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Case Study of Undergraduate Research Projects in Vector Analysis

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Case Studies for Undergraduate Research Projects in Vector Calculus

Abstract. This paper presents two examples of undergraduate research projects in vector calculus accomplished at a community college. The author whose name appears first on this paper supervised the projects. The projects were completed by students pursuing an associate degree in engineering in their sophomore year. One project aimed to obtain explicit formulas for the curvature of a curve defined implicitly in rectangular or polar coordinates in a plane. The second project attempted to develop an alternative procedure for finding potential function for a vector field in space based on simultaneous integration. Participation in these projects had a positive impact on the students' academic preparedness, as well as empowering them as individuals.

Keywords. Undergraduate research projects, Vector analysis, Curvature, Potential function

1. Introduction

With the growing robotization of production and management processes in the modern workplace, only high quality education and training will lead workers towards enjoying secure employment and a stable income. Orienting towards a future that features extensive automation, pursuing STEM disciplines (Science, Technology, Engineering and Mathematics) is highly promising and should be rewarding. It is important to note that regardless of whichever STEM discipline a student pursues, mathematics is a must as it is the gateway to all STEM careers.

In this situation, two-year colleges play a special role. They allow an individual relatively quick and inexpensive entry to a new area that is attractive, ensures good wages, and fits the desires and abilities of the individual. We may recall Steve Jobs saying: ". . . A way <should> be found to train more American engineers. Apple had 700,000 factory workers employed in China … and that was because it needed 30,000 engineers on-site to support those workers. . . These factory engineers did not have to be PhDs or geniuses; they simply needed to have basic engineering skills for manufacturing. Tech schools, community colleges, or trade schools could train them," Isaacson (2011, p. 546). Research experience adds even more value to the education and training.

We conducted this research at a two-year community college that is an integral part of a large city university. The parent university serves more than 500,000 students who attend its 24 colleges. This college offers 29 associate degree programs and five certificate programs. Its students can continue to associated four-year colleges or baccalaureate studies at other institutions with ease, keeping all of their earned credits. The college has specially designed programs to support the students' education, and to provide mentorship as well as financial aid. One of these is the City University of New York Research Scholars Program (CRSP) that provides funded laboratory experiences for students on track towards their associate. Another program is the Collegiate Science and Technology Entry Program (CSTEP). It provides support in STEM disciplines, helps students acquire research experiences and opportunities to participate in conferences, and offers academic advisement, as well as career and financial counseling.

The research participants were selected on a voluntary basis from the pool of the first author's students who took part in these programs. These students expressed interest in mathematics and were willing to study beyond the requirements of the curriculum. They were pursuing varying engineering disciplines, and by the time they joined our research, they were either taking calculus III (multiple integrals and the Green's, Stokes' and the Divergence theorems) or continuing their mathematics preparation via courses in vector analysis and differential equations. During the project, the participants were asked to study relevant theoretical material and prepare a collection of solved problems. As participants in the CRSP or STEP programs, they received scholarships and were asked to present their work at conferences.

This paper will address two projects. Project #1, completed by Ms. Mendez, aimed to obtain an explicit formula for the curvature of a curve in plane defined implicitly in rectangular or polar coordinates. Project #2, completed by Mr. Baez Lara and Mr. Diao, attempted to develop an alternative procedure for finding potential function in space based on simultaneous integration. This project won the first prize in the "Technology and Mathematics" category at the 25th Annual CSTEP Statewide Student Conference, held on April 8, 2017 at the Sagamore Resort, NY.

The paper is organized as follows. Sections 2 and 3 present the Curvature and the Potential function projects, respectively. Section 4 contains conclusive remarks.

