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Models for Decision-Making

Steven Cosares
CUNY La Guardia Community College

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Situations Requiring a Decision

Atlas Bookbinders

It costs Atlas Bookbinders about $80,000 to prepare for production the new edition of their popular Business Math textbook. This includes the payment to the authors and editors, the costs of setting the presses, adding the book to their catalog, and designing the book's cover. These are one-time "fixed" costs that are incurred, even if only one copy of the textbook is printed. Materials, marketing, pressing and binding cost a total of about $44.00 for each book. These are called "variable" costs because the total depends on the number of units that will be produced. Any number of books can be produced at that price. Copies of the book are sold to campus bookstores for $60 each. The total demand for this book is not known exactly, but some estimates are available.

Would it be a good idea for Atlas to spend the $80,000 to initiate production of the textbook? If so, how many books should Atlas make in its first production run?

To keep its operations running smoothly and profitably, a business like Atlas Bookbinders needs to have managers who make intelligent decisions about how to utilize resources, design facilities, schedule activities, produce items, deploy personnel, and invest capital. It is the role of these operations managers to establish the procedures for the day-to-day operations of the business, to determine how resources like time, money, materials, and people, are distributed among the different business activities, and to make decisions that address new situations as they arise. We will henceforth refer to these people using the more general term "decision-makers" because at different times any employee of a company may have to take on this role. The decision making activities are performed so that the business can achieve objectives like maximizing profits, minimizing financial risk, minimizing its ecological footprint, having effective resource utilization, and/or maintaining a lean manufacturing process that keeps idle inventories low.

As illustrated with Atlas Bookbinders, the decision-maker encounters a situation that gives rise to questions that need to be answered or mathematically-based problems that need to be solved. The decision-maker in this situation, most likely a publisher, would weigh the consequences associated with different alternatives: whether to incur the fixed costs by initiating production of the textbook; if so, how many to produce in the first pressing; whether a second pressing might be necessary. The objective in this case would be to make enough in total revenue to offset the costs of production. Common sense dictates that if the demand for the book is likely to be large, then the revenue generated would more than offset the fixed costs; otherwise it is better not to initiate production.

Examples of such decision-making situations abound. A retailer's inventory manager must determine which items to carry, how many of each to keep on hand, and how often to replenish the supplies. The objective of the decision-maker in this case is to minimize the total costs due to storage, unsold items, and shipping. An investment analyst must determine if a mutual fund or portfolio of stocks is profitable, and if so, how much money to invest in it. The objective is to maximize the total return on investments. A taxi dispatcher must determine, for each shift, how many drivers to have available and how to assign drivers to the calls as they arise. The objective is to maximize the total fares and to fairly distribute the workload among the drivers. When a company builds a new factory, the operations manager must determine how the facility is to be laid out. The objective is to maximize the production capacity while minimizing the construction cost. Drawing from experience, expertise, intuition, and computerized decision-support tools, decision-makers hope to find the best course of action and then see to it that their decisions are implemented.

What makes the job of a decision-maker particularly difficult is that the situations they encounter usually have both *quantitative* and *qualitative* components. The *quantitative* components take the form of questions that include phrases like: "how many?" or "how often?" or “at what distance?” or "at what price?” or "for how long?" Such questions can usually be answered through the use of analytical methods like those provided in Mathematics, Engineering, Statistics, Management Science, and Computer Science. The situation described for Atlas Books has been presented predominantly in this quantitative form.

The *qualitative* components of a situation, on the other hand, often defy direct measurement. They include issues like public relations, employee morale, image, politics, corporate culture, government regulation, and the perceptions of customers. Such issues may not be easily represented with numbers. They are usually addressed using intuition, relevant past experience, political savvy, and common sense. Fields of study like Political Science, Psychology, Sociology, and Business Administration help to prepare a decision-maker to determine policies for predominantly qualitative situations. It often turns out that such qualitative factors compel a decision-maker to select a course of action that is not necessarily an optimal one in terms of costs or profits. For example, the publisher at Atlas Books might be more likely to print the Business Math textbook if he values his relationship with the authors, even if the sales might be lower than he would hope for.

