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On the Casimir interaction between holes

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We study the leading long-distance attractive force between two holes in a plate arising from a scalar field with Dirichlet boundary conditions on the plate. We use a formalism in which the interaction is governed by a non-local field theory which lives on the two holes. The interaction energy is proportional to $Q_1 Q_2 / r^7$ at large separation r, where Q_1 and Q_2 are certain charges associated with the holes. We compute these charges for round and rectangular holes. We show that the $1/r^7$ behavior is universal for separations large compared to the linear dimensions of the holes, irrespective of the spin or interactions of the bosonic field. We also study the interaction between two long thin slits, for which the energy falls off as $1/r^6$.

I. INTRODUCTION

The original Casimir effect described the interaction between two parallel conducting plates due to vacuum fluctuations of the electromagnetic field [1]. Since the pioneering work of Casimir many variants of this effect have been studied. For a recent review see [2].

In the present paper we consider a single infinite plate with two holes in it. We consider a scalar field with Dirichlet boundary conditions on the plate, and study the way in which the holes modify the ground state energy of the field. We do this using a formalism developed in [3], where one first integrates out the scalar field in the bulk to obtain a non-local field theory that lives in the holes. This description makes it easy to study the leading long-distance interaction between holes. We find that the force between the holes is attractive and that the interaction energy scales as $1/r^7$, where r is the distance between the holes. The interaction energy is also proportional to the product $Q_1 Q_2$ of certain charges associated with the holes. These charges have units of (length)^3 and depend on the geometry of the holes.

An outline of this paper is as follows. In section II we set up the general formalism and extract the distance dependence. We then compute the charge for round and rectangular holes. In section III we study the interaction between two long thin parallel slits, using a mix of numerical and perturbative techniques. Section IV gives the argument for the universality of the $1/r^7$ interaction. We conclude with some remarks in section V.

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II. INTERACTION BETWEEN HOLES

Consider a scalar field ϕ in the presence of an infinite plate with two holes in it, as shown in Fig. 1. We first work in three Euclidean dimensions with a scalar field of mass μ. We introduce a pair of coordinates $\mathbf{x}$ along the plate and a single transverse coordinate y. We may take the field ϕ to be confined to a large cubical box with the plate of interest partitioning this box into a left region and a right region. The location of the plate is at $y = 0$. We impose a Dirichlet condition on the plate.

The approach of [3] begins by integrating out the scalar field in the bulk to write a non-local effective action for the field in the holes. One way to think about this effective action is to regard y as Euclidean time. The ground state wavefunctional for the field takes the standard form [4, 5]

$$\Psi_0[\phi] = \text{const.} \exp\left(-\int d^2 x \, \frac{1}{2} \phi \sqrt{-\nabla^2 + \mu^2} \, \phi \right) . \quad (1)$$

Here $\phi(\mathbf{x})$ is the value of the field on a slice of fixed y and ∇^2 is the Laplacian on $\mathbb{R}^2$. The three-dimensional partition function of the field in the presence of the plate at $y = 0$ is then

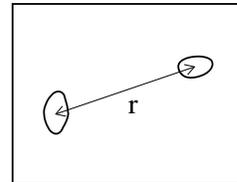$$Z_{3d} = \int \mathcal{D}\phi_0 \, \Psi_0^*[\phi_0] \, \Psi_0[\phi_0] \quad (2)$$

FIG. 1: An infinite plate with two holes separated by a distance r. The field vanishes at the location of the plate. In the holes we denote the fluctuating value of the field by ϕ_0.

where ϕ_0 is the value of the field at $y = 0$. The boundary conditions imposed by the plate are taken into account by requiring that ϕ_0 vanishes outside the holes. This formalism was used in [3] to obtain the diffractive edge effects for the Casimir interaction between a plate and half-plate parallel to it, as well as for a plate and another plate perpendicular to it at a finite separation. This gave results in agreement with numerical calculations [6] for both cases and with a special calculation for the case of perpendicular plates [7].

