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Sign Patterns of the Liouville Function and Mobius Function over the Integers

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Abstract

Let $x \geq 1$ be a large number, and let $0 \leq a_0 < a_1 < \cdots < a_{k-1} \leq x$ be an integer k-tuple. In this note it shown that the k-sign patterns

$$\lambda(n+a_0) = \pm 1, \lambda(n+a_1) = \pm 1, \ldots, \lambda(n+a_{k-1}) = \pm 1$$

of the Liouville function $\lambda(n) \in \{-1, 1\}$ are equidistributed on the interval $[1, x]$. Similarly, the k-sign patterns

$$\mu(n+a_0) = \pm 1, \mu(n+a_1) = \pm 1, \ldots, \mu(n+a_{k-1}) = \pm 1$$

of the closely related Mobius function $\mu(n) \in \{-1, 0, 1\}$ are equidistributed on the interval $[1, x]$.

Contents

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1 Introduction

Given an integer $n = p_1^{v_1} p_2^{v_2} \cdots p_w^{v_w}$, where the $p_i \geq 2$ are primes, and $v_i \geq 1$, the Liouville function is defined by

$$\lambda(n) = (-1)^{v_1+v_2+\cdots+v_w}. \tag{1.1}$$

The closely related Mobius function $\mu : \mathbb{N} \longrightarrow \{-1, 0, 1\}$ is defined by

$$\mu(n) = \begin{cases} (-1)^w & n = p_1 p_2 \cdots p_w \\ 0 & n \neq p_1 p_2 \cdots p_w. \end{cases} \tag{1.2}$$

Let $k \geq 1$ be an integer. Define the integer k-tuple

$$\boldsymbol{a} = (a_0, a_1, \ldots, a_{k-1}), \tag{1.3}$$

where $0 \leq a_0 < a_1 < \cdots < a_{k-1} \leq x$, and the sign pattern k-tuple

$$\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}), \tag{1.4}$$

where $\epsilon_i \in \{-1, 1\}$. The signs of a vector of Liouville function values

$$\lambda(n + a_0) = \pm 1, \ \lambda(n + a_1) = \pm 1, \ \ldots, \ \lambda(n + a_{k-1}) = \pm 1 \tag{1.5}$$

is usually identified with the sign pattern k-tuple

$$(\lambda(n + a_0), \lambda(n + a_1), \ldots, \lambda(n + a_{k-1})) = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}), \tag{1.6}$$

where $n \in [1, x]$. The corresponding k-sign pattern counting function has the form

$$\mathcal{N}_\lambda(\boldsymbol{a}, \boldsymbol{\epsilon}, x) = \#\{n \leq x : (\lambda(n + a_0), \ldots, \lambda(n + a_{k-1})) = (\epsilon_0, \ldots, \epsilon_{k-1})\}. \qquad (1.7)$$

This note proposes the following equidistribution results, these include new results, and simpler proofs of the current available results in the literature.

Theorem 1.1. *If x is a large number, then, the k-sign patterns of the Liouville function $\lambda : \mathbb{N} \longrightarrow \{-1, 1\}$ are equidistributed on the interval $[1, x]$, and the counting function for each k-sign pattern has the asymptotic formula*

$$\mathcal{N}_\lambda(\boldsymbol{a}, \boldsymbol{\epsilon}, x) = \frac{1}{2^k}[x] + O\left(\frac{x}{(\log x)^c}\right).$$

In particular, the density of each each k-sign pattern has the asymptotic formula

$$\delta_\lambda(\boldsymbol{a}, \boldsymbol{\epsilon}, x) = \frac{1}{x} \sum_{\substack{n \leq x \\ \lambda(n+a_0)=\pm 1,\, \ldots, \lambda(n+a_{k-1})=\pm 1}} 1 = \frac{1}{2^k} + O\left(\frac{1}{(\log x)^c}\right),$$

where $c > 0$ is an arbitrary constant.

Similarly, the signs of the vector of Mobius values

$$\mu(n + a_0) = \pm 1, \; \mu(n + a_1) = \pm 1, \; \ldots, \; \mu(n + a_{k-1}) = \pm 1 \qquad (1.8)$$

is usually identified with the k-sign pattern vector

$$(\mu(n + a_0), \mu(n + a_1), \ldots, \mu(n + a_{k-1})) = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}). \qquad (1.9)$$

The corresponding k-sign pattern counting function has the form

$$\mathcal{N}_\mu(\boldsymbol{a}, \boldsymbol{\epsilon}, x) = \#\{n \leq x : (\mu(n + a_0), \ldots, \mu(n + a_{k-1})) = (\epsilon_0, \ldots, \epsilon_{k-1})\}. \qquad (1.10)$$

The related result for the Mobius function is the following.

Theorem 1.2. *If x is a large number, then, the k-sign patterns of the Mobius function $\mu : \mathbb{N} \longrightarrow \{-1, 0, 1\}$ are equidistributed on the interval $[1, x]$, and the counting function for each k-sign pattern has the asymptotic formula*

$$\mathcal{N}_\mu(\boldsymbol{a}, \boldsymbol{\epsilon}, x) = \frac{s_k}{2^k} x + O\left(\frac{x}{(\log x)^c}\right),$$

where $s_k > 0$ is a constant, and $c > 0$ is an arbitrary constant. In particular, the density of each each k-sign pattern has the asymptotic formula

$$\delta_\mu(\boldsymbol{a}, \boldsymbol{\epsilon}, x) = \frac{1}{x} \sum_{\substack{n \leq x \\ \mu(n+a_0)=\pm 1,\, \ldots, \mu(n+a_{k-1})=\pm 1}} 1 = \frac{s_k}{2^k} + O\left(\frac{1}{(\log x)^c}\right).$$

The proofs of these results, partially conjectured in [9], are almost independent of each other, and are presented in two separate sections. The simpler proof of Theorem 1.1 for the Liouville function is presented Subsection 2.7. The slight more difficult proof of Theorem 1.2 for the Mobius function appears in Subsection 3.8.

2 Result for the Liouville Function over the Integers

The essential foundational materials are covered in Subsection 2.1 to Subsection 2.5.

2.1 Single-Sign Patterns Liouville Characteristic Functions

The analysis of single-sign pattern characteristic function

$$\lambda^{\pm}(n)=\left(\frac{1\pm\lambda(n)}{2}\right)=\begin{cases}1 & \text{if } \lambda(n)=\pm 1,\\ 0 & \text{if } \lambda(n)\neq\pm 1,\end{cases} \tag{2.1}$$

of the subset of integers

$$\mathcal{N}_{\lambda}^{\pm 1}=\{n\geq 1:\mu(n)=\pm 1\} \tag{2.2}$$

is well known. Here, the same idea is extended to the shifted primes.

Lemma 2.1. *Let $t\neq 0$ be an integer, and let $\lambda(n)\in\{-1,1\}$ be the Liouville function. Then,*

$$\begin{aligned}\lambda^{\pm}(t,n) &= \left(\frac{1\pm\lambda(n+t)}{2}\right) \\ &= \begin{cases}1 & \textit{if } \lambda(n+t)=\pm 1,\\ 0 & \textit{if } \lambda(n+t)\neq\pm 1,\end{cases}\end{aligned} \tag{2.3}$$

are the characteristic functions of the subset of primes

$$\mathcal{N}_{\lambda}^{\pm}(t)=\{n\geq 1:\lambda(n+t)=\pm 1\}. \tag{2.4}$$

2.2 Double-Sign Patterns Liouville Characteristic Functions

The analysis of single-sign pattern characteristic functions is extended here to the double-sign patterns

$$\lambda(n)=\pm 1 \quad \text{and} \quad \lambda(n+t)=\pm 1, \tag{2.5}$$

where $t\neq 0$ is a small integer, and $n\geq 1$ is an integer. Some of the research on multiple sign patterns appear in [8], [13], [26, Corollary 1.7], [20], [23], and similar literature.

Lemma 2.2. *Let $t\neq 0$ be small fixed integer, and let $\lambda(n)\in\{-1,1\}$ be the Liouville function. Then,*

$$\begin{aligned}\lambda^{\pm\pm}(t,p) &= \left(\frac{1\pm\lambda(n)}{2}\right)\left(\frac{1\pm\lambda(n+t)}{2}\right) \\ &= \begin{cases}1 & \textit{if } \lambda(n)=\pm 1, \mu(n+t)=\pm 1,\\ 0 & \textit{if } \lambda(n)\neq\pm 1, \mu(n+t)\neq\pm 1,\end{cases}\end{aligned} \tag{2.6}$$

are the characteristic functions of the subset of integers

$$\mathcal{N}_{\lambda}^{\pm\pm}(t)=\{n\geq 1:\lambda(n)=\pm 1, \lambda(n+t)=\pm 1\}. \tag{2.7}$$

2.3 k-Signs Patterns Liouville Characteristic Functions

The general sign patterns of a vector of values of Liouville functions is consider here. A slight change in notation to simplify the formulas is introduced in this subsection.