2. Project 1. Curvature of implicitly defined curve in plane

This section presents explicit formulas for finding the curvatures of the curves in plane defined as implicit functions in rectangular or polar coordinates. Intuitively, the curvature measures the amount of curving as a point is tracing the path, thus presenting to what extent a path of a curve differs from the straight line. The curvature is an inverse of the radius of the osculating circle that is tangent to the curve and provides a second order of contact with a curve (a tangent line provides just a first order contact). The curvature, together with the speed, determines the normal acceleration and thus, the centripetal and centrifugal forces occurring in the process of the motion. These properties of the curvature explain the importance of enhancement of the techniques of its computations for different ways of the curve's definition.

Geometrically, the curvature may be defined as

$$k = \left|\frac{d\phi}{ds}\right|, \qquad (1)$$

where k is curvature, ϕ is the angle of inclination of the tangent line, ds is the differential of the length and symbol $|\cdot|$ stands for the absolute value, see Stewart (2012) for details. Thus, the curvature determines the rate of rotation of the tangent line.

The precise definition of the curvature is as follows

$$k = \left\|\frac{d\boldsymbol{T}}{ds}\right\|, \qquad (2)$$

where $\boldsymbol{T}$ is the unit tangent vector, and symbol $\|\cdot\|$ is the Euclidian norm of a vector. Recall that

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}, \qquad (3)$$

where $\boldsymbol{r}(t)$ is the position vector, and t is a parameter tracing the curve. We assume that both $\boldsymbol{r}(t)$ and $\boldsymbol{r}'(t)$ are continuous functions of parameter t and $\|\boldsymbol{r}'(t)\|\neq\boldsymbol{0}$, so that the curve is tracing in one direction.

For the curves in plane, the following formulas for curvature are in use, Thomas et al. (2014). If a curve is defined parametrically as

$$x=x(t),\ y=y(t),\ a \leq t \leq b, \qquad (4)$$

then

$$k = \frac{\left|x''y' - x'y''\right|}{((x')^2 + (y')^2)^{3/2}}, \qquad (5)$$

where differentiation is with regard to parameter *t*.

If a curve is a graph of a continuously differentiable function *y=f(x)*, then

$$k = \frac{\left|f''(x)\right|}{(1 + (f'(x))^2)^{3/2}}. \qquad (6)$$

Finally, if a curve is defined in polar coordinates as

$$r = g(\theta), \qquad (7)$$

then we can think of it as a parametrically defined curve with parameter θ

$$x = g(\theta)\cos(\theta),\ y = g(\theta)\sin(\theta),\ \alpha \le \theta \le \beta, \qquad (8)$$

and apply formula (5). By doing so, we get

$$k = \frac{\left|2(g'(\theta))^2 + (g(\theta))^2 - g(\theta)g''(\theta)\right|}{\left((g(\theta))^2 + (g'(\theta))^2\right)^{3/2}} = \frac{\left|2(r')^2 + r^2 - rr''\right|}{\left(r^2 + (r')^2\right)^{3/2}}, \qquad (9)$$

see http://mathonline.wikidot.com/the-curvature-of-plane-polar-curves for details.

It may be mentioned that the formulas (5), (6), and (9) follow directly from a more general case for the curves in space, considered in Vaninsky (2016). Our goal in this paper is to obtain the formulas for the curvature in cases when a curve is given implicitly, as an equation in two variables in either rectangular or polar coordinates.

Consider the rectangular coordinates first. In this case, a curve is a set of points satisfying an equation

$$F(x,y) = 0, \qquad (10)$$

where *F(x,y)* is twice continuously differentiable function with

$$F_x^2 + F_y^2 \ne 0, \qquad (11)$$

where lower indexes *x* and *y* stand for partial derivatives with respect to *x* or *y*, correspondingly.

Due to condition (11), in the neighborhood of each point *(x,y)* equation (10) defines either *y* as a function of *x* – if $F'_y \ne 0$, or *x* as a function of *y*, if $F'_x \ne 0$, see Kaplan (1993) for details.