For working out the functional integral in (2) explicitly for the present case of two holes on the plate, it is useful to introduce projection operators P_1, P_2 on the two holes. These are operators on $L^2(\mathbb{R}^2)$, defined by

$$P_i f(x) = \begin{cases} f(x) & \text{if } x \in \text{hole } i \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The Gaussian functional integral in (2) then gives

$$-\log Z_{3d} = \frac{1}{2} \operatorname{Tr} \log \begin{pmatrix} \mathcal{O}_{11} & \mathcal{O}_{12} \\ \mathcal{O}_{21} & \mathcal{O}_{22} \end{pmatrix} \tag{4}$$

where $\mathcal{O}_{ij} = P_i \sqrt{-\nabla^2 + \mu^2} P_j$. To extend this result to

four dimensions we think of the mass μ as arising from Kaluza-Klein momentum around an additional periodic coordinate with period $\beta \to \infty$. This means the four-dimensional partition function is

$$-\log Z_{4d} = \beta \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} \left(-\log Z_{3d}\right). \tag{5}$$

The interaction between holes is clearly due to the off-diagonal entries $\mathcal{O}_{12}$, $\mathcal{O}_{21}$. If the holes are widely separated these off-diagonal entries will be small and we can treat them perturbatively. The zeroth order term in this perturbation series is independent of the separation distance, and the first order term vanishes identically. So the leading contribution to the interaction energy is

$$E_{\text{int}} = -\frac{1}{2\pi} \int_0^{\infty} d\mu \operatorname{Tr} \left[(\mathcal{O}_{11})^{-1} \mathcal{O}_{12} (\mathcal{O}_{22})^{-1} \mathcal{O}_{21}\right]. \tag{6}$$

In this equation, for the inverses, one first constructs the diagonal projected operators $\mathcal{O}_{ii}$, then inverts them in the subspace corresponding to hole i. Writing the trace out in position space we get

$$E_{\text{int}} = -\frac{1}{2\pi} \int_0^{\infty} d\mu \int d^2x_1 d^2x_1' d^2x_2 d^2x_2' \langle \mathbf{x}_1| (\mathcal{O}_{11})^{-1} |\mathbf{x}_1'\rangle \langle \mathbf{x}_1'|\mathcal{O}_{12}|\mathbf{x}_2\rangle \langle \mathbf{x}_2| (\mathcal{O}_{22})^{-1} |\mathbf{x}_2'\rangle \langle \mathbf{x}_2'|\mathcal{O}_{21}|\mathbf{x}_1\rangle \tag{7}$$

where $\mathbf{x}_1$, $\mathbf{x}_1'$ correspond to points in the first hole and $\mathbf{x}_2$, $\mathbf{x}_2'$ to points in the second.

For widely-separated holes there are further simplifications we can make. Large separation necessarily means the separation distance r is large compared to the size of the holes, so we can pull the off-diagonal matrix elements out of the integrals. Indeed we have (the projection operators are immaterial for this)

$$\begin{aligned} \langle \mathbf{x}_1|\mathcal{O}_{12}|\mathbf{x}_2\rangle &\approx \langle 0|\sqrt{-\nabla^2 + \mu^2}|r\rangle \\ &= -\frac{1}{2\sqrt{\pi}} \int_0^{\infty} \frac{ds}{s^{3/2}} \langle 0|e^{-s(-\nabla^2+\mu^2)}|r\rangle \\ &= -\frac{1}{2\pi r^3}(1 + \mu r)e^{-\mu r} \end{aligned} \tag{8}$$

where we introduced an integral representation for the square root and used the heat kernel

$$\langle 0|e^{s\nabla^2}|r\rangle = \frac{1}{4\pi s} e^{-r^2/4s}. \tag{9}$$

In principle the remaining matrix elements $\langle \mathbf{x}_1| (\mathcal{O}_{11})^{-1} |\mathbf{x}_1'\rangle$, $\langle \mathbf{x}_2| (\mathcal{O}_{22})^{-1} |\mathbf{x}_2'\rangle$ depend on μ. But for small holes we can neglect this dependence. (Retaining it would give corrections down by powers of (size of hole)$/r$.) So using (8) in (7) and integrating over

the explicit μ dependence gives the leading long-distance interaction between small holes,

$$E_{\text{int}} = -\frac{5\,Q_1 Q_2}{32\pi^3 r^7} \tag{10}$$

where the charge associated with each hole is

$$Q_i = \int_{\text{hole } i} d^2x d^2x' \langle \mathbf{x}| (\mathcal{O}_{ii})^{-1} |\mathbf{x}'\rangle. \tag{11}$$