Let $k \geq 1$ be an integer. Define the integer k-tuple

$$\boldsymbol{a} = (a_0, a_1, \ldots, a_{k-1}), \tag{2.8}$$

where $0 \leq a_0 < a_1 < \cdots < a_{k-1} \leq x$, and the k-sign pattern

$$\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}), \tag{2.9}$$

where $\epsilon_i \in \{-1, 1\}$. The same principle used for single-sign and double-sign patterns is extended to the general k-sign patterns characteristic function of k-tuple of Liouville function values

$$(\lambda(n + a_0), \lambda(n + a_1), \ldots, \lambda(n + a_{k-1})) = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}). \tag{2.10}$$

Lemma 2.3. *Let $n \in \mathbb{N}$ be an integer, and let $\lambda(n) \in \{-1, 1\}$ be the Liouville function. If $\boldsymbol{a}$ is an integer k-tuple, and $\boldsymbol{\epsilon}$ is a k-sign pattern, then,*

$$\begin{aligned} \lambda(\boldsymbol{a}, \boldsymbol{\epsilon}, n) &= \prod_{0 \leq i < k} \left(\frac{1 \pm \lambda(n + a_i)}{2} \right) \qquad (2.11) \\ &= \begin{cases} 1 & \text{if } \lambda(n + a_0) = \epsilon_0, \ldots, \ \lambda(n + a_{k-1}) = \epsilon_{k-1}, \\ 0 & \text{if } \lambda(n + a_0) \neq \epsilon_0, \ldots, \ \lambda(n + a_{k-1}) \neq \epsilon_{k-1}, \end{cases} \end{aligned}$$

is the characteristic functions of the subset of integers

$$\mathcal{N}_\lambda(\boldsymbol{a}, \boldsymbol{\epsilon}) = \{n \geq 1 : \lambda(n) = \epsilon_0, \ldots \lambda(n + k - 1) = \epsilon_{k-1}\}. \tag{2.12}$$

2.4 Single-Sign Pattern Liouville Counting Functions

The single-sign patterns $\lambda(n) = 1$ and $\lambda(n) = -1$ single-sign pattern counting functions over the integers have the simplest analysis.

$$\mathcal{N}_\lambda^+(x) = \sum_{\substack{n \leq x \\ \lambda(n)=1}} 1 = \sum_{n \leq x} \left(\frac{1 + \lambda(n)}{2} \right) = \frac{1}{2}[x] + O\left(x e^{-c\sqrt{\log x}} \right), \tag{2.13}$$

and

$$\mathcal{N}_\lambda^-(x) = \sum_{\substack{n \leq x \\ \lambda(n)=-1}} 1 = \sum_{n \leq x} \left(\frac{1 - \lambda(n)}{2} \right) = \frac{1}{2}[x] + O\left(x e^{-c\sqrt{\log x}} \right), \tag{2.14}$$

where $[x]$ is the largest integer function, respectively. In terms of the single-sign pattern counting functions, the summatory function has the asymptotic formula

$$\mathcal{N}_\lambda(x) = \sum_{n \leq x} \lambda(n) = \mathcal{N}_\lambda^+(x) - \mathcal{N}_\lambda^-(x) = O\left(x e^{-c\sqrt{\log x}} \right). \tag{2.15}$$

Basically, it is a different form of the Prime Number Theorem

$$\pi(x) = \text{li}(x) + O\left(xe^{-c\sqrt{\log x}}\right), \tag{2.16}$$

where $\text{li}(x) = \int_2^2 (\log t)^{-1} dt$ is the logarithm integral, and $c > 0$ is an absolute constant, see [5, Eq. 27.12.5], [6, Theorem 3.10], et alii.

2.5 Double-Sign Patterns Counting Functions

The counting functions for the single-sign patterns $\lambda(n) = 1$ and $\lambda(n) = -1$ are extended to the counting functions for the double-sign patterns

$$(\lambda(n), \lambda(n+t)) = (\pm 1, \pm 1). \tag{2.17}$$

The double-sign patterns counting functions are defined by

$$\mathcal{N}_\lambda^{++}(t,x) = \sum_{\substack{n \le x \\ \lambda(n)=1,\ \lambda(n+t)=1}} 1 = \sum_{\substack{n \le x \\ n \in \mathcal{N}_\lambda^{++}(t)}} 1, \tag{2.18}$$

$$\mathcal{N}_\lambda^{+-}(t,x) = \sum_{\substack{n \le x \\ \lambda(n)=1,\ \lambda(n+t)=-1}} 1 = \sum_{\substack{n \le x \\ n \in \mathcal{N}_\lambda^{+-}(t)}} 1, \tag{2.19}$$

$$\mathcal{N}_\lambda^{-+}(t,x) = \sum_{\substack{n \le x \\ \lambda(n)=-1,\ \lambda(n+t)=1}} 1 = \sum_{\substack{n \le x \\ p \in \mathcal{N}_\lambda^{-+}(t)}} 1, \tag{2.20}$$

$$\mathcal{N}_\lambda^{--}(t,x) = \sum_{\substack{n \le x \\ \lambda(n)=-1,\ \lambda(n+t)=-1}} 1 = \sum_{\substack{n \le x \\ p \in \mathcal{N}_\lambda^{--}(t)}} 1. \tag{2.21}$$

The double-sign patterns counting functions (2.18) to (2.21) are precisely the counting functions of the subsets of integers

1. $\mathcal{N}_\lambda^{++}(t) \subset \mathbb{N}$,
2. $\mathcal{N}_\lambda^{+-}(t) \subset \mathbb{N}$,
3. $\mathcal{N}_\lambda^{-+}(t) \subset \mathbb{N}$,
4. $\mathcal{N}_\lambda^{--}(t) \subset \mathbb{N}$,

defined in (2.7).

Lemma 2.4. *Let x be a large number, and let $t \ne 0$ be a fixed integer. If $\lambda : \mathbb{Z} \longrightarrow \{-1, 1\}$ is the Liouville function, then,*

$$\mathcal{N}_\lambda^{\pm\pm}(t,x) = \frac{1}{4}[x] + O\left(\frac{x}{(\log x)^{2c}}\right),$$

where $c > 0$ is an arbitrary constant.

Proof. Without loss in generality, consider the pattern $(\lambda(n), \lambda(n+t)) = (+1, +1)$. Now, use Lemma 2.2 to express the double-sign pattern counting function as

$$\begin{aligned} 4\mathcal{N}_\lambda^{++}(t,x) &= \sum_{n \leq x} \lambda^{++}(t,n) \\ &= \sum_{n \leq x} (1 + \lambda(n))(1 + \lambda(n+t)) \\ &= \sum_{n \leq x} (1 + \lambda(n) + \lambda(n+t) + \lambda(n)\lambda(n+t)) \qquad (2.22) \\ &= \sum_{n \leq x} 1 + \sum_{n \leq x} \lambda(n) + \sum_{n \leq x} \lambda(n+t) + \sum_{n \leq x} \lambda(n)\lambda(n+t) \\ &\geq 0. \end{aligned}$$

The first three finite sums on the last line have the following evaluations or estimates.

1. $\displaystyle\sum_{p \leq x} 1 = [x]$,
2. $\displaystyle\sum_{n \leq x} \lambda(n) = O\left(xe^{-c\sqrt{\log x}}\right)$, see Theorem 4.6.
3. $\displaystyle\sum_{n \leq x} \lambda(n+t) = O\left(xe^{-c\sqrt{\log x}}\right)$, see Theorem 4.6,
4. $\displaystyle\sum_{n \leq x} \lambda(n)\lambda(n+t) = O\left(\frac{x}{(\log x)^{2c}}\right)$, see Theorem 4.10,

where $[x]$ is the largest integer function, $c > 0$ is an absolute constant. Summing these evaluations or estimates verifies the claim for $\mathcal{N}_\lambda^{++}(t,x) \geq 0$. The verifications for the next three double-sign pattern counting functions $\mathcal{N}_\lambda^{+-}(t,x) \geq 0$, $\mathcal{N}_\lambda^{-+}(t,x) \geq 0$, and $\mathcal{N}_\lambda^{--}(t,x) \geq 0$ are similar. ■

2.6 k-Sign Patterns Counting Functions

The k-sign pattern of each quasi-consecutive values of the Liouville function

$$(\lambda(n+a_0), \lambda(n+a_1), \ldots, \lambda(n+a_{k-1})) = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}), \qquad (2.23)$$

where $\epsilon_i \in \{-1, 1\}$, and $n \in [1, x]$, has a k-tuple counting function of the form

$$\mathcal{N}_\lambda(\mathbf{a}, \boldsymbol{\epsilon}, x) = \#\{n \leq x : (\lambda(n+a_0), \ldots, \lambda(n+a_{k-1})) = (\epsilon_0, \ldots, \epsilon_{k-1})\}. \qquad (2.24)$$

The precise asymptotic formula for these counting functions has a lengthy derivation provided below.

Theorem 2.1. *Let x be a large number, and let $\lambda : \mathbb{Z} \longrightarrow \{-1, 1\}$ be the Liouville function. If $\boldsymbol{a}$ is an integer k-tuple, and $\boldsymbol{\epsilon}$ is a k-sign pattern, then,*

$$\mathcal{N}_\lambda(\boldsymbol{a}, \boldsymbol{\epsilon}, x) = \frac{1}{2^k}[x] + O\left(\frac{x}{(\log x)^{2c}}\right),$$

where $c > 0$ is an arbitrary constant.

Proof. Fix an integer $k \ll \log x$, and an integer k-tuple $\boldsymbol{a} = (a_0, a_1, \ldots, a_{k-1})$. Without loss in generality consider the k-sign pattern $\boldsymbol{\epsilon} = (+1, +1, \ldots, +1)$. Now, use Lemma 2.3 to express the double-sign pattern counting function as

$$\begin{aligned} 2^k \mathcal{N}_\lambda(\mathbf{a}, \boldsymbol{\epsilon}, x) &= \sum_{n \leq x} \lambda(\mathbf{a}, \boldsymbol{\epsilon}, n) \\ &= \sum_{n \leq x,\, 0 \leq i < k} \prod \left(\frac{1 + \lambda(n + a_i)}{2} \right) \\ &= \sum_{n \leq x} 1 + \sum_{0 \leq i < k,} \sum_{n \leq x} \lambda(n + a_i) + \sum_{\substack{0 \leq i,j < k, \\ i \neq j}} \sum_{n \leq x} \lambda(n + a_i)\lambda(n + a_j) \\ &\quad + \sum_{\substack{0 \leq h,i,j < k, \\ h \neq i \neq j}} \sum_{n \leq x} \lambda(n + a_h)\lambda(n + a_i)\lambda(n + a_j) \\ &\quad + \cdots + \sum_{n \leq x} \lambda(n + a_0)\lambda(n + a_1) \cdots \lambda(n + a_{k-1}) \\ &\geq 0. \end{aligned} \tag{2.25}$$

For the integer k-tuple $0 = a_0 < a_1 < \cdots < a_{k-1} < x$, these finite sums have the following evaluations or estimates.