Barbara and Richard Starr and their family have just relocated to a new city and must decide between renting an apartment in the city and buying a house in the suburbs.

What are the quantitative and qualitative factors that affect their decision? Give some reasons why the Starr family would decide to buy the house, even if it is not the cheaper alternative.

You can probably see that the quantitative components in the Starr’s situation are not expected to be the sole factors involved in making the decision. If you have ever decided to buy a house, you will know that saving money is not the reason! Qualitative issues like the desire for personal space, the availability of parking, and the reputation of the school district play as much a part in the decision as do measurable factors like rent, inflation, interest rates, taxes, depreciation, and the cost of houses. Quantitative analysis serves mainly to identify the more economical courses of action and to measure the costs and benefits associated with making a choice.

In this text, we provide many examples of how decision making occurs in ways that takes into consideration both the quantitative and qualitative aspects of a situation. Basically, the decision making process we describe requires organizing and manipulating the data so that the most reasonable alternatives can be evaluated and compared using quantitative methods. Important qualitative factors are then considered in conjunction with these evaluations before a final decision is made.

A key component in the decision-making process is a *quantitative model,* which is an abstract representation of the quantitative components of a decision. Quantitative models identify the key quantities and their inter-relationships so that they can be measured and analyzed. Building a model for a situation can include such simple tasks as drawing a picture, or jotting down numbers on a sheet of paper, or making charts and graphs. A more sophisticated model may include spreadsheets, computer graphics, mathematical functions, probability distributions, or computer simulation programs to represent the situation. A good model can make the job of the decision-maker easier because it demonstrates, in an organized fashion, the important quantitative factors that affect a decision.

Quantitative models focus on measurable entities like amounts of money, lengths of time, costs, prices, distances, resource levels, customer demands, production rates and capacities. They enable efficient and effective decision making by organizing the relevant information in a situation, measuring the consequences of the available decision alternatives, and establishing a documented rationale to evaluate or justify the decisions that are made.

The task of finding or developing a good model to represent a particular situation may be quite difficult and could require a fair amount of mathematical sophistication and modeling experience. In many cases, the decision-maker might enlist the services of some experienced expert model builder to perform this task. For many situations, the model builder is aware of an appropriate existing model, which we call a *template model,* that has already been developed and studied and, with some modifications, can be used to support the decision-making process for the situation at hand. As we shall see, for many commonly occurring business situations, there are appropriate template models. It is not unusual to find that different organizations in different industries are faced with the same basic quantitative problems, e.g., maximize profit out of limited resources, ship goods between locations as cheaply and reliably as possible, or find the best location for a new warehouse given the locations of the terminals. *Hence, a pair of seemingly different situations might require the same template model!* The only differences would be in the size of the model and the particular values involved.

The advantage of using a template model is that it is common enough to have been deeply studied by modeling professionals. Hence, it is likely that good techniques are available for manipulating the model and answering the associated decision questions. The fields of Operations Research and Management Science share the responsibility of developing and maintaining template models and the methodologies for their solution. They also educate the decision making public about the existence and applicability of these models.

It is necessary for decision-makers in a business to continually examine their operations to make sure that: a) the right activities are being performed, (i.e., they are *effective*); and b) the activities are being performed without wasting time or money, (i.e., they are *efficient*). For example, the manager of a sneaker factory must be assured that the production process is well designed, and that the resources are being allocated to the most profitable items. A portfolio manager must make sure that allocations are quickly adjusted in response to imbalances caused by fluctuations in prices. If a business fails to continually monitor and upgrade its operations, it could experience a loss in its competitive edge and an erosion of its profitability.

When a new situation arises that requires attention or some “triggering event” raises an alarm in the operation, it then becomes necessary for the operations manager to identify what decisions must be made and what data is available to measure the impacts of different alternatives. The manager may decide to hire a consultant, e.g., to build models to help determine the best course of action. So it is necessary for the manager to state, in clear terms, the nature of the situation and the associated background information available. This information is also required to keep directors of the organization well informed. Hence, the first part of the decision making process is to develop a clear description of the situation requiring that some be made.