Assume for the sake of brevity that $F'_y \ne 0$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y},$$

$$\frac{d^2y}{dx^2} = \frac{2F_{xy}F_xF_y - F_{xx}(F_y)^2 - F_{yy}(F_x)^2}{(F_y)^3}. \qquad (12)$$

Substituting these expressions into formula (6), we get

$$k = \frac{\left|2F_{xy}F_xF_y - F_{xx}(F_y)^2 - F_{yy}(F_x)^2\right|}{(F_x^2 + F_y^2)^{3/2}}. \qquad (13)$$

This formula holds also if we assume $F'_x \neq 0$ that follows directly from its symmetry with regard to the variables x and y, and the known fact that $F_{xy} = F_{yx}$.

Consider now case of a curve in plane given in polar coordinates by equation

$$G(r,\theta) = 0, \qquad (14)$$

where $G(r,\theta)$ is twice continuously differentiable function with

$$G_r^2 + G_\theta^2 \neq 0. \qquad (15)$$

In this paper we focus on a more practically important case $G'_r \neq 0$, when the equation (14) defines r as a function of θ in the neighborhood of each point (r,θ), as given by formula (7).

Proceeding in this case in the same way as above by using formulas (12), we get

$$\frac{dr}{d\theta} = -\frac{G_\theta}{G_r},$$

$$\frac{d^2r}{d\theta^2} = \frac{2G_{r\theta}G_rG_\theta - G_{\theta\theta}(G_r)^2 - G_{rr}(G_\theta)^2}{(G_r)^3}. \qquad (16)$$

Substitute these expressions into formula (9) to obtain

$$k = \frac{\left|2G_rG_\theta(G_\theta - rG_{r\theta}) + rG_{rr}(G_\theta)^2 + r^2(G_r)^3 + rG_{\theta\theta}(G_r)^2\right|}{(r^2(G_r)^2 + (G_\theta)^2)^{3/2}}. \qquad (17)$$

This formula provides an explicit expression for the curvature of a curve defined implicitly by the equation (14).

For the case $G'_r = 0$, we consider a situation when

$$\frac{\partial G(r,\theta)}{\partial r} = 0, \quad a < r < b, \qquad (18)$$

including the possibilities for $a = -\infty$ or $b = \infty$. In such cases, for any r_1 and r_2 inside the line segment (a,b), by the Mean Value theorem

$$G(r_2,\theta) - G(r_1,\theta) = \frac{\partial G(\varsigma,\theta)}{\partial r}(r_2 - r_1) = 0, \qquad (19)$$

where partial derivative with respect to r is taken at some intermediate value of r in the range (a,b). From this expression, we get

$$G(r_1,\theta) = G(r_2,\theta) \tag{20}$$

for any r_1 *and* r_2 in the range (a,b). As follows from this observation, for the values of the variable θ satisfying condition (18), the function $G(r,\theta)$ may be written as $G(\theta)$, without the variable r. In this situation, equation (14) becomes

$$G(\theta)=0. \tag{21}$$

Assume that this equation has finite number of roots $\theta_1, \theta_2, \ldots, \theta_k$. Then the corresponding parts of the curve are the parts of the rays forming the angles $\theta_1, \theta_2, \ldots, \theta_k$ with the x-axis and ranging from a to b from the origin. Since they are line segments, their curvature is zero.

Consider examples of applications of the obtained formulas.

Example 1. Following Stewart (2012, p. 159) consider the folium of Descartes with parameter $a = 2$ and find a formula for the curvature. An equation of this curve is

$$x^3 + y^3 = 6xy. \tag{22}$$

In this case, formula (10) is as follows

$$F(x,y) = x^3 + y^3 - 6xy = 0, \tag{23}$$

so that

$$F_x = 3x^2 – 6y,$$

$$F_y = 3y^2 – 6x,$$

$$F_{xx} = 6x,$$

$$F_{yy} = 6y,$$

$$F_{xy} = -6. \tag{24}$$

By substituting these expressions into formula (13) we get

$$k = \frac{\left|2xy(x^3 - 6xy + y^3 + 8)\right|}{(x^4 + y^4 - 4x^2y - 4xy^2 + 4x^2 + 4y^2)^{3/2}}. \tag{25}$$