Note that Q_i is simply the matrix element of $(P_i \sqrt{-\nabla^2} P_i)^{-1}$ between functions that are constant (equal to one) in the hole.

A. Round holes

The charge Q defined in (11) depends on the geometry of the hole in question but is independent of the separation distance. Here we study it for a round hole of radius R.

The approach we will use is quite simple: we adopt a lattice discretization of the operator $P\sqrt{-\nabla^2} P$ and compute the relevant matrix element numerically. In doing this we can restrict to rotationally-invariant functions,

meaning we only need to keep track of the radial dependence. Introducing a radial lattice spacing a we identify

$$r \quad \leftrightarrow \quad \hat{r} = a \begin{pmatrix} 1/2 & & & \\ & 3/2 & & \\ & & 5/2 & \\ & & & \ddots \end{pmatrix} \quad (12)$$

$$\frac{d}{dr} \quad \leftrightarrow \quad \frac{1}{2a} \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & 1 & \\ & -1 & 0 & 1 \\ & & & \ddots \end{pmatrix} \quad (13)$$

(note that we begin the lattice half-a-spacing from the origin, and that our discretization of d_r preserves antisymmetry). The projection operator

$$P \quad \leftrightarrow \quad \begin{pmatrix} \mathbb{1}_{N \times N} & 0 \\ 0 & 0 \end{pmatrix} \quad (14)$$

where the rank of P is related to the size of the hole by $R = aN$. Finally we identify the inner product on rotationally-invariant functions

$$\int d^2x\, \phi_1^* \phi_2 \quad \leftrightarrow \quad \phi_1^\dagger \hat{r}\, \phi_2 \quad (15)$$

and the state of interest

$$\int d^2x\, |\mathbf{x}\rangle \quad \leftrightarrow \quad \sqrt{2\pi a} \left.\begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix}\right\} N \quad (16)$$

(the coefficient is fixed by requiring that the two states have the same norm). From these ingredients we can construct $P\sqrt{-\frac{1}{r}d_r r d_r}\, P$ numerically as an $N \times N$ matrix,

invert it, and take the matrix element between the state (16). Extrapolating to the continuum limit $N \to \infty$, $a \to 0$ with R fixed, we find that the charge for a round hole is

$$Q = 1.28\, R^3 \qquad \text{(round hole)} \qquad (17)$$

Note that the charge has units of $(\text{length})^3$, as could have been guessed on dimensional grounds. But one should be careful – if two holes merge to form a single hole the charges should not be expected to be additive.

B. Rectangular holes

In order to gain insight into how the charge depends on the shape of the hole, we now consider a rectangular hole of size $L_1 \times L_2$. Rather than use a position-space lattice, we will work in momentum space with a mode cutoff.

The charge we wish to compute is

$$Q = \int d^2x\, d^2x'\, \langle \mathbf{x} | (P\sqrt{-\nabla^2}P)^{-1} | \mathbf{x}' \rangle .$$

Inserting complete sets of functions which vanish outside the hole, namely

$$\langle \mathbf{x} | mn \rangle = \begin{cases} \frac{2}{\sqrt{L_1 L_2}} \sin\left(\frac{m\pi x_1}{L_1}\right) \sin\left(\frac{n\pi x_2}{L_2}\right) & \text{in the hole} \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

with $m, n = 1, 2, 3, \ldots$ we have

$$\frac{Q}{64 L_1 L_2} = \sum_{\text{odd } m,n,m',n'} \frac{1}{m\pi} \frac{1}{n\pi} \langle mn | \mathcal{O}^{-1} | m'n' \rangle \frac{1}{m'\pi} \frac{1}{n'\pi} . \quad (19)$$