1. $\displaystyle\sum_{n \leq x} 1 = [x]$,

2. $\displaystyle\sum_{0 \leq i < k,} \sum_{n \leq x} \lambda(n + a_i) = O\left(\binom{k}{1} x e^{-c\sqrt{\log x}} \right)$, see Theorem 4.6.

3. $\displaystyle\sum_{\substack{0 \leq i,j < k, \\ i \neq j}} \sum_{n \leq x} \lambda(n + a_i)\lambda(n + a_j) = O\left(\binom{k}{2} \frac{x}{(\log x)^{2c}} \right)$, see Theorem 4.10,

4. $\displaystyle\sum_{\substack{0 \leq h,i,j < k, \\ h \neq i \neq j}} \sum_{n \leq x} \lambda(n + a_h)\lambda(n + a_i)\lambda(n + a_j) = O\left(\binom{k}{3} \frac{x}{(\log x)^{2c}} \right)$,

$\vdots \qquad \vdots \qquad \vdots$

5. $\displaystyle\sum_{n \leq x} \lambda(n + a_0)\lambda(n + a_1) \cdots \lambda(n + a_{k-1}) = O\left(\frac{x}{(\log x)^{2c}} \right)$,

where $[x]$ is the largest integer function, $c > 0$ is an arbitrary constant. There are $k + 1$ single and double finite sums, in equation (2.25), and in the list. Moreover, the binomial coefficient satisfies the inequality

$$(k + 1)\binom{k}{i} \leq 2^k \tag{2.26}$$

for $i \leq k$. Therefore, the total sum of the exact evaluations or estimates listed as (1) to (6) has the asymptotic formula

$$0 \leq 2^k \mathcal{N}_\lambda(\mathbf{a}, \boldsymbol{\epsilon}, x) = [x] + O\left((k+1)\binom{k}{k/2}\frac{x}{(\log x)^{2c}}\right) \tag{2.27}$$

as claimed. The verifications for all the other k-sign patterns $\boldsymbol{\epsilon} \neq (+1, +1, \ldots, +1)$ are similar. ■

2.7 Equidistribution of Double-Sign Patterns

The nontrivial result for the summatory Liouville function

$$\sum_{n \leq x} \lambda(n) \quad = \quad \mathcal{N}_\lambda^+(x) - \mathcal{N}_\lambda^-(x) = O\left(xe^{-c\sqrt{\log x}}\right) \tag{2.28}$$

has no main term. It vanishes because the number of single-sign patterns $\mathcal{N}_\lambda^+(x) = \#\{n \leq x : \lambda(n) = 1\}$ and $\mathcal{N}_\lambda^-(x) = \#\{n \leq x : \lambda(n) = -1\}$ have the same cardinality. This implies that the single-sign patterns are equidistributed on the interval $[1, x]$, and each has the natural density $\delta_\lambda^\pm = 1/2$, see (2.13) to (2.16) for more detail. This idea is extended in the proof of the equidistribution of the double-sign patterns.

Recall that $\mathcal{N}_\lambda^{\pm\pm}(t, x) = \#\{n \leq x : \lambda(n) = \pm 1, \lambda(n+t) = \pm 1\}$, and the natural density of a double-sign pattern is defined by

$$\delta_\lambda^{\pm\pm}(t) = \lim_{x \to \infty} \frac{\#\{n \leq x : \lambda(n) = \pm 1, \lambda(n+t) = \pm 1\}}{x}. \tag{2.29}$$

Theorem 2.2. *Let x be a large number, and let $t \neq 0$ be a fixed integer. Then, the double-sign patterns $++$, $+-$, $-+$, and $--$ of the Liouville pair $\lambda(n), \lambda(n+t)$ are equidistributed on the interval $[1, x]$. In particular, each double-sign pattern has the natural density*

$$\delta_\lambda^{\pm\pm}(t) = \frac{1}{4}.$$

Proof. Observe that for large $x \geq 1$, $\mathcal{N}_\lambda^{\pm\pm}(t, x) > 0$, this follows from Lemma 2.4. Next, compute the limit of the proportion of double-sign pattern

$$\begin{aligned} \delta_\lambda^{\pm\pm}(t) &= \lim_{x \to \infty} \frac{\#\{n \leq x : \lambda(n) = \pm 1, \lambda(n+t) = \pm 1\}}{x} \\ &= \lim_{x \to \infty} \frac{[x] + O\left(x(\log x)^{-2c}\right)}{4x} \\ &= \frac{1}{4}. \end{aligned}$$

This proves that the double-sign patterns $++$, $+-$, $-+$, and $--$ are equidistributed on the interval $[1, x]$ as $x \to \infty$. ■

Example 2.1. Let $t = 1$. By Theorem 2.2, in any sufficiently large interval $[1, x]$, the number of any double-sign pattern $\lambda(n) = \pm 1, \lambda(n+1) = \pm 1$ is

$$\mathcal{N}_{\lambda}^{\pm\pm}(t, x) = \delta_{\lambda}^{\pm\pm}(t)x + o(x) = \frac{1}{4}x + O\left(\frac{x}{(\log x)^{2c}}\right). \tag{2.30}$$

The numerical data for $x = 10^4$, shows that the actual value of the autocorrelation function is

$$\sum_{n \leq x} \lambda(n)\lambda(n+1) = 112, \tag{2.31}$$

and the actual values of the double-sign pattern counting functions are tabulated below.

$\lambda(n)$	$\lambda(n+1)$	Actual Count	Expected $R^{\pm\pm}(1, x)$
$+1$	$+1$	$9924/4$	$10000/4 + o(x)$
$+1$	-1	$9888/4$	$10000/4 + o(x)$
-1	$+1$	$9888/4$	$10000/4 + o(x)$
-1	-1	$10300/4$	$10000/4 + o(x)$

Given the small scale of this experiment, $x = 10^4$, the actual data fits the prediction very well. The tiny differences among the actual values, (in third column), and the prediction by the double-sign pattern counting functions $\mathcal{N}_{\lambda}^{\pm\pm}(1, x)$ seem to be properties of the races between the different subsets of integers $\mathcal{N}_{\lambda}^{\pm\pm}(t)$ attached to the double-sign patterns, see (2.7). For an introduction to the literature in comparative number theory, prime number races, and similar topics, see [7], et cetera.

2.8 Equidistribution of k-Sign Patterns

The nontrivial result for the summatory Liouville function

$$\sum_{n \leq x} \lambda(n) \quad = \quad \mathcal{N}_{\lambda}^{+}(x) - \mathcal{N}_{\lambda}^{-}(x) = O\left(xe^{-c\sqrt{\log x}}\right) \tag{2.32}$$

has no main term. It vanishes because the number of single-sign patterns $\mathcal{N}_{\lambda}^{+}(x) = \#\{n \leq x : \lambda(n) = 1\}$ and $\mathcal{N}_{\lambda}^{-}(x) = \#\{n \leq x : \lambda(n) = -1\}$ have the same cardinality. This implies that the single-sign patterns are equidistributed on the interval $[1, x]$, and each has the natural density $\delta_{\lambda}^{\pm} = 1/2$, see (2.13) to (2.16) for more detail. This idea is extended in the proof of the equidistribution of the double-sign patterns, triple-sign patterns, ..., and in general to k-sign patterns.

Recall that for any fixed $\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1})$, where $\epsilon_i \in \{-1, 1\}$, the k-sign pattern counting function is defined by

$$\mathcal{N}_{\lambda}(\mathbf{a}, \boldsymbol{\epsilon}, x) = \#\{n \leq x : (\lambda(n+a_0), \ldots, \lambda(n+a_{k-1})) = (\epsilon_0, \ldots, \epsilon_{k-1})\}, \tag{2.33}$$

and the natural density of a k-sign pattern is defined by

$$\delta_{\lambda}(\mathbf{a}, \boldsymbol{\epsilon}) = \lim_{x \to \infty} \frac{\#\{n \leq x : (\lambda(n+a_0), \ldots, \lambda(n+a_{k-1})) = (\epsilon_0, \ldots, \epsilon_{k-1})\}}{x}. \tag{2.34}$$

Theorem 2.3. *Let x be a large number, and let $\lambda : \mathbb{Z} \longrightarrow \{-1, 1\}$ be the Liouville function. If $\boldsymbol{a} = (a_0, a_1, \ldots, a_{k-1})$, and $\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1})$, where $\epsilon_i \in \{-1, 1\}$, then, the k-sign pattern of the Liouville function λ are equidistributed on the interval $[1, x]$. In particular, each k-sign pattern has the natural density*

$$\delta_\lambda(\boldsymbol{a}, \boldsymbol{\epsilon}) = \frac{1}{2^k}.$$

Proof. Observe that for large $x \geq 1$, $\mathcal{N}_\lambda(\mathbf{a}, \boldsymbol{\epsilon}, x) > 0$, this follows from Theorem 2.1. Next, compute the limit of the proportion of k-sign pattern

$$\begin{aligned} \delta_\lambda(\mathbf{a}, \boldsymbol{\epsilon}) &= \lim_{x \to \infty} \frac{\#\{n \leq x : (\lambda(n + a_0), \ldots, \lambda(n + a_{k-1})) = (\epsilon_0, \ldots, \epsilon_{k-1})\}}{x} \\ &= \lim_{x \to \infty} \frac{[x] + O\left(x(\log x)^{-2c}\right)}{2^k x} \\ &= \frac{1}{2^k}. \end{aligned}$$

This proves that the k-sign patterns $\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1})$ are equidistributed on the interval $[1, x]$ as $x \to \infty$. ■

3 Results for the Mobius Function over the Integers

The essential foundational materials are covered in Subsection 4.4 to Subsection 3.6.