Consider as an example point (3,3) on the curve. By formula (25), the curvature at this point is

$$k(3,3) = \frac{4\sqrt{2}}{3}. \tag{26}$$

Example 2. Following Thomas et al. (2014, p. 677), consider a curve defined in polar coordinates as

$$r^2 = 4cos(\theta) \qquad (27)$$

and find a formula for its curvature. Geometrically, this curve is a union of two loops located horizontally and intersecting at the origin. The curves intersect the x-axis at points (-2,0) and (2,0), corresponding to the polar coordinates (2, π) and (2,0), correspondingly. The loops are the reflections of each other about the y-axis and in the origin. Also, both loops are symmetric about the x-axis.

In this example, formula (14) takes a form

$$G(r,\theta) = r^2 - 4cos(\theta) = 0, \qquad (28)$$

so that

$$G_r = 2r,$$

$$G_\theta = 4sin(\theta),$$

$$G_{rr} = 2,$$

$$G_{\theta\theta} = 4cos(\theta),$$

$$G_{r\theta} = 0. \qquad (29)$$

By using formula (17), we get

$$k = \frac{\left| r(r^4 + 2r^2 cos(\theta) + 12 sin^2(\theta)) \right|}{(r^4 + 4sin^2(\theta))^{3/2}}. \qquad (30)$$

Since on the curve $r^2 = 4cos(\theta)$, formula (30) may be transformed to

$$k = \frac{3r\left(cos^2\theta + 1\right)}{2\left(3cos^2\theta + 1\right)^{3/2}} \qquad (31)$$

The values of the curvatures at the points of x-intercept are equal to each other: k(2,0) = k(2, π) = 3/4. The curvature at the origin, $k(0,0) = 0$, since $r = 0$. The left and the right loops of the curve may be written as $r = -2cos(\theta)^{1/2}$ and as $r = 2cos(\theta)^{1/2}$, correspondingly, so that for each of the loops formula (17) reduces to formula (9), so that formula (30) becomes

$$k = \frac{3\sqrt{|cos\theta|}\left(cos^2\theta + 1\right)}{\left(3cos^2\theta + 1\right)^{3/2}}. \qquad (32)$$

The coincidence of the formulas for the curvatures of the two loops stems from the fact that the equations of the loops differ just in the range of the values of parameter θ.

As follows from the formula (32), the curvature of the curve equals to 0 if and only if *cos(θ)*=0. At this point, $r = 0$ as well, see (28), so this is the origin. The graph of the function for the curvature (32) reveals four points of maximum located at $\theta = \pm 1.2538$ rad and $\theta = \pm 4.3954$ rad, where the curvature $k = 1.2521$. This means that when the curve is traced, the normal acceleration and corresponding normal component of the force are maximal at these points. This fact may be of importance, for example, for the design of highway intersections or development of software for driverless cars.

3. Project 2. Finding potential function by simultaneous integration

This section presents an alternative approach to finding a potential function of a vector field in space. Recall that a vector field has a potential function, if and only if it is the gradient fields. The latter means that the vector $\boldsymbol{F}(x,y,z)$ acting at the point $P(x,y,z)$ is a gradient of some scalar valued function $f(x,y,z)$:

$$\boldsymbol{F}(x,y,z) = M(x,y,x)\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + N(x,y,z)\boldsymbol{k} = \nabla \boldsymbol{f}(x,y,z), \tag{33}$$

where symbol ∇ stands for the gradient.

Condition (33) means that

$$M(x,y,z) = \frac{\partial f}{\partial x},\ N(x,y,z) = \frac{\partial f}{\partial y},\ P(x,y,z) = \frac{\partial f}{\partial z}, \tag{34}$$

see Thomas et al. (2014) for detail. The scalar valued function $f(x,y,z)$ is called a potential function for the vector field $\boldsymbol{F}(x,y,z)$. It is defined up to an arbitrary additive constant.