It is a popular belief among individuals concerned with quality assurance that in order to remain competitive, one must be diligent in continually finding components of the business operation that are in need of some improvement and that such opportunities for improvement *always* exist. Hence this step is a crucial part of the decision-making process. Skills in finding these opportunities are developed from many personal experiences and also by the “virtual experiences” of performing simulations or reading case studies about the decisions made in business operations of all types.

Quantitative Model Components

Most of the situations addressed here will make use of some type of quantitative model. Many of these can be facilitated by use of a *spreadsheet,* which involves the use of a technology where the relevant data can be placed in tabular form, e.g., as in a ledger book, and any relationships between data items can be easily managed. In other cases, *simulation* software could be employed, which provides the means by which a decision-maker could observe a situation under a variety of scenarios to develop a better understanding of the impacts of different alternatives. *Optimization* software exploits the mathematical relationships between different components to calculate what appears to be the “best” decision given the specific data values.

For optimization models, an *objective* represents some goal that the decision-maker would like to achieve. Objectives usually take on forms like, "achieve the highest possible profit level" or "minimize total costs". For example, an objective for Atlas Bookbinding Company might be to maximize the likelihood of making a profit. One of the objectives for the Starr family when deciding on living arrangements might be to choose the one that requires minimal total expense over a five-year horizon. Such objectives are often represented by mathematical functions whose value must be as large (or as small) as possible. (These are called *objective functions).* Any decisions that are ultimately suggested as a result of using the optimization model by maximizing (or minimizing) the objective function should help the decision-maker come as close as possible to achieving his or her objectives.

Constraints are rules that are placed into a model to make it realistic. Constraints prevent the model from suggesting impossible or unrealistic courses of action. Examples include assurances that time spans, distance values or production quantities be non-negative, and that an organization does not use up more of a resource, like money or time, than it has available. When customers demand a certain amount of a product or service from the organization, corresponding constraints are added to the model to make sure they are satisfied. Chapter 4 features many problems with constraints that are presented as mathematical expressions.

Parameters are the relevant quantities (data) in a situation whose values affect how the different decision alternatives compare. They are used in defining objective functions and the mathematical expressions that define constraints. They often appear as coefficients in these expressions. The values for the parameters are usually determined as part of the process wherein the decision-maker describes the situation that needs to be addressed. They are usually fixed by the situation at hand and cannot be changed by the decision-maker. For example, the Starrs might base their "rent vs. buy" decision on the values of parameters like the amount of rent they would have to pay vs. the price of the houses they saw, the prevailing mortgage interest rates, the amount of money they have saved, and so on. The relevant parameters for Atlas Bookbinding are the particular values of the fixed costs, the variable costs, and the selling price of the textbook. In many models, the values of the parameters are obtained by collecting data, aggregating information, making calculations or converting between units. For instance, since mortgage interest may be tax deductible, the rate used in the Starrs' comparison should be reflective of this discount based on what they calculate to be their effective rate of income taxation.

A *decision* is the selection of a particular course of action to address a situation. Examples include determining whether to produce an item, scheduling the deliveries from a milk truck, selecting an investment portfolio into which to invest money, or choosing the particular section of a required class. For a question that takes the form, "How many?" the decision would be a value that represents the answer. In a mathematical model, such decisions are represented as variables in the expressions representing objectives and constraints. (These are called *decision variables).*

Algebraic Decision Model Example: Cost-Volume-Profit (CVP)

Recall that it costs Atlas Bookbinders $80,000 in fixed costs to prepare the new edition of their popular Business Math textbook. The variable cost is $44.00 per book. The book generates revenue of $60 for each copy sold. The model for this situation would consist of mathematical functions that represent the total cost of production and amount of revenues obtained from sales. With data regarding the expected demand for the album, it would be possible to quantify the profit and determine whether it is worthwhile to go into production. In a situation of this type, the following components are identified. Together they comprise what we call the *Cost-Volume-Profit Model.*

We use the letter x to represent the decision variable for the number of units of the CD that Atlas will produce, should they decide to print the book. We hope that our model will help us determine an appropriate value for this variable. Ideally, the demand will be high enough and x will be exactly equal number of units demanded by the public. If we produce less that we can sell then some sales opportunity will be missed; if we produce more than can be sold then Atlas will incur the cost of producing textbooks that are not offset by sales revenue. In order to more easily perform the required analysis, the Cost-Volume-Profit (CVP) model includes the "simplifying assumption" that production can be immediately adjusted to meet demand. This implies that the value of x represents both the number produced and the number demanded. This value may not be known, but there may be some probability information available.