3.1 Single-Sign Patterns Mobius Characteristic Functions

The analysis of single-sign pattern characteristic function is well known.

Lemma 3.1. *If $\mu(n) \in \{-1, 1\}$ is the Mobius function, then,*

$$\begin{aligned} \mu^{\pm}(n) &= \mu^2(n) \left(\frac{1 \pm \mu(n)}{2} \right) \\ &= \begin{cases} 1 & \text{if } \mu(n) = \pm 1, \\ 0 & \text{if } \mu(n) \neq \pm 1, \end{cases} \end{aligned} \tag{3.1}$$

are the characteristic functions of the subset of primes

$$\mathcal{N}_\mu^{\pm} = \{n \geq 1 : \mu(n) = \pm 1\}. \tag{3.2}$$

3.2 Double-Sign Patterns Characteristic Functions

The principle of single-sign pattern characteristic function is extended to the double-sign patterns $(\mu(n + a), \mu(n + b)) = (\pm 1, \pm 1)$, where $a, b \in \mathbb{Z}$ is a pair of small integers such that $a \neq b$. Other sign patterns are topics of current research, confer [8], [13], [26, Corollary 1.7], [20], and similar literature, for details. A new and different approach to the analysis of double-sign patterns, triple-sign patterns, et cetera, based on elementary methods, is provided here.

Lemma 3.2. *Let $t \neq 0$ be an integer, and let $\mu(n) \in \{-1, 0, 1\}$ be the Mobius function. Then,*

$$\begin{aligned} \mu^{\pm\pm}(n,t) &= \mu^2(n)\mu^2(n+t)\left(\frac{1 \pm \mu(n)}{2}\right)\left(\frac{1 \pm \mu(n+t)}{2}\right) \\ &= \begin{cases} 1 & \text{if } \mu(n) = \pm 1, \mu(n+t) = \pm 1, \\ 0 & \text{if } \mu(n) \neq \pm 1, \mu(n+t) \neq \pm 1, \end{cases} \end{aligned} \tag{3.3}$$

are the characteristic functions of the subset of integers

$$\mathcal{N}_\mu^{\pm\pm}(t) = \{n \geq 1 : \mu(n) = \pm 1, \mu(n+t) = \pm 1\}. \tag{3.4}$$

3.3 k-Signs Patterns Mobius Characteristic Functions

Due to the presence of the extra term $\mu(n) \in \{-1, 0, 1\}$, the characteristic functions of sign patterns of Mobius functions are more complex than the characteristic functions of sign patterns of Liouville functions. In Subsections 3.1 and 3.2, respectively, the restrictions to $\epsilon_i \in \{-1, 1\}$ is used to construct the characteristic functions for single-sign and double-sign patterns. This idea is extended to the general k-sign patterns of Mobius function.

This idea is extended to the general k-sign patterns characteristic function of k-tuple of Mobius function values

$$(\mu(n + a_0), \mu(n + a_1), \ldots, \mu(n + a_{k-1})) = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}). \tag{3.5}$$

Lemma 3.3. *Let $n \in \mathbb{N}$ be an integer, and let $\mu(n) \in \{-1, 0, 1\}$ be the Mobius function. If $\boldsymbol{a} = (a_0, a_1, \ldots, a_{k-1})$ is an integer k-tuple, and $\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1})$ is a k-sign pattern, then,*

$$\begin{aligned} \mu(\boldsymbol{a}, \boldsymbol{\epsilon}, n) &= \prod_{0 \leq i < k} \left(\frac{1 \pm \mu(n + a_i)}{2}\right) \mu^2(n + a_i) \\ &= \begin{cases} 1 & \text{if } \mu(n + a_0) = \epsilon_0, \ldots, \ \lambda(n + a_{k-1}) = \epsilon_{k-1}, \\ 0 & \text{if } \mu(n + a_0) \neq \epsilon_0, \ldots, \ \mu(n + a_{k-1}) \neq \epsilon_{k-1}, \end{cases} \end{aligned} \tag{3.6}$$

is the characteristic functions of the subset of integers

$$\mathcal{N}_\mu(\boldsymbol{a}, \boldsymbol{\epsilon}) = \{n \geq 1 : \mu(n + a_0) = \epsilon_0, \mu(n + a_1) = \epsilon_1, \ldots \ \mu(n + a_{k-1}) = \epsilon_{k-1}\}. \tag{3.7}$$

The restricted k-sign pattern characteristic function automatically vanish on any sequence of Mobius values that has one or more zeros values. For example, if $\boldsymbol{\epsilon} = (\epsilon_0 = 0, \epsilon_1 = \pm 1, \ldots, \epsilon_{k-1} = \pm 1)$, then $\mu(\mathbf{a}, \boldsymbol{\epsilon}, n) = 0$.

3.4 Single-Sign Patterns Counting Functions

The single-sign patterns $\mu(n) = 1$ and $\mu(n) = -1$ have single-sign pattern counting functions have the simplest analysis.

Lemma 3.4. *If $\mu(n) \in \{-1,1\}$ is the Mobius function, then,*

$$\sum_{\substack{n\leq x\\ \mu(n)=1}} 1 = \frac{1}{2}\frac{x}{\zeta(2)} + O\left(xe^{-c\sqrt{\log x}}\right),$$

and

$$\sum_{\substack{n\leq x\\ \mu(n)=-1}} 1 = \frac{1}{2}\frac{x}{\zeta(2)} + O\left(xe^{-c\sqrt{\log x}}\right).$$

Proof. Use the indicator function, Lemma 3.1, to write the single-sign pattern counting functions as

$$\begin{aligned} \mathcal{N}_\mu^{\pm}(x) &= \sum_{\substack{n\leq x\\ \mu(n)=\pm1}} 1 \\ &= \sum_{n\leq x}\left(\frac{1\pm\mu(n)}{2}\right)\mu^2(n) \\ &= \frac{1}{2}\sum_{n\leq x}\mu^2(n) \pm \frac{1}{2}\sum_{n\leq x}\mu(n), \end{aligned} \tag{3.8}$$

since $\mu^{2k+1}(n) = \mu(n)$ for any $k \in \mathbb{Z}$. The final asymptotic formulas follow from Theorem 4.3, and Theorem 4.5. ■

In terms of these functions, the summatory Mobius function has the asymptotic formula

$$\begin{aligned} \mathcal{N}_\mu(x) &= \sum_{n\leq x}\mu(n) \\ &= \mathcal{N}_\mu^{+}(x) - \mathcal{N}_\mu^{-}(x) \\ &= \left(\left(\frac{1}{2}\frac{x}{\zeta(2)}\right) + O\left(xe^{-c\sqrt{\log x}}\right)\right) - \left(\left(\frac{1}{2}\frac{x}{\zeta(2)}\right) + O\left(xe^{-c\sqrt{\log x}}\right)\right) \\ &= O\left(xe^{-c\sqrt{\log x}}\right). \end{aligned} \tag{3.9}$$

Basically, it is a different form of the Prime Number Theorem, confer (2.13) to (2.16) for some details.

The same principle is applied to the double-sign patterns $(\mu(n), \mu(n+t)) = (\pm1, \pm1)$ to derive the extended results provided here.

3.5 Trivial Double-Sign Patterns Counting Functions

Let $\epsilon_i \in \{-1,0,1\}$, and $(\mu(n), \mu(n+t) = (\epsilon_0, \epsilon_1)$ denote an arbitrary double-sign pattern. The trivial double-sign patterns

$$(-1,0), \quad (0,-1), \quad (0,0), \quad (0,1), \quad (1,0), \tag{3.10}$$

do not contribute to the autocorrelation function. Nevertheless, it is sometimes required to quantify the natural densities of these double-sign patterns.

Lemma 3.5. *For a large number x, the zero double-sign pattern $(\epsilon_0, \epsilon_1) = (0, 0)$ counting function is given by*

$$\mathcal{N}_\mu^{00}(t,x) = \sum_{\substack{n \le x \\ \mu(n)\neq 0,\ \mu(n+t)\neq 0}} 1 = s_2 x + O\left(x^{2/3}\right),$$

where $s_2 = 1 - 2\zeta(2)^{-1} + s_1 > 0$ is a constant. Further, the natural density of the double zero pattern $(0, 0)$ is

$$\delta^{00}(t) = 1 - 2\zeta(2)^{-1} + s_1.$$

***Proof*.** The counting function for the zero double-sign pattern $(\epsilon_0, \epsilon_1) = (0, 0)$ is

$$\begin{aligned} \mathcal{N}_\mu^{00}(t,x) &= \sum_{n \le x} \left(1-\mu^2(n)\right)\left(1-\mu^2(n+t)\right) \\ &= \sum_{n \le x} \left(1-\mu^2(n)-\mu^2(n+t)+\mu^2(n)\mu^2(n+t)\right) \\ &= \sum_{n \le x} 1 - \sum_{n \le x} \mu^2(n) - \sum_{n \le x} \mu^2(n+t) + \sum_{n \le x} \mu^2(n)\mu^2(n+t) \\ &\geq 0. \end{aligned} \tag{3.11}$$

The last four finite sums have the following evaluations or estimates.