Gradient fields are conservative in the sense that the total of potential and kinetic energy remains constant when a point is moving in such a field. A line integral along a curve connecting the points A and B in space does not depend on the curve itself, but only on its initial and end points. A line integral along a closed curve C equals to zero. Some important vector fields, for example, the gravitational field and electrostatic field - are gradient.

The Component Test theorem provides necessary and sufficient conditions for a vector field to be potential, Thomas et al. (2014):

Theorem 1. Let a vector field $\boldsymbol{F} = \langle M,N,P \rangle$ be defined on an open simply connected domain, and its component functions M, N, and P have continuous first partial derivatives. Then the field $\boldsymbol{F}$ is potential if and only if the following conditions are met:

$$\frac{\partial P}{\partial y}=\frac{\partial N}{\partial z},\ \ \frac{\partial M}{\partial z}=\frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}. \tag{35}$$

In this paper below we assume that the condition (35) is held and focus on finding the potential function *f(x,y,z)*. A conventional approach suggests doing this task by applying a series of partial differentiations and integrations. Not only such approach is unnecessary lengthy, but it also puts in question the independence of the final result from the order of operations. (This is true, though not self-evident.) What we suggest in this paper, is application of just one operation – integration, to each of the components of the vector field and finding the potential function by observation, comparison and adjustment of the obtained results. This is usually a simpler and shorter way.

The suggested approach is based on the following theorem.

Theorem 2. Let a function *f(x,y,z)* be presented in the following three ways

$$f(x,y,z) = p(x,y,z) + h(y,z) = p(x,y,z) + g(x,z) = p(x,y,z) + q(x,y). \tag{36}$$

Then

$$h(y,z) = g(x,z) = q(x,y) = C, \tag{37}$$

where *C* is an arbitrary constant.

Proof. Let a function

$$r(x,y,z) = f(x,y,z) - p(x,y,z). \tag{38}$$

Then, as follows from (36),

$$r(x,y,z) = h(y,z) = g(x,z) = q(x,y). \tag{39}$$

Taking partial derivatives of *r(x,y,z)* with respect to *x, y*, and *z*, we get

$$\frac{\partial r}{\partial x}=\frac{\partial h}{\partial x}=0,\quad \frac{\partial r}{\partial y}=\frac{\partial g}{\partial y}=0,\quad \frac{\partial r}{\partial z}=\frac{\partial q}{\partial z}=0 \quad . \tag{40}$$

Consider any two points (x_1,y_1,z_1) and (x_2,y_2,z_2) in the domain of the vector field $\boldsymbol{F}(x,y,z)$. Then, by the Mean Value theorem and formulas (40),

$$r(x_2,y_2,z_2) - r(x_1,y_1,z_1) =$$

$$(r(x_2,y_1,z_1) - r(x_1,y_1,z_1)) + (r(x_2, y_2, z_1) - r(x_2,y_1,z_1)) + r(x_2,y_2,z_2) - r(x_2,y_2,z_1) =$$

$$\frac{\partial r(\hat{x},y_1,z_1)}{\partial x}(x_2-x_1)+\frac{\partial r(x_2,\hat{y},z_1)}{\partial y}(y_2-y_1)+\frac{\partial r(x_2,y_2,\hat{z})}{\partial z}(z_2-z_1)=0\,, \tag{41}$$

where partial derivatives are taken at the intermediate points, correspondingly.

As follow from (41),

$r(x_2,y_2,z_2) - r(x_1,y_1,z_1) = 0$ (42)

for any two arbitrary points in space, so that $r(x,y,z) = C$, an arbitrary constant. ■

For vector fields in plane, this result was obtained in Johnson and Vaninsky (2007).

Application of this theorem is as follows. Begin with the validation of the Component Test theorem to confirm that the field is potential. After that integrate each of the components with respect to x, y, or z, correspondingly, keeping a constant of integration in the form $h(y,z)$, $g(x,z)$ or $q(x,y)$, respectively. Next, adjust the functions $h(\ )$, $g(\ )$, and $q(\ )$ to satisfy conditions (36) for some function $p(x,y,z)$. Now, by the theorem 2, the potential function is $p(x,y,z)$, the common part of the adjusted functions.

Following Thomas et al. (2014, p. 975), consider an example of application.

Example 3. Given a vector field

$$\boldsymbol{F}(x,y,z) = (e^x cos(y) + yz)\boldsymbol{i} + (xz - e^x sin(y))\boldsymbol{j} + (xy + z)\boldsymbol{k}, \tag{43}$$

show that it is potential and find the potential function.

Solution. In this example,

$$M(x,y,z) = e^x cos(y) + yz,$$

$$N(x,y,z) = xz - e^x sin(y),$$

$$P(x,y,z) = xy + z. \tag{44}$$

Show that the conditions (35) are met:

$$\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y}, \tag{45}$$

so that the field is the gradient one.

Now, integrate each component with respect to x, y, or z, respectively, to obtain the potential function $f(x,y,z)$ in three different forms:

$$f(x,y,z) = \int M(x,y,z)dx = \int (e^x cos(y) + yz)dx = e^x cos(y) + xyz + h(y,z)\,, \tag{46}$$

$$f(x,y,z) = \int N(x,y,z)dy = \int (xz - e^x sin(y))dy = yxz + e^x cos(y) + g(x,z)\,, \tag{47}$$

$$f(x,y,z) = \int P(x,y,z)dz = \int (xy + z)dz = zxy + \frac{z^2}{2} + q(x,y)\,. \tag{48}$$

Observing the expressions (46), (47) and (48), we see that they have a common term xyz and some other additive terms that are different. Make the adjustments, as suggested by the theorem 2.

Formulas (46) and (47) have a common term $e^x cos(y)$ that is missing in (48). Following theorem 2, take in formula (48)

$$q(x,y) = e^x cos(y) + q_1(x,y). \tag{49}$$

Now, we can see that formula (48) has a term $z^2/2$ that is not present in both (46) and (47). Make the adjustments as follows:

$$g(y,z) = \frac{z^2}{2} + g_1(y,z)\,, \tag{50}$$

$$q(x,z) = \frac{z^2}{2} + q_1(x,z)\,. \tag{51}$$

After these adjustments, the expressions (46) – (48) become as follows

$$f(x,y,z) = e^x cos(y) + xyz + \frac{z^2}{2} + h_1(y,z)\,, \tag{52}$$

$$f(x,y,z) = xyz + e^x cos(y) + \frac{z^2}{2} + g_1(x,z)\,, \tag{53}$$

$$f(x,y,z) = xyz + e^x cos(y) + \frac{z^2}{2} + q_1(x,y)\,. \tag{54}$$

As follows from theorem 2,

$$h_1(y,z) = g_1(x,z) = q_1(x,y) = C, \tag{55}$$

and the potential function is

$$f(x,y,z) = xyz + e^x cos(y) + \frac{z^2}{2} + C\,. \tag{56}$$

The value of the additive arbitrary constant C is determined by assigning a point in space having the potential equal to zero.

4. Conclusion

Participating in undergraduate mathematics research projects helps students to further develop logical thinking, and to acquire experience in mathematical reasoning and conducting long chains of transformations. It also inspires them to use computer algebra systems, and encourages them to recognize the relationship between formal categories and practical problems. Studying material beyond the standard curriculum teaches them to work independently and to seek relevant information in different sources. Working in groups on the actual research and presentation improves their communication skills. Taken together, all of these factors make the students better prepared for further academic study and professional careers.

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