In order for a model to be reused in a variety of similar situations, we use letters to represent parameters. In this model, the letter F represents the value of the fixed cost in dollars, which is 80,000 in this case. The letter v represents the total variable costs in dollars, which is $44 in this situation. The letter r represents the sales revenue in dollars per unit sold, which is $60 for Atlas. Letters given to these quantities also allow for flexibility in the analysis. For example, if one parameter value changes, then most of the work performed can still be utilized, without having to start from the beginning. In addition, the model developed can be used in similar situations. The difference between this situation and others having the same form is that the specific values of F, v and r will differ.

The total gain or loss from the operations can be represented as a function of x. The total profit is equal to the revenue from sales less the fixed and variable costs incurred. This relationship can be represented algebraically as follows:

Revenue from selling x units = $R(x) = r\,x$

Cost of producing x units = $C(x) = F + v\,x$

Profit is defined as the difference between revenue and cost, so:

Profit from selling x units = $P(x) = R(x) - C(x) = (r - v)\,x - F$

The *break-even point* is the number of units for which total costs equals total revenue; i.e., the value of x satisfying $P(x) = 0$. Clearly, one should go into production only if one expects to make a positive profit. So if the company is likely to sell more than the break-even quantity, represented by x_{BEP}, then production would be profitable.

In terms of the parameters from the model, we have the following formula.

Break-even point = $x_{BEP} = F / (r - v)$

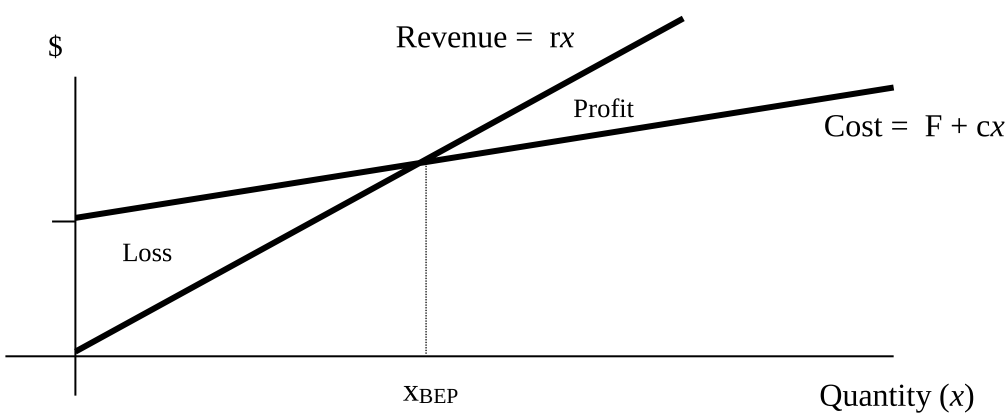
So the number of textbooks that need to be sold in order for Atlas Bookbinding to break even is:

$$x_{BEP} = F / (r - v) = 80{,}000 / (60-44) = 5{,}000 \text{ units.}$$

The profit function also tells us that if we want the particular value of x that results in some desired profit level P, the formula is:

Number of units to sell to obtain P dollars of profit = $x_P = (F + P) / (r - v)$

Some decision-makers may not decide to go into production unless they are likely to make at least some positive profit level. This formula is helpful to them. The following graph provides a picture-based model for the situation. It provides a clear illustration of how the profit or loss depends on the quantity that will be sold.



When the publisher at Atlas is confronted with the decision of whether to produce the textbook, he or she may decide to use some *probabilistic* decision criterion. For example, he or she may produce and market the item only if the probability of breaking even is greater than some desired percentage. In order to do this, probability information regarding the demand for the product is required.