Here $\mathcal{O} = P\sqrt{-\nabla^2}\, P$. The matrix elements of $\mathcal{O}$ in this basis are (assuming odd m and n)

$$\begin{aligned} \langle mn | \mathcal{O} | m'n' \rangle &= \langle mn | \sqrt{-\nabla^2} | m'n' \rangle \\ &= 64 L_1 L_2 \int \frac{d^2k}{(2\pi)^2} \sqrt{k_1^2 + k_2^2}\, \cos^2(k_1 L_1/2)\cos^2(k_2 L_2/2) f_1(m) f_2(n) f_1(m') f_2(n') \end{aligned} \quad (20)$$

where

$$f_1(m) = \frac{m\pi}{k_1^2 L_1^2 - m^2\pi^2}, \quad f_2(n) = \frac{n\pi}{k_2^2 L_2^2 - n^2\pi^2} \quad (21)$$

In the first line of (20), we used the fact that the projection operators do not matter, and in the second line we made use of a Fourier integral representation of the basis functions (18).

To compute the charge we introduce mode cutoffs

$m \leq 2M - 1$, $n \leq 2N - 1$. Then we can construct the operator $\mathcal{O}$ numerically as an $(MN) \times (MN)$ matrix, invert it, and evaluate the sums in (19). Extrapolating to the continuum limit $M, N \to \infty$ then gives the charge. For instance, for a square hole with $L_1 = L_2 = L$, this procedure gives the charge

$$Q = 0.228\, L^3 \qquad \text{(square hole)} \qquad (22)$$

By comparison, the charge of a round hole (17) is

$0.230 \,(\text{area})^{3/2}$. So a square hole has slightly less charge than a round hole of equal area.

Another simple case is a rectangle with $L_1 \gg L_2$. In this limit the matrix elements (20) go over to

$$\langle mn|\mathcal{O}|m'n'\rangle = \delta_{mm'} \langle n|\sqrt{-\frac{d^2}{dx_2^2}}|n'\rangle \qquad (23)$$

i.e., it reduces to a one-dimensional problem. (The easiest way to see this is to rescale $k_1 \to k_1/L_1$, $k_2 \to k_2/L_2$ and send $L_1 \to \infty$.) Using this in (19) we find that the charge for a long thin rectangle is

$$Q = L_1 Q_{\text{slit}} \qquad (\text{thin rectangle}) \qquad (24)$$

where Q_{slit} is the charge for a slit of width L_2. This is something we study in section III. Borrowing the result (37), the charge for a long thin rectangle is

$$Q = 0.393 \, L_1 L_2^2 \qquad \text{for } L_1 \gg L_2 \qquad (25)$$

Note that, for a given area, a long thin rectangle has less charge than a square. This is consistent with the observation above, that the value of $Q/(\text{area})^{3/2}$ is slightly smaller for a square hole than for a round hole. It's tempting to speculate that round holes have the largest charge for a given area.

C.   Square hole: Perturbation theory

The charge Q in (19) can also be calculated using the perturbation method developed in [3] where $\mathcal{O}$ is split

into two parts: a term diagonal in the basis (18) and a nondiagonal term. The nondiagonal part is then treated in a perturbation series.

The k_1-integration in (20) can be done by moving the integration contour slightly above the real k_1 axis and writing $4\cos^2(k_1 L_1/2) = e^{ik_1 L_1} + e^{-ik_1 L_1} + 2$. For each term in this decomposition, the integration contour can be completed in the upper or lower half-plane, appropriately. We then pick up pole contributions at $\pm(m\pi/L_1) + i\epsilon$, $\pm(m'\pi/L_1) + i\epsilon$ and cut contributions from the square root term $\sqrt{k_1^2 + k_2^2}$. The pole contributions cancel out unless $m = m'$. For the cut contributions it is simpler to use the integral representation

$$\sqrt{k_1^2 + k_2^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\gamma \, \frac{k_1^2 + k_2^2}{k_1^2 + k_2^2 + \gamma^2} \qquad (26)$$

and evaluate contributions at the imaginary poles $k_1 = \pm i\sqrt{k_2^2 + \gamma^2}$.