1. $\sum_{n \le x} 1 = [x]$,
2. $\sum_{n \le x} \mu^2(n) = \zeta(2)^{-1}x + O\left(x^{1/2}\right)$, see Theorem 4.3.
3. $\sum_{n \le x} \mu^2(n+t) = \zeta(2)^{-1}x + O\left(x^{1/2}\right)$, see Theorem 4.3.
4. $\sum_{n \le x} \mu^2(n)\mu^2(n+t) = s_1 x + O\left(x^{2/3}\right)$, see Theorem 4.4,

where $[x]$ is the largest integer function, and $s_1 = s_1(t) > 0$ is a constant. Summing these evaluations or estimates verifies the claim for $\mathcal{N}_\mu^{00}(t,x) \geq 0$. ■

Lemma 3.6. *For a large number x, the zero double-sign pattern $(\epsilon_0, \epsilon_1) = (0, 0)$ counting function is given by*

$$\mathcal{N}_\mu^{01}(t,x) = \sum_{\substack{n \le x \\ \mu(n)=0,\ \mu(n+t)=1}} 1 = s_3 x + O\left(x^{2/3}\right), \tag{3.12}$$

where $s_3 = (\zeta(2)^{-1} - s_1)/2 > 0$, and $c > 0$ is an absolute constant. Further, the natural density of the double zero pattern $(0, 1)$

$$\delta^{01}(t) = (\zeta(2)^{-1} - s_1)/2.$$

Proof. The counting function for double-sign pattern $(\epsilon_0, \epsilon_1) = (0, 1)$ is

$$\begin{aligned} 2\mathcal{N}_\mu^{01}(t,x) &= \sum_{n \le x} \left(1-\mu^2(n)\right)\left(1+\mu(n+t)\right)\mu^2(n+t) \qquad (3.13) \\ &= \sum_{n \le x} \mu^2(n+t) - \sum_{n \le x} \mu^2(n)\mu^2(n+t) \\ &\qquad + \sum_{n \le x} \mu^3(n+t) - \sum_{n \le x} \mu^2(n)\mu^3(n+t) \\ &\ge 0. \end{aligned}$$

The last four finite sums have the following evaluations or estimates.

1. $\sum_{n \le x} \mu^2(n+t) = \zeta(2)^{-1}x + O\left(x^{1/2}\right),$ see Theorem 4.3,
2. $\sum_{n \le x} \mu^2(n)\mu^2(n+t) = s_0 x + O\left(x^{2/3}\right),$ see Theorem 4.4,
3. $\sum_{n \le x} \mu^3(n+t) = O\left(xe^{-c\sqrt{\log x}}\right),$ see Theorem 4.1,
4. $\sum_{n \le x} \mu^2(n)\mu^3(n+t) = O\left(xe^{-c\sqrt{\log x}}\right),$ see Lemma 4.1,

where $c > 0$ is an absolute constant, and where $s_0 = s_0(t) > 0$ is a constant. Summing these evaluations or estimates verifies the claim for $\mathcal{N}_\mu^{01}(t,x) \ge 0$. ■

The same counting function and natural density given in Lemma 3.6 applies to any of the trivial double-sign patterns $(\mu(n), \mu(n+t)) = (0,-1)$, $(1,0)$,$(-1,0)$. The verification is similar, mutatis mutandis. The numerical value of these densities are the followings.

1. $s_1 = 0.322634616605433963 47\ldots,$ computed in Example 4.1,
2. $s_2 = 1 - 2\zeta(2)^{-1} + s_1 = 0.106780412897381\ldots,$
3. $s_3 = (\zeta(2)^{-1} - s_1)/2 = 0.142646242624296\ldots.$

3.6 Double-Sign Patterns Counting Functions

The double-sign pattern counting functions are defined by

$$\mathcal{N}_\mu^{++}(t,x) = \sum_{\substack{n \le x \\ \mu(n)=1,\, \mu(n+t)=1}} 1 = \sum_{\substack{n \le x \\ n \in \mathcal{N}_\mu^{++}(t)}} 1, \qquad (3.14)$$

$$\mathcal{N}_\mu^{+-}(t,x) = \sum_{\substack{n \le x \\ \mu(n)=1,\, \mu(n+t)=-1}} 1 = \sum_{\substack{n \le x \\ n \in \mathcal{N}_\mu^{+-}(t)}} 1, \qquad (3.15)$$

$$\mathcal{N}_\mu^{-+}(t,x) = \sum_{\substack{n\le x\\ \mu(n)=-1,\ \mu(n+t)=1}} 1 = \sum_{\substack{n\le x\\ n\in\mathcal{N}_\mu^{-+}(t)}} 1, \tag{3.16}$$

$$\mathcal{N}_\mu^{--}(t,x) = \sum_{\substack{n\le x\\ \mu(n)=-1,\ \mu(n+t)=-1}} 1 = \sum_{\substack{n\le x\\ n\in\mathcal{N}_\mu^{--}(t)}} 1. \tag{3.17}$$

The double-sign pattern counting functions (3.14) to (3.17) are precisely the counting functions associated with the subsets of integers

1. $\mathcal{N}_\mu^{++}(t) \subset \mathbb{N}$,
2. $\mathcal{N}_\mu^{+-}(t) \subset \mathbb{N}$,
3. $\mathcal{N}_\mu^{-+}(t) \subset \mathbb{N}$,
4. $\mathcal{N}_\mu^{--}(t) \subset \mathbb{N}$,

defined in (3.4). In terms of the double-sign pattern counting functions, the Mobius autocorrelation function has form

$$\begin{aligned} \mathcal{N}_\mu(t,x) &= \sum_{n\le x}\mu(n)\mu(n+t) \\ &= \mathcal{N}_\mu^{++}(t,x) - \mathcal{N}_\mu^{+-}(t,x) + \mathcal{N}_\mu^{--}(t,x) - \mathcal{N}_\mu^{-+}(t,x). \end{aligned} \tag{3.18}$$

The next result is required to complete the analysis of the asymptotic formula for the density constants $\delta_\mu^{\pm\pm}(t)$, which is completed in the next section.

Lemma 3.7. *Let x be a large number, and let $t \ne 0$ be a fixed integer. Then,*

$$\mathcal{N}_\mu^{\pm\pm}(t,x) = \frac{1}{4}s_1 x + O\left(\frac{x}{(\log x)^c}\right),$$

where $s_1 = s_1(t) > 0$, and $c > 0$ are constants.

Proof. Without loss in generality, consider the double-sign pattern $(\mu(n), \mu(n+t)) = (+1,+1)$. Now, use Lemma 3.2 to express the double-sign pattern counting function as

$$\begin{aligned} 4\mathcal{N}_\mu^{++}(t,x) &= \sum_{n\le x}\mu^{++}(t,n) \\ &= \sum_{n\le x}\mu^2(n)\mu^2(n+t)\left(1+\mu(n)\right)\left(1+\mu(n+t)\right) \\ &= \sum_{n\le x}\mu^2(n)\mu^2(n+t)\left(1+\mu(n)+\mu(n+t)+\mu(n)\mu(n+t)\right) \\ &= \sum_{n\le x}\mu^2(n)\mu^2(n+t) + \sum_{n\le x}\mu^3(n)\mu^2(n+t) \\ &\qquad + \sum_{n\le x}\mu(n)^2\mu^3(n+t) + \sum_{n\le x}\mu^3(n)\mu^3(n+t) \\ &\ge 0. \end{aligned} \tag{3.19}$$

Since $\mu^{2k+1}(n) = \mu(n)$ for $k \ge 0$, the last four finite sums have the following evaluations or estimates.

1. $\sum_{n \le x} \mu^2(n)\mu^2(n+t) = s_0(t)x + O\left(x^{2/3}\right),$ see Theorem 4.4,

2. $\sum_{n \le x} \mu^3(n)\mu^2(n+t) = O\left(\frac{x}{(\log x)^c}\right),$ see Lemma 4.1,

3. $\sum_{n \le x} \mu^2(n)\mu^3(n+t) = O\left(\frac{x}{(\log x)^c}\right),$ see Lemma 4.1,

4. $\sum_{n \le x} \mu^3(n)\mu^3(n+t) = \sum_{n \le x} \mu(n)\mu(n+t) = O\left(\frac{x}{(\log x)^c}\right),$ see Theorem 4.10

where $c > 0$ is an arbitrary constant, and $s_1 = s_1(t) > 0$ is a constant, Summing these evaluations or estimates verifies the claim for $\mathcal{N}_\mu^{++}(t,x) \ge 0$. The verifications for the next three double-sign pattern counting functions $\mathcal{N}_\mu^{+-}(t,x) \ge 0$, $\mathcal{N}_\mu^{-+}(t,x) \ge 0$, and $\mathcal{N}_\mu^{--}(t,x) \ge 0$ are similar. ■

3.7 k-Sign Patterns Counting Functions

The k-sign pattern of each quasi-consecutive values of the Mobius function

$$(\mu(n+a_0), \mu(n+a_1), \ldots, \mu(n+a_{k-1})) = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}), \tag{3.20}$$

where $\epsilon_i \in \{-1, 1\}$, and $n \in [1, x]$, has a k-tuple counting function of the form

$$\mathcal{N}_\mu(\boldsymbol{a}, \boldsymbol{\epsilon}, x) = \#\{n \le x : (\mu(n+a_0), \ldots, \mu(n+a_{k-1})) = (\epsilon_0, \ldots, \epsilon_{k-1})\}. \tag{3.21}$$

The precise asymptotic formula for these counting functions has a lengthy derivation provided below.