Such information may take on the following form:

Event	**Probability**
Sales Exceed 1000 units	0.95
Sales Exceed 2500 units	0.75
Sales exceed 5000 units	0.60
Sales exceed 10000 units	0.10

This shows that the company is very likely to sell over 2500 books, but not very likely to sell more than 10,000. With the parameter values and the quantitative model available, it is possible to perform experiments and analyses that help the decision-maker improve his or her understanding of the situation and make an informed decision about whether to go into production. For Atlas the profit function is

$$P(x) = (60 - 44)x - 80{,}000 = 16x - 80{,}000$$

This gives the profit or loss that would result from total sales of x units. The break-even point is

$$x_{BEP} = 80{,}000 / (60 - 44) = 5{,}000 \text{ units.}$$

This implies that if the sales are expected to exceed 5,000, then it would be profitable to produce the textbook. Notice that for different values of F, r and v, the location of the break-even point would change. For example, if the per-unit revenue from the album were to increase from $60 to $65, then the break-even point would move down to 80,000 / (65 – 44) = 3809.52 or, more realistically, 3810 units. However, demand would likely be smaller at that new price. The publisher could use the model help determine a price that would provide a reasonable level of overall profitability.

If the fixed costs in the process were to increase from $80,000 to $85,000, the break-even point would have to move up to 85,000 / (60 – 44) = 5312.5 or 5513 units. Notice that when we measure the impact of a change to the data, we revert to the original values of the other parameters. So the model also shows the relationship between production costs and profitability.

With all of the experimentation on the model completed, the decision-maker is poised to make intelligent decisions. For example, after having determined the relative likelihood of achieving various profit levels, the publisher at Atlas Bookbinding Company can finally determine whether the textbook should be printed. Suppose she has decided that she will produce the album only if the chances of at least breaking even are at least 75%. Then the book would not be produced since probability that demand exceeds the break-even point of 5000 is only 60%. This particular decision criterion is somewhat subjective (qualitative); a different decision-maker might be less conservative, so would be willing to print the book if the chance of breaking even is some lower value, like 50%. In that case he or she would decide to print the textbook.

The Cost-Volume-Profit model and the analysis performed for Atlas Bookbinding can be reused in a variety of similar situations in which one is trying to determine whether the revenue generated by an activity is enough to offset the costs. It could also be used in situations where a decision-maker must select a production facility, where each option has a different fixed and variable cost structure.

Example 1: Suppose that the selling price of a small lamp is $7.50, its variable cost is $4.50, and the fixed cost for production is $15,000. Suppose we are given the following information regarding demand:

Event	Probability
Sales Exceed 2500 units	0.90
Sales Exceed 5000 units	0.80
Sales exceed 7500 units	0.60
Sales exceed 10000 units	0.20

a) What is the break-even point? Is it likely make a profit?

b) How likely is it to achieve a profit of $10,000 or more?

c) Suppose you are the manager. Would you decide to produce lamps? Why or why not?

Solution: a) The break-even point based on these values is $x_{BEP} = \frac{15000}{7.5 - 4.5}$ = 5000 units. The probability of at least breaking even is 80%.

b) The required sales volume is $x = \frac{15,000+10,000}{7.5-4.5}$ = 8333 units. The likelihood of achieving this number of sales is somewhere between 20% and 60%, (maybe about 50%).

Example 2: Alpha Cosmetics company has developed a new perfume that they want to market. If Alpha goes into small-scale production its annual fixed costs will be $400,000 and it can produce for a variable cost of $5 per unit. If it goes into large-scale production, its annual fixed costs are $1,050,000 and it can produce for a variable cost of $3 per unit. The selling price will be $10 per unit regardless of the scale of production.

a) Determine the break-even points for small-scale and large-scale production.

b) Suppose the expected demand is for 150,000 units. Which production method would be more profitable?

c) Suppose the expected demand is for 500,000 units. Which production method would be more profitable?

d) Find the number of units for which one would be indifferent between small-scale and large-scale production, i.e., the number of units for which the costs are equal.

Solution: a) The break-even point for small-scale production is $\frac{400,000}{10-5}$ = 80,000 units. For large-scale production, the break-even point is $\frac{1,050,000}{10-3}$ = 150,000 units.

b) To determine the profits we can use the formula P (x) = (r - v) x – F. The profit for small scale is given by P(150,000) = (10 – 5)150,000 – 400,000 = 350,000.