We follow the same procedure for the k_2-integration. For a square hole, with $L_1 = L_2 = L$, we then find

$$\langle mn|\mathcal{O}|m'n'\rangle = \langle mn|\mathcal{O}_{\text{pole}} + \mathcal{O}_{\text{cut}}|m'n'\rangle \qquad (27)$$

where

$$\langle mn|\mathcal{O}_{\text{pole}}|m'n'\rangle = \frac{1}{L}\sqrt{(m\pi)^2 + (n\pi)^2} \, \delta_{mm'}\delta_{nn'} \qquad (28)$$

is the diagonal term arising from the real-pole contributions in k_1 and k_2 integrations, and

$$
\begin{aligned}
\langle mn|\mathcal{O}_{\text{cut}}|m'n'\rangle = & -\frac{4}{\pi L}(m\pi)^2 \int_1^{\infty} dy \, \sqrt{y^2-1}\,(1+e^{-m\pi y}) \frac{n\pi}{(m\pi y)^2 + n^2\pi^2} \frac{n'\pi}{(m\pi y)^2 + n'^2\pi^2} \, \delta_{mm'} \\
& -\frac{16}{\pi^2 L} \int_1^{\infty} dy \int_0^{\infty} dk \, k^2 \sqrt{y^2-1}\,(1+e^{-ky})(1+\cos k) \frac{n\pi}{k^2 - n^2\pi^2} \frac{n'\pi}{k^2 - n'^2\pi^2} \frac{m\pi}{k^2 y^2 + m^2\pi^2} \frac{m'\pi}{k^2 y^2 + m'^2\pi^2}
\end{aligned}
\tag{29}
$$

Treating $\mathcal{O}_{\text{cut}}$ as a perturbation we have

$$\mathcal{O}^{-1} = \mathcal{O}_{\text{pole}}^{-1} - \mathcal{O}_{\text{pole}}^{-1}O_{\text{cut}}\mathcal{O}_{\text{pole}}^{-1} + \cdots \qquad (30)$$

with a corresponding series for the charge,

$$Q = Q^{(0)} + Q^{(1)} + \cdots .$$

For a square hole, denoting

$$h(m,n) = 1/\sqrt{(m\pi)^2 + (n\pi)^2} \, ,$$

we find

$$
\begin{aligned}
Q^{(0)} &= 64L^3 \sum_{\text{odd } m,n} \frac{h(m,n)}{(m\pi)^2(n\pi)^2} = 0.170 \, L^3 \\
Q^{(1)} &= -64L^3 \sum_{\text{odd } m,n,m',n'} \frac{\langle mn|\mathcal{O}_{\text{cut}}|m'n'\rangle h(m,n)h(m',n')}{(m\pi)(n\pi)(m'\pi)(n'\pi)} \\
&= 0.039 \, L^3
\end{aligned}
\tag{31}
$$

These first two terms capture, respectively, 75% and 17% of the lattice result (22), strongly suggesting the validity of the perturbation expansion.
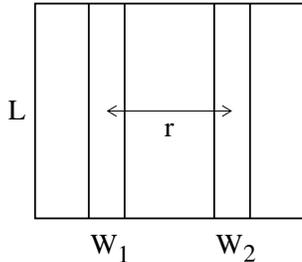
FIG. 2: Slits of width W_1 and W_2 and length L, separated by a distance r.

III.   INTERACTION BETWEEN SLITS

As a simple setting which nicely illustrates our formalism, we conclude by studying a plate with two long thin slits in it. We denote the widths of the slits by W and the separation between slits by r. We periodically identify both Euclidean time with period $\beta \to \infty$ and the dimension along the slit with period $L \to \infty$. The geometry is shown in Fig. 2.

The basic result for the interaction energy (7) still applies, with the following modifications.