Theorem 3.1. *Let x be a large number, and let $\mu : \mathbb{Z} \longrightarrow \{-1, 0, 1\}$ be the Mobius function. If $\boldsymbol{a}$ is an integer k-tuple, and $\boldsymbol{\epsilon}$ is a k-sign pattern, then,*

$$\mathcal{N}_\mu(\boldsymbol{a}, \boldsymbol{\epsilon}, x) = \frac{s_k}{2^k}x + O\left(\frac{x}{(\log x)^{2c}}\right),$$

where $c > 0$ is an arbitrary constant.

Proof. Fix an integer $k \ll \log x$, and an integer k-tuple $\boldsymbol{a} = (a_0, a_1, \ldots, a_{k-1})$. Without loss in generality consider the k-sign pattern $\boldsymbol{\epsilon} = (+1, +1, \ldots, +1)$. Now, use Lemma 3.3 to express the double-sign pattern counting function as

$$
\begin{aligned}
2^k \mathcal{N}_\mu(\boldsymbol{a}, \boldsymbol{\epsilon}, x) &= \sum_{n \leq x} \mu(\boldsymbol{a}, \boldsymbol{\epsilon}, n) \qquad (3.22)\\
&= \sum_{n \leq x,\, 0 \leq i < k} \prod \left(\frac{1 + \mu(n + a_i)}{2} \right) \\
&= \sum_{n \leq x} 1 + \sum_{0 \leq i < k,\, n \leq x} \sum \mu(n + a_i) + \sum_{\substack{0 \leq i,j < k,\, n \leq x \\ i \neq j}} \sum \mu(n + a_i)\mu(n + a_j) \\
&\quad + \sum_{\substack{0 \leq h,i,j < k,\, n \leq x \\ h \neq i \neq j}} \sum \mu(n + a_h)\mu(n + a_i)\mu(n + a_j) \\
&\quad + \cdots + \sum_{n \leq x} \mu(n + a_0)\mu(n + a_1) \cdots \mu(n + a_{k-1}) \\
&\geq 0.
\end{aligned}
$$

For the integer k-tuple $0 \leq a_0 < a_1 < \cdots < a_{k-1} < x$, these finite sums have the following evaluations or estimates.

1. $\displaystyle \sum_{n \leq x} 1 = [x],$

2. $\displaystyle \sum_{0 \leq i < k,\, n \leq x} \sum \mu(n + a_i) = O\left(\binom{k}{1} x e^{-c\sqrt{\log x}} \right),$ see Theorem 4.5.

3. $\displaystyle \sum_{\substack{0 \leq i,j < k,\, n \leq x \\ i \neq j}} \sum \mu(n + a_i)\mu(n + a_j) = O\left(\binom{k}{2} \frac{x}{(\log x)^{2c}} \right),$ see Theorem 4.10,

4. $\displaystyle \sum_{\substack{0 \leq h,i,j < k,\, n \leq x \\ h \neq i \neq j}} \sum \mu(n + a_h)\lambda(n + a_i)\mu(n + a_j) = O\left(\binom{k}{3} \frac{x}{(\log x)^{2c}} \right),$

$\vdots \qquad \vdots \qquad \vdots$

5. $\displaystyle \sum_{n \leq x} \mu(n + a_0)\mu(n + a_1) \cdots \mu(n + a_{k-1}) = O\left(\frac{x}{(\log x)^{2c}} \right),$

where $[x]$ is the largest integer function, $c > 0$ is an arbitrary constant. There are $k+1$ single and double finite sums, in equation (3.22), and in the list. Moreover, the binomial coefficient satisfies the inequality

$$(k+1)\binom{k}{i} \leq 2^k \qquad (3.23)$$

for $i \leq k$. Therefore, the total sum of the exact evaluations or estimates listed as (1) to (6) has the asymptotic formula

$$0 \leq 2^k \mathcal{N}_\mu(\boldsymbol{a}, \boldsymbol{\epsilon}, x) = [x] + O\left((k+1)\binom{k}{k/2} \frac{x}{(\log x)^{2c}} \right) \qquad (3.24)$$

as claimed. The verifications for all the other k-sign patterns $\boldsymbol{\epsilon} \neq (+1, +1, \ldots, +1)$ are similar. ■

3.8 Equidistribution of Double-Sign Patterns

The nontrivial result for the summatory Mobius function

$$\sum_{n \leq x} \mu(n) = \mathcal{N}_\mu^+(x) - \mathcal{N}_\mu^-(x) = O\left(xe^{-c\sqrt{\log x}}\right) \tag{3.25}$$

has no main term. It vanishes because the number of single-sign patterns $\mathcal{N}_\mu^+(x) = \#\{n \leq x : \mu(n) = 1\}$ and $\mathcal{N}_\mu^-(x) = \#\{n \leq x : \mu(n) = -1\}$ have the same cardinality. This implies that the single-sign patterns are equidistributed on the interval $[1, x]$, and each has the natural density $\delta_\mu^\pm = 3/\pi^2$, see (3.14) to (3.17) for more detail. This idea is extended in the proof of the equidistribution of the double-sign patterns.

Recall that $\mathcal{N}_\mu^{\pm\pm}(t, x) = \#\{n \leq x : \mu(n) = \pm 1, \mu(n + t) = \pm 1\}$, and the natural density of a double-sign pattern is defined by

$$\delta_\mu^{\pm\pm}(t) = \lim_{x \to \infty} \frac{\#\{n \leq x : \mu(n) = \pm 1, \mu(n + t) = \pm 1\}}{x}. \tag{3.26}$$

Theorem 3.2. *Let x be a large number, and let $t \neq 0$ be a fixed integer. Then, the double-sign patterns $++$, $+-$, $-+$, and $--$ of the Mobius pair $\mu(n), \mu(n + t)$ are equidistributed on the interval $[1, x]$. In particular, each double-sign pattern has the natural density*

$$\delta_\mu^{\pm\pm}(t) = \frac{1}{4} s_1,$$

where $s_1 = s_1(t) > 0$ is a constant.

Proof. By Lemma 3.7, $\mathcal{N}_\mu^{\pm\pm}(t, x) = s_1 x/4 + O(x(\log x)^{-c})$. Accordingly, the limit of the proportion of double-sign pattern

$$\begin{aligned} \delta_\mu^{\pm\pm}(t) &= \lim_{x \to \infty} \frac{\#\{n \leq x : \mu(n) = \pm 1, \mu(n + t) = \pm 1\}}{x} \\ &= \lim_{x \to \infty} \frac{s_1 x + O(x(\log x)^{-c})}{4x} \\ &= \frac{1}{4} s_1. \end{aligned}$$

■

This proves that the double-sign patterns $++$, $+-$, $-+$, and $--$ are equidistributed on the interval $[1, x]$ as $x \to \infty$.

Example 3.1. Let $t = 1$. The constant $s_1 = s_1(1) = 0.322634\ldots$ for the double-sign patterns $\mu(n) = \pm 1, \mu(n+1) = \pm 1$ is computed in Example 4.1. Thus, by Theorem 3.2, in any sufficiently large interval $[1, x]$, the number of double-sign patterns

$$\mathcal{N}_\mu^{\pm\pm}(t, x) = \delta_\mu^{\pm\pm}(t)x + o(x) = \frac{0.322634\ldots}{4} x + O\left(\frac{x}{(\log x)^c}\right). \tag{3.27}$$

Consequently, as $x \to \infty$, the main term of the autocorrelation function

$$\begin{aligned}\sum_{n\le x}\mu(n)\mu(n+1) &= \mathcal{N}_\mu^{++}(t,x)-\mathcal{N}_\mu^{+-}(t,x)+\mathcal{N}_\mu^{--}(t,x)-\mathcal{N}_\mu^{-+}(t,x)\\ &= O\left(\frac{x}{(\log x)^c}\right). \qquad (3.28)\end{aligned}$$

vanished. For $x = 10^4$, the actual value of the autocorrelation function is

$$\sum_{n\le x}\mu(n)\mu(n+1) = 12, \qquad (3.29)$$

and the actual values of the double-sign counting functions are tabulated below.

$\mu(n)$	$\mu(n+1)$	$\mathcal{N}_\mu^{\pm\pm}(1,x)$
$+1$	$+1$	$3228/4$
$+1$	-1	$3152/4$
-1	$+1$	$3282/4$
-1	-1	$3256/4$

The differences among the double-sign counting functions $\mathcal{N}_\mu^{\pm\pm}(1,x)$ seem to be properties of the biases toward the different double-sign patterns. The comparative analysis of the double-sign patterns can be a topic of future research.

3.9 Equidistribution of k-Sign Patterns

The nontrivial result for the summatory Mobius function

$$\sum_{n\le x}\mu(n) = \mathcal{N}_\mu^{+}(x)-\mathcal{N}_\mu^{-}(x) = O\left(xe^{-c\sqrt{\log x}}\right) \qquad (3.30)$$

has no main term. It vanishes because the number of single-sign patterns $\mathcal{N}_\mu^{+}(x) = \#\{n\le x : \mu(n)=1\}$ and $\mathcal{N}_\mu^{-}(x) = \#\{n\le x : \mu(n)=-1\}$ have the same cardinality. This implies that the single-sign patterns are equidistributed on the interval $[1,x]$, and each has the natural density $\delta_\mu^{\pm} = 1/2$, see Lemma 3.4 for more detail. This idea is extended in the proof of the equidistribution of the double-sign patterns, triple-sign patterns, ..., and in general to k-sign patterns.