The profit for large scale is given by P(150,000) = (10 – 3)150,000 – 1,050,000 = 0. We conclude that small scale is better.

c) The profit for small scale is given by P(500,000) = (10 – 5)500,000 – 400,000 = 2,100,000.

The profit for large scale is given by P(500,000) = (10 – 3)500,000 – 1,050,000 = 2,450,000. So now, large scale is better.

d) One is indifferent if the costs are the same, so this is the point where

$$400,000 + 5x = 1,050,000 + 3x.$$

Moving the 3x to the left side via subtraction and moving the 400,000 to the right side, also via subtraction gives 2x = 650,000

Dividing both sides by 2 gives the answer x = $\frac{1,050,000-400,000}{5-3}$ = 325,000 units. This implies that sales have to exceed 325,000 units for large-scale production to be more profitable.

Managing Risk

In the previous section we discussed a model that examines profitability based on the estimates for the quantity that can be sold. Typically, these are not the only quantitative factors involved in a decision to produce an item. In these, as well as other decisions about making an investment in the hopes of eventually making a profit, the decision maker should assess the risks associated with a loss. Often, the decision-maker will place constraints on the model and any decisions it recommends to limit the likelihood and/or the severity of a loss. We refer to this as *risk management.* After all of the potential risks have been assessed, the decision-maker can consider the trade-off between potential gains and potential losses, (analyzing *risk vs. reward*).

When analyzing risk, one of the first places to begin is consider the worst possible outcome that can happen, and how likely is it. We shall call this a *worst case analysis*. For example, if Atlas decides to invest $520,000 to print 10,000 copies of the Business Math textbook, then the worst case is that no one will buy the book, which would result in a total loss of $520,000. Even though this is very unlikely, given the information about potential demand, risk analysis requires consideration of this possibility. Once the worst case and other, less severe possibilities are understood, a risk analyst can evaluate the complete range of possible outcomes along with their associated probabilities.

Suppose, for example, that an investor is offered the opportunity to pay $5,000 for a security whose return (profit) after one year could be any one of the following possibilities:

Return ($)	1,000	500	300	70	50	–500
Probability	0.20	0.50	0.15	0.10	0.04	0.01

We call this chart a *distribution* of potential returns. The probabilities represents the relative likelihood of each potential outcome. Notice that there is a .01 probability or 1% chance that the investment would lose money by yielding a return of -$500. There is an 85% chance that the investment will yield $300 in profit or more. The investor would have to weigh these possibilities before deciding whether to accept this opportunity or to pursue a different opportunity.

Suppose the investor is offered an alternative opportunity to pay the same $5000 for an investment that provides the following distribution of returns:

Return ($)	5,000	1,000	500	–500	–2000
Probability	0.35	0.35	0.25	0.04	0.01

It should be clear that this second choice is much riskier than the first. There is a small chance of losing a lot of money. However, it is more likely that the investment will return a profit that is much greater than that offered in the first investment. As we shall see, there is no objectively better choice. The decision made by the investor would need to reflect the weight she gives to the likely returns against the level of risk she is willing to accept.

Exercises

1. A cosmetic company has developed a new after-shave lotion. If the company goes into small-scale production its annual fixed costs will be $200,000 and it can produce the product for a variable cost of $3 per unit. If the company goes into large-scale production its annual fixed costs will be $400,000 and it can produce the product for a variable cost of $1.25 per unit. Assume that the selling price is $7 per unit.

    a) Determine the break-even points for small scale and large scale production.
    b) If the expected sales are for 500,000 units, which production process is more profitable: small-scale or large-scale? Determine profits for both cases.
    c) Find the number of units for which one would be indifferent between small-scale and large-scale production.
    d) If the expected sales are for 100,000 units, use the answer from part c) to determine which production process is more profitable?

2. Cyber-Space Unlimited is trying to decide if it should sell a new type of computer product. Fixed costs associated to the production of the product are estimated to be $10,000. The product will sell for $60 with variable cost of $35. The sales manager believes that sales are normally distributed with an average of 550 annually, and a standard deviation of 100.
    a) What is the break-even point?
    b) How many units must be sold to earn a profit of $20,000?