1. The integral over Kaluza-Klein momentum becomes two dimensional,

$$\beta L \int \frac{d^2\mu}{(2\pi)^2} = \beta L \int_0^\infty \frac{\mu d\mu}{2\pi}$$

2. The operator $\mathcal{O}$ involves a one-dimensional Laplacian, and the matrix element (8) changes to[1]

$$\langle x|\sqrt{-d_x^2 + \mu^2}|x'\rangle = -\frac{\mu}{\pi r}K_1(\mu r)$$

This means that the interaction energy per unit length between narrow, widely-separated slits is

$$
\begin{aligned}
\frac{E_{\text{int}}}{L} &= -\frac{1}{4\pi}\int_0^\infty \mu d\mu \left(\frac{\mu}{\pi r}K_1(\mu r)\right)^2 Q_1 Q_2 \\
&= -5.375 \times 10^{-3}\,\frac{Q_1 Q_2}{r^6}
\end{aligned}
\tag{32}
$$

where the charge associated with each slit is

$$Q = \int dx dx'\, \langle x|\left(P\sqrt{-d_x^2}\,P\right)^{-1}|x'\rangle. \tag{33}$$

---

[1] For d dimensional plates the heat kernel (9) becomes $(4\pi s)^{-d/2}\exp(-r^2/4s)$ and the matrix element (8) becomes $-2\left(\frac{\mu}{2\pi r}\right)^{\frac{d+1}{2}} K_{(d+1)/2}(\mu r)$.

A.   Lattice approach

The charge (33) can be computed in a variety of ways. We begin with a lattice approach, in which we introduce a lattice spacing a and discretize the operators via

$$-\frac{d^2}{dx^2} \quad \leftrightarrow \quad \frac{1}{a^2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & & \ddots \end{pmatrix} \tag{34}$$

$$P = \begin{pmatrix} 0 & & \\ & \mathbb{1}_{N\times N} & \\ & & 0 \end{pmatrix} \tag{35}$$

The rank of P is related to the width of the slit by $W = aN$. Finally we identify

$$\int dx\,|x\rangle \quad \leftrightarrow \quad \sqrt{a}\begin{pmatrix} 0 \\ 1_N \\ 0 \end{pmatrix} \tag{36}$$

Computing the matrix element and extrapolating to the continuum limit gives the charge for a slit,

$$Q = 0.393\,W^2. \tag{37}$$

The same coefficient appears in the charge of a long thin rectangle (25).

B.   Perturbation theory

An alternative approach to computing the charge of a slit is to use perturbation theory. Inserting a complete set of states

$$\langle x|n\rangle = \sqrt{\frac{2}{W}}\,\sin\left(\frac{n\pi x}{W}\right) \qquad n = 1,2,3,\ldots \tag{38}$$

the charge (33) becomes

$$Q = 8W \sum_{m,\,n\text{ odd}} \frac{1}{m\pi}\mathcal{O}_{mn}^{-1}\frac{1}{n\pi} \tag{39}$$

where $\mathcal{O} = P\sqrt{-d_x^2}\,P$. The matrix elements of this operator were studied in [3], where it was found that they could be decomposed into 'pole' and 'cut' contributions, $\mathcal{O} = \mathcal{O}^{\text{pole}} + \mathcal{O}^{\text{cut}}$. The matrix elements for odd m and n are[2]

$$\mathcal{O}_{mn}^{\text{pole}} = \frac{m\pi}{W}\delta_{mn} \tag{40}$$

$$\mathcal{O}_{mn}^{\text{cut}} = -\frac{4}{\pi W}\int_0^\infty dy\, y\left(1 + e^{-y}\right)\frac{m\pi}{m^2\pi^2 + y^2}\frac{n\pi}{n^2\pi^2 + y^2}$$

---

[2] These are the even-parity modes of [3] equations (29), (32) with the dictionary $p = m/2$, $q = n/2$, $a = W/2$.