Recall that for any fixed $\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1})$, where $\epsilon_i \in \{-1, 1\}$, the k-sign pattern counting function is defined by

$$\mathcal{N}_\mu(\boldsymbol{a},\boldsymbol{\epsilon},x) = \#\{n\le x : (\mu(n+a_0),\ldots,\mu(n+a_{k-1})) = (\epsilon_0,\ldots,\epsilon_{k-1})\}, \qquad (3.31)$$

and the natural density of a k-sign pattern is defined by

$$\delta_\mu(\boldsymbol{a},\boldsymbol{\epsilon}) = \lim_{x\to\infty}\frac{\#\{n\le x : (\mu(n+a_0),\ldots,\mu(n+a_{k-1})) = (\epsilon_0,\ldots,\epsilon_{k-1})\}}{x}. \qquad (3.32)$$

Theorem 3.3. *Let x be a large number, and let $\mu : \mathbb{Z} \longrightarrow \{-1, 0, 1\}$ be the Mobius function. If $\boldsymbol{a} = (a_0, a_1, \ldots, a_{k-1})$, and $\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1})$, where $\epsilon_i \in \{-1, 1\}$, then, the k-sign pattern of the Mobius function μ are equidistributed on the interval $[1, x]$. In particular, each k-sign pattern has the natural density*

$$\delta_\mu(\boldsymbol{a}, \boldsymbol{\epsilon}) = \frac{s_k}{2^k},$$

where $s_k > 0$ is a constant.

Proof. Observe that for large $x \geq 1$, $\mathcal{N}_\mu(\boldsymbol{a}, \boldsymbol{\epsilon}, x) > 0$, this follows from Theorem 3.1. Next, compute the limit of the proportion of k-sign pattern

$$\begin{aligned} \delta_\mu(\boldsymbol{a}, \boldsymbol{\epsilon}) &= \lim_{x \to \infty} \frac{\#\{n \leq x : (\mu(n + a_0), \ldots, \mu(n + a_{k-1})) = (\epsilon_0, \ldots, \epsilon_{k-1})\}}{x} \\ &= \lim_{x \to \infty} \frac{s_k x + O\left(x(\log x)^{-2c}\right)}{2^k x} \\ &= \frac{s_k}{2^k}. \end{aligned}$$

This proves that the k-sign patterns $\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1})$ are equidistributed on the interval $[1, x]$ as $x \to \infty$. ■

4 Appendix

4.1 Average Orders and Mean Value of Mobius Function

Theorem 4.1. *If $\mu : \mathbb{N} \longrightarrow \{-1, 0, 1\}$ is the Mobius function, then, for any large number x, the following statements are true.*

(i) $\displaystyle\sum_{n \leq x} \mu(n) = O\left(x e^{-c\sqrt{\log x}}\right)$, *unconditionally,*

(ii) $\displaystyle\sum_{n \leq x} \mu(n) = O\left(x^{1/2+\varepsilon}\right)$, *conditional on the RH,*

where $c > 0$ is an absolute constant, and $\varepsilon > 0$ is an arbitrarily small number.

Proof. See [4, p. 6], [21, p. 182], et alii. ■

There are many sharp bounds of the summatory function of the Mobius function, say, $O(xe^{-c(\log x)^\delta})$, and the conditional estimate $O(x^{1/2+\varepsilon})$ presupposes that the nontrivial zeros of the zeta function $\zeta(\rho) = 0$ in the critical strip $\{0 < \Re e(s) < 1\}$ are of the form $\rho = 1/2 + it, t \in \mathbb{R}$. However, the simpler notation will be used whenever it is convenient.

Theorem 4.2. *Let x be a large number, and let $q \ll (\log x)^B$, where $B \geq 0$ is an arbitrary constant. If $1 \leq a < q$ are relatively prime integers, then,*

$$\sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \mu(n) = O\left(\frac{x}{(\log x)^C}\right),$$

where $C = C(B) > 0$ is a constant.

Proof. A sketch of the proof appears in [21, p. 385]. ■

There are several mean values and equidistribution results for arithmetic function μ over arithmetic progressions of level of distribution $\theta < 1/2$. The best known case is the Bombieri-Vinogradov theorem, see [4, Theorem 15.4], the case for the Mobius function is proved in [27, Theorem 1] and [24] states the following.

Corollary 4.1. ([24, Corollary 1] *Let $a \geq 1$ be a fixed parameter, and let $x \geq 1$ be a large number. If $C > 0$ is a constant, then*

$$\sum_{q \leq x^{1/2}/\log^B x} \max_{a \bmod q} \max_{z \leq x} \left| \sum_{\substack{n \leq z \\ n \equiv a \bmod q}} \mu(n) \right| \ll \frac{x}{(\log x)^C}, \tag{4.1}$$

where the constant $B > 0$ depends on C.

4.2 Nonlinear Autocorrelation Functions Results

The number of squarefree integers have the following asymptotic formulas.

Theorem 4.3. *Let $\mu : \mathbb{Z} \longrightarrow \{-1, 0, 1\}$ be the Mobius function. Then, for any sufficiently large number $x \geq 1$,*

$$\sum_{n \leq x} \mu^2(n) = \frac{6}{\pi^2} x + O\left(x^{1/2}\right).$$

Proof. Use identity $\mu(n)^2 = \sum_{d^2 | n} \mu(d)$, and other elementary routines, or confer to the literature. ■

The constant coincides with the density of squarefree integers. Its approximate numerical value is

$$\frac{6}{\pi^2} = \prod_{p \geq 2} \left(1 - \frac{1}{p^2}\right) = 0.607988295164627617135754\ldots, \tag{4.2}$$

where $p \geq 2$ ranges over the primes. The remainder term

$$E(x) = \sum_{n \leq x} \mu^2(n) - \frac{6}{\pi^2} x \tag{4.3}$$

is a topic of current research, its optimum value is expected to satisfies the upper bound $E(x) = O(x^{1/4+\varepsilon})$ for any small number $\varepsilon > 0$. Currently, $E(x) = O\left(x^{1/2} e^{-c\sqrt{\log x}}\right)$ is the best unconditional remainder term.

Assuming $t \neq 0$, the earliest result for the autocorrelation of the squarefree indicator function $\mu^2(n)$ appears to be

$$\sum_{n \leq x} \mu^2(n) \mu^2(n+t) = cx + O\left(x^{2/3}\right), \tag{4.4}$$

where $c > 0$ is a constant, this is proved in [16]. Except for minor adjustments, the generalization to the k-tuple autocorrelation function has nearly the same structure.

Theorem 4.4. *Let $q \neq 0$, $a_0, a_1, \ldots, a_{k-1}$ be small integers, such that $0 \leq a_0 < a_1 < \cdots < a_{k-1}$. Let x be a large number, and let $\mu : \mathbb{Z} \longrightarrow \{-1, 0, 1\}$ be the Mobius function. Then,*

$$\sum_{n \leq x} \mu^2(n + a_0)\mu^2(n + a_1) \cdots \mu^2(n + a_{k-1}) = s_k x + O\left(x^{2/3+\varepsilon}\right),$$

where the constant is given by the convergent product

$$s_k = \prod_{p \geq 2} \left(1 - \frac{\varpi(p)}{p^2}\right) > 0, \tag{4.5}$$

and

$$\varpi(p) = \#\{m \leq p^2 : qm + a_i \equiv 0 \bmod p^2 \text{ for } i = 0, 1, 2, ..., k-1\}. \tag{4.6}$$

The small number $\varepsilon > 0$ and the implied constant depends on $q \neq 0$.

Proof. Consult [16], [15, Theorem 1.2], and the literature for additional details. ■

Example 4.1. For the parameters $q = 2$, $a_0 = 0$, and $a_1 = 1$, the number of solutions of the system of equations is $\varpi(p) = \#\{m \leq p^2 : qm + a_i \equiv 0 \bmod p^2 = 2$ for any prime $p \geq 2$, so constant $s_1 = s_1(t)$ has the numerical value, (using $p \leq 10^5$),

$$s_1 = \prod_{p \geq 2} \left(1 - \frac{\varpi(p)}{p^2}\right) = \prod_{p \geq 2} \left(1 - \frac{2}{p^2}\right) = 0.322634616605433963 47 \ldots. \tag{4.7}$$

Lemma 4.1. *Let x be a large number, and let $\mu : \mathbb{Z} \longrightarrow \{-1, 0, 1\}$ be the Mobius function. If $t \neq 0$ is a fixed integer, then,*

$$\sum_{n \leq x} \mu(n)^2 \mu(n + t) = O\left(\frac{x}{(\log x)^c}\right),$$

where $c > 0$ is an arbitrary constant.

Proof. Substitute the identity $\mu(n)^2 = \sum_{d^2 | n} \mu(d)$, and switching the order of summation yield

$$\begin{aligned} \sum_{n \leq x} \mu(n)^2 \mu(n + t) &= \sum_{n \leq x} \mu(n + t) \sum_{d^2 | n} \mu(d) && (4.8) \\ &= \sum_{d^2 \leq x} \mu(d) \sum_{\substack{n \leq x \\ d^2 | n}} \mu(n + t) \\ &= \sum_{d^2 \leq x^{2\varepsilon}} \mu(d) \sum_{\substack{n \leq x \\ d^2 | n}} \mu(n + t) + \sum_{x^{2\varepsilon} < d^2 \leq x} \mu(d) \sum_{\substack{n \leq x \\ d^2 | n}} \mu(n + t), \end{aligned}$$

where $\varepsilon \in (0, 1/4)$. Applying Corollary 4.1 to the first subsum in the partition yields

$$\begin{aligned} \sum_{d^2 \leq x^{2\varepsilon}} \mu(d) \sum_{\substack{n \leq x \\ d^2 | n}} \mu(n + t) &\leq \sum_{q \leq x^{\varepsilon}} \left| \mu(d) \sum_{\substack{n \leq x \\ m \equiv b \bmod q}} \mu(m) \right| && (4.9) \\ &= O\left(\frac{x}{(\log x)^c}\right), \end{aligned}$$

where $q = d^2$. An estimate of the second subsum in the partition yields

$$\begin{aligned} \sum_{x^{2\varepsilon} < d^2 \leq x} \mu(d) \sum_{\substack{n \leq x \\ d^2 \mid n}} \mu(n+t) &\leq \sum_{x^{2\varepsilon} < d^2 \leq x} \sum_{\substack{n \leq x \\ d^2 \mid n}} 1 \\ &\ll x \sum_{d^2 \leq x} \frac{1}{d^2} \\ &\ll x^{1-\varepsilon}. \end{aligned} \tag{4.10}$$

Summing (4.9) and (4.10) completes the verification. ■

4.3 Results for the Liouville and Mobius Functions

Some standard results required in the proofs of the correlations functions are recorded in this section.