3. RMC Training is trying to decide if it should sell a new type of training product. Fixed costs associated to the production of the product are estimated to be $300,000. The product will sell for $30 with variable cost of $15. The sales director of marketing believes that sales will be normally distributed with an average of 40,000 annually, with a standard deviation of 15,000.
    a) What is the breakeven point?
    b) What is the profit/loss when 60,000 units are sold?
    c) How many units must be sold to earn a profit of $150,000?
    d) What is the probability of selling 50,000 units or more?
    e) What is the probability of earning a profit of $100,000 or more?

4. Suppose that $10,000 is invested into a mutual fund. After an investment period of ten years, the value of the investment is expected to follow a normal distribution with mean $23,000 with standard deviation $6,000.

    a) Find the percentile profile for this investment.
    b) Find the value at risk for this investment.
    c) Find the probability of a loss.
    d) Find the probability of losing more than $2,000.

Payoff Table Models

Carter Coffee Company

Carter Coffee is considering a number of different ways to increase the production capacity at its Capital Hills factory. They may either: 1) build a new, larger facility, 2) expand their present facility, or 3) lease a subcontractor's facility. Based on the information provided in a recent marketing survey, the following table lists the potential increase in profits (in thousands) for each decision alternative (i.e. choice), depending on the amount of demand that will possibly arise next year.

Possible States:

	High Demand	Moderate Demand	Low Demand
Build	500	250	-150
Expand	350	300	100
Lease	300	300	200

The chart that Carter Coffee built to describe their decision-making situation is a model that is often called a *payoff table*. It lists the set of available decision alternatives as the rows of the chart. The columns correspond to an exhaustive list of the possible outcomes (or *states*) that might occur which would in turn determine how much profit or loss might be experienced for each of the alternatives. The table gives, for each alternative-state combination, the payoff, (which usually in the form of dollars, although a point-system called a *utility* might be used instead). If the information is available, then the table might also include values representing the probability associated with each of the potential states.

The objective of the decision-maker is to select the particular alternative that has the most desirable set of potential payoffs (or points). Each alternative is evaluated and ranked in a way that is reflective of the values of the payoffs, the relative likelihood of the states, and the preferences of the decision-maker.

Observe that the payoff table associated with the Carter Coffee situation shows that the decision-maker may choose to build, or expand, or lease. If we believe that the demand will turn out to be high, then building a new facility would be most profitable alternative, but if the demand turns out to be low, then this alternative results in the largest loss, (as indicated by a negative payout). The table also shows that the alternative to lease is least risky because the payoffs for that alternative do not vary much. Actually, there is reasonable justification for selecting *any* of the three alternatives. The final decision depends on the preferences of the decision maker and the likelihood of the states.

In order to build a useful payoff table for a given situation, it is first necessary to identify the feasible alternatives that are available to the decision-maker. Then all of the potential states must be listed. These are defined to be the potential events that would affect the payoff resulting from the alternatives. Any information about the relative likelihood of the states should be incorporated. Usually this information comes in the form of a discrete probability distribution. An important rule of thumb that helps one construct a payoff table is: The decision maker has control over which alternatives to select from, but has no control over the set of states or their relative likelihood of occurring.

Example 1: Alpha Cosmetics has developed a new perfume that they want to market. Their analysts have determined that there is a 30% probability that the first year's sales will be about 750,000 units, a 20% chance that the sales will be closer to 500,000 units, a 20% chance that the annual sales will be about 300,000 units, and a 30% chance that they will sell only 100,000 units. If Alpha goes into small-scale production its annual fixed costs will be $400,000 and it can produce for a variable cost of $8 per unit. If the company goes into large-scale production, its annual fixed costs are $1,000,000 and it can produce for a variable cost of $6.50 per unit. The selling price will be $10 per unit regardless of the scale of production. Construct a payoff table to model the situation.

Solution: The payoff table is as follows. Each payoff values represents the revenue generated from sales minus the fixed and variable costs associated with each production process, (in thousands of dollars).

	