) \leq \int_S |f_v(r\zeta)|^2 |f_\lambda(r\zeta)|^{p-2} d\sigma(\zeta).$$

We now use the Hölder inequality with exponents $p/2$ and $p/(p - 2)$ and we have

$$\int_S |f_\lambda(r\zeta)|^p d\sigma(\zeta) \leq A \left(\int_S |f_v(r\zeta)|^p d\sigma(\zeta) \right)^{\frac{2}{p}} \left(\int_S |f_\lambda(r\zeta)|^p d\sigma(\zeta) \right)^{\frac{p-2}{p}}$$

Dividing both sides by $\left(\int_S |f_\lambda(r\zeta)|^p d\sigma(\zeta) \right)^{\frac{p-2}{p}}$ we have

$$\int_S |f_\lambda(r\zeta)|^p d\sigma(\zeta) \leq A^{p/2} \int_S |f_v(r\zeta)|^p d\sigma(\zeta)$$

We use the inequality $(a + b)^{2/p} \geq a^{2/p} + b^{2/p}$ to $f_\lambda = (1 - \lambda)f_v + \lambda f$ and octonion algebra,

$$\begin{aligned} x &= \sum_{j=0}^7 x_j e_j \\ x^* &= x_0 e_0 - \sum_{j=1}^7 x_j e_j \\ xx^* &= \sum_{j=0}^7 |x_j|^2 \end{aligned}$$

From this it follows that $f_v^* = -f_v$ and $f^* = u_0 e_0 - f_v$ so $|f_\lambda|^2 = |(1 - \lambda)f_v + \lambda f|^2 = (1 - \lambda)^2 |f_v|^2 + \lambda^2 |f|^2$ and we have—

$$\begin{aligned} A^{p/2} \int_S |f_v(r\zeta)|^p d\sigma(\zeta) &\geq \int_S |f_\lambda|^p d\sigma \\ &= \int_S |f_\lambda|^p d\sigma = \int_S |(1 - \lambda)f_v + \lambda f|^p d\sigma \\ &= \int_S ((1 - \lambda)^2 |f_v|^2 + \lambda^2 |f|^2)^{p/2} d\sigma \\ &\geq (1 - \lambda)^p \int_S |f_v|^p d\sigma + |\lambda|^p \int_S |f|^p d\sigma \end{aligned}$$

Finally,

$$\int_S |f(r\zeta)|^p d\sigma(\zeta) \leq \frac{A^{p/2} - (1 - \lambda)^p}{|\lambda|^p} \int_S |f_v(r\zeta)|^p d\sigma(\zeta)$$

which completes the proof.

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