The pole contribution corresponds to putting a Dirichlet boundary condition on the edges of the slit, while the cut contribution captures diffractive effects. Once again, treating $\mathcal{O}^{\text{cut}}$ as a perturbation as in (30), the first few terms in the series for the charge are

$$Q^{(0)} = \frac{7\zeta(3)}{\pi^3}\, W^2 = 0.2714\, W^2$$
$$Q^{(1)} = 0.0773\, W^2 \qquad\qquad (41)$$
$$Q^{(2)} = 0.0268\, W^2$$

All terms in this perturbation series are positive. Note that the perturbation series appears to be nicely convergent. Compared to the lattice result (37), the first three terms capture 69%, 20% and 7% of the exact result, respectively, for a total of 96%.

IV.   UNIVERSALITY OF $1/r^7$

The behavior of the potential between two holes in a plate is very reminiscent of the van der Waals interaction between neutral atoms of zero intrinsic dipole moment. In that case, there is an argument due to Feinberg and Sucher [8] which shows that the behavior of the potential is universal at large distances. In modern language the argument for this low energy theorem is that the effective action is of the form $\frac{1}{2}(\alpha_E E^2 + \alpha_B B^2)$, where E and B are the electric and magnetic fields, respectively, and α_E, α_B are the corresponding polarizabilities. If one introduces a field Φ representing the creation and annihilation of the atom as a whole, then the effective action is of the form

$$S_{\text{eff}} = g_1 \partial_\mu \Phi \partial_\nu \Phi F^{\mu\alpha} F^\nu_\alpha \; + \; g_2 \Phi^2 F^2 \; + \cdots \qquad (42)$$

For simplicity, we take the atom to be represented by a scalar field, the final result is not sensitive to this choice. The terms displayed in (42) are those with the lowest dimension; these are the relevant ones for low energy or long distance behavior. (The coefficients g_1 and g_2 can be related to the polarizabilities, but this is not relevant for our argument.) This effective action can be used to calculate the potential between two atoms at large separation; it is given by the two-photon exchange generated by (42). The computation of this process leads to the $1/r^7$ behavior of the van der Waals potential, in the long-distance regime where retardation effects are important [8]. This is the universality of the van der Waals interaction.

A similar low energy theorem applies to the potential between two holes on a plate. For this purpose, we first note that the boundary action (the action on the holes) can be obtained from the quantum effective action $\Gamma[\chi]$ which generates the 1PI diagrams. For this, recall that the boundary action is defined by

$$\exp\left(-S[\phi_0]\right) = \int [d\phi]\; \exp\left(-S[\chi + \phi]\right) \qquad (43)$$

where χ obeys the condition $\chi \to \phi_0$ as one approaches the boundary. We defined χ as a special solution of the

(free) equations of motion with this boundary condition. Explicitly, for fields on the left side of the partition of the box in which the theory is defined, we could take

$$\chi(x) = \int d^{d-1}x'\; \phi_0(x')\, n \cdot \partial' G(x|x') \qquad (44)$$

where $G(x|x')$ is the propagator to the left (and right) of the plate with Dirichlet boundary conditions. We do not need to make this particular choice, χ is any specific field with the boundary behavior $\chi \to \phi_0$, so that ϕ in (43) can be taken to obey Dirichlet boundary conditions.

Consider now the calculation of $\Gamma[\chi]$ for the left side of the partition of the theory in a box. From the Legendre transformation of the generating functional for connected Green's functions, it is easily seen that we can write

$$\exp\left(-\Gamma[\chi]\right) = \int [d\phi]\; \exp\left(-S[\chi + \phi] + \int \frac{\delta\Gamma}{\delta\chi}\phi\right) \quad (45)$$

where χ is, for the moment, an arbitrary field. If we choose χ to be a solution of the quantum equations of motion, $\delta\Gamma/\delta\chi = 0$, obeying the condition $\chi \to \phi_0$ on the boundary, we see that the functional integral on the right side of (45) becomes the defining integral (43) for the boundary action. In other words,

$$S[\phi_0] = \Gamma[\chi], \qquad \frac{\delta\Gamma}{\delta\chi} = 0, \qquad \chi \to \phi_0 \;\; \text{on the boundary} \tag{46}$$

This gives the boundary action for any theory, including the effect of interactions. It is the quantum effective action evaluated on its critical point with the boundary condition $\chi \to \phi_0$.