4.4 Average Orders of Mobius Functions

Theorem 4.5. *If $\mu : \mathbb{N} \longrightarrow \{-1, 0, 1\}$ is the Mobius function, then, for any large number x, the following statements are true.*

(i) $\displaystyle\sum_{n \leq x} \mu(n) = O\left(xe^{-c\sqrt{\log x}}\right)$, *unconditionally,*

(ii) $\displaystyle\sum_{n \leq x} \frac{\mu(n)}{n} = O\left(e^{-c\sqrt{\log x}}\right)$, *unconditionally,*

where $c > 0$ is an absolute constant.

Proof. See [4, p. 6], [21, p. 182], et alii. ■

4.5 Average Orders of Liouville Functions

Theorem 4.6. *If $\lambda : \mathbb{N} \longrightarrow \{-1, 1\}$ is the Liouville function, then, for any large number x, the following statements are true.*

(i) $\displaystyle\sum_{n \leq x} \lambda(n) = \zeta(1/2)x^{1/2} + O\left(xe^{-c\sqrt{\log x}}\right)$, *unconditionally,*

(ii) $\displaystyle\sum_{n \leq x} \frac{\lambda(n)}{n} = O\left(e^{-c\sqrt{\log x}}\right)$, *unconditionally,*

where $c > 0$ is an absolute constant.

The most recent research on the summatory Liouville function seems to [19].

4.6 Twisted Exponential Sums

One of the earliest result for exponential sum with multiplicative coefficients is stated below.

Theorem 4.7. ([3]) *If $\alpha \neq 0$ is a real number, and $c > 0$ is an arbitrary constant, then*

(i) $$\sup_{\alpha\in\mathbb{R}} \sum_{n\leq x} \mu(n)e^{i2\pi\alpha n} < \frac{c_1 x}{(\log x)^c}, \qquad \textit{unconditionally,}$$

(ii) $$\sup_{\alpha\in\mathbb{R}} \sum_{n\leq x} \frac{\mu(n)}{n}e^{i2\pi\alpha n} < \frac{c_2}{(\log x)^c}, \qquad \textit{unconditionally,}$$

where $c_1 = c_1(c) > 0$ and $c_2 = c_2(c) > 0$ are constants depending on c, as the number $x \to \infty$.

The same results are also valid for the Liouville function.

Theorem 4.8. ([3]) *If $\alpha \neq 0$ is a real number, and $c > 0$ is an arbitrary constant, then*

(i) $$\sup_{\alpha\in\mathbb{R}} \sum_{n\leq x} \lambda(n)e^{i2\pi\alpha n} < \frac{c_3 x}{(\log x)^c}, \qquad \textit{unconditionally,}$$

(ii) $$\sup_{\alpha\in\mathbb{R}} \sum_{n\leq x} \frac{\lambda(n)}{n}e^{i2\pi\alpha n} < \frac{c_4}{(\log x)^c}, \qquad \textit{unconditionally,}$$

where $c_3 = c_3(c) > 0$ and $c_4 = c_4(c) > 0$ are constants depending on c, as the number $x \to \infty$.

Advanced, and recent works on these exponential sums with multiplicative coefficients, and the more general exponential sums

$$\sum_{n\leq x} f(n)e^{i2\pi\alpha n} \tag{4.11}$$

where $f : \mathbb{N} \longrightarrow \mathbb{C}$ is a function, are covered in [22], [11], [1], [18], et alii.

4.7 Double Twisted Exponential Sums

Various results on discrete Fourier transform are effective in producing estimates of the autocorrelation of arithmetic functions. The specific case of the Mobius function is estimated here.

Theorem 4.9. *If x is a large integer, then*

(i) $$\sum_{s\leq x,}\sum_{n\leq x} \mu(n)\mu(n+s)e^{i2\pi ns/x} \ll \frac{x^2}{(\log x)^{2c}}, \qquad \textit{unconditionally,}$$

(ii) $$\sum_{s\leq x,}\sum_{n\leq x} \frac{\mu(n)\mu(n+s)}{n}e^{i2\pi ns/x} \ll \frac{x}{(\log x)^{2c}}, \qquad \textit{unconditionally,}$$

where $c > 0$ is an absolute constant.

4.8 k-Tuple Autocorrelation Functions

The results for the general cases of k-tuple autocorrelation functions are recorded here. The estimates for $k = 2$ have asymptotic formulae of the forms

$$\sum_{n<x} \mu(n)\mu(n+t) = O\left(\frac{x}{\sqrt{\log\log x}}\right), \tag{4.12}$$

and

$$\sum_{n<x} \lambda(n)\lambda(n+t) = O\left(\frac{x}{\sqrt{\log\log x}}\right), \tag{4.13}$$

or weaker, where $t \neq 0$ is a fixed integer, see [10, Corollary 1.5].

Theorem 4.10. *Let $\mu : \mathbb{N} \longrightarrow \{-1, 0, 1\}$ be the Mobius function, and let x be a large integer. If $a_0 < a_1 < \cdots < a_{k-1} < x$ is an integer k-tuple then,*

$$\sum_{n \leq x} \mu(n + a_0)\mu(n + a_1) \cdots \mu(n + a_{k-1}) = O\left(\frac{x}{(\log x)^{2c}}\right),$$

where $c > 0$ is an arbitrary constant.

Proof. A complete proof appears in [2, Theorem 1.1]. ■

Precisely the same result is valid for the Liouville function. Except for minor notational changes, and implied constant, everything is exactly the same.

Theorem 4.11. *Let $\lambda : \mathbb{N} \longrightarrow \{-1, 1\}$ be the Liouville function, and let x be a large integer. If $a_0 < a_1 < \cdots < a_{k-1} < x$ is an integer k-tuple then,*

$$\sum_{n \leq x} \lambda(n + a_0)\lambda(n + a_1) \cdots \lambda(n + a_{k-1}) = O\left(\frac{x}{(\log x)^{2c}}\right),$$

where $c > 0$ is an arbitrary constant.

Proof. A complete proof appears in [2, Theorem 1.2]. ■

4.9 Distribution Functions

For any fixed integer $k > 4$, the k-tuples $\mu(n + a_0), \mu(n + a_1), \ldots, \mu(n + a_{k-1})$ of Mobius values are not random, but pseudorandom or quasirandom. For example, the k-tuple

$$-1, -1, -1, -1, \mu(n + a_4), \mu(n + a_1), \ldots, \mu(n + a_{k-1}), \tag{4.14}$$

and infinitely many other similarly structured k-tuples are not possible. This property seems to preempt the effect of the Linear Independence Conjecture on the summatory Mobius function. Assuming, the LI, the limits

$$\liminf_{x\to\infty} \frac{\sum_{n\leq x} \mu(n)}{x^{1/2}} = -\infty \quad \text{and} \quad \liminf_{x\to\infty} \frac{\sum_{n\leq x} \mu(n)}{x^{1/2}} = \infty \tag{4.15}$$

were proved in [12], and refinements in [17]. This conjecture seems to imply that $\sum_{n\leq x}\mu(n) = O\left(x^{1/2+\varepsilon}\right)$, where $\varepsilon > 0$. But, the Simple Zero Conjecture seems to imply that $\sum_{n\leq x}\mu(n) = O\left(x^{1/2}\right)$, see [25, Theorem 14.29] for details. The numerical data are given in [14] is not conclusive, and it supports many different conjectures on the Mertens sum.

For any fixed integer $k > 4$, the k-tuples $\lambda(n + a_0), \lambda(n + a_1), \ldots, \lambda(n + a_{k-1})$ of Liouville function values appears to be random, there are no known obstacles. For example, the k-tuple

$$-1, -1, -1, -1, \lambda(n + a_4), \lambda(n + a_1), \ldots, \lambda(n + a_{k-1}), \tag{4.16}$$

and infinitely many other similarly structured k-tuples are possible. Thus, the law of iterated logarithm for sequences of independent random variables seem to imply that

$$\liminf_{x\to\infty} \frac{\sum_{n\leq x}\lambda(n)}{\sqrt{2x\log\log x}} = -\infty \quad \text{and} \quad \liminf_{x\to\infty} \frac{\sum_{n\leq x}\lambda(n)}{\sqrt{2x\log\log x}} = \infty. \tag{4.17}$$

In synopsis, the Linear Independence Conjecture does seem to apply to the summatory Liouville function. In particular, $\sum_{n\leq x}\lambda(n) = \zeta(1/2)x^{1/2} + O\left(x^{1/2+\varepsilon}\right)$, where $\varepsilon > 0$.

The information in (4.14), (4.15), and the information in (4.16), (4.17) seems to imply that the random or pseudorandom variables

$$L(x) = \sum_{n\leq x}\lambda(n) \qquad \text{and} \qquad M(x) = \sum_{n\leq x}\mu(n) \tag{4.18}$$

have different distribution functions.

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