The long-distance interaction between holes comes from terms in the effective action which are quadratic in χ. For these quadratic terms, the low energy or derivative expansion of $\Gamma[\chi]$ follows the familiar pattern. For a scalar field, the lowest order term is the mass term. Since we are considering a massless theory, we can take the renormalized mass to be zero, so there is no χ^2 term in $\Gamma[\chi]$. The lowest nontrivial term in $\Gamma[\chi]$ is then the kinetic term,

$$\Gamma[\chi] = \frac{1}{2} \int \partial_\mu \chi\, \partial^\mu \chi \; + \cdots \qquad (47)$$

This term has been canonically normalized by an appropriate wavefunction renormalization. It leads to the boundary action we used above and gives $1/r^7$ behavior for the hole-hole potential on the plate. Higher dimension terms in $\Gamma[\chi]$ will not contribute to the long distance behavior of the potential. This argument makes it evident that we will get the same behavior for other kinds of fields as well. For example, for the electromagnetic field, $\Gamma[A]$ has similar behavior, with the transverse potentials contributing. Hence it will lead to the same $1/r^7$ potential between holes. It would be interesting to extend this result to spinor fields.

Thus we have obtained a low energy theorem, analogous to the Feinberg-Sucher proof of universality of the

van der Waals force: The long distance potential between two holes in a plate behaves as $1/r^7$ for all massless fields for which the leading kinetic operator is the Laplacian. (This result holds for a four-dimensional theory.)

V. DISCUSSION

In this paper we have analyzed the Casimir interaction between two holes on a conducting plate (a plate on which the fields obey Dirichlet conditions) using the formalism of the non-local field theory developed in [3]. For a separation distance r which is large compared to the linear dimensions of the holes, the interaction energy is proportional to $Q_1 Q_2/r^7$, where Q_1, Q_2 are charges associated with the holes. The $1/r^7$ form is universal, given by a low energy theorem. For any number of holes, from equation (6), the result is the same with a pairwise interaction between holes. (For long slits, because of an additional integration along the length of the slits, the interaction energy scales as $1/r^6$.)

There are clearly interesting questions for further analysis. The charge Q for each hole depends on the geometry of the hole. While we have considered round and rectangular holes and long slits explicitly, it is interesting to seek a more general understanding of how Q is related to the geometry of the hole.

The $1/r^7$ behavior of the two-hole interaction is for large distances. What is the behavior of the interaction energy as the holes come closer? While we do not have a definitive answer, it is worth noting that we are considering the mutual interaction energy between holes; the self-energies of the holes do not appear, or, equivalently, they have been subtracted out. But when the holes merge, the entire energy is the self-energy. It would be inter-esting to study the approach and merger of two holes in more detail. An intriguing possibility is that, as the holes come closer, the potential could reach a minimum value at some finite separation, similar to the Lennard-Jones potential between neutral atoms.

A large number of small mobile holes on a plate would behave like a gas of particles with a pairwise (two-particle) interaction which is attractive and goes like $1/r^7$ at long separations. It would be interesting to study the thermodynamics of this "Casimir gas".

Our analysis was done using a non-local field theory on the holes. There are other methods of analysis, notably the world-line formalism, which has been used before for studying diffractive effects on Casimir forces [6]. It would be interesting, and an independent check, to reproduce our results in this formalism.

Finally, one may ask about experimental observation of this interaction. Casimir energies resulting directly from fundamental forces, such as the electrodynamic Casimir energy, are small and very difficult to measure [9]. However the thermodynamic analogue of the Casimir effect, which can occur in liquid mixtures near a critical point, can be appreciable because, for this effect, temperature plays the role of Planck's constant. In fact, there have been recent observations of such a thermodynamic analogue of the Casimir effect [10]. It could be that the interaction between holes would be observable in such a setting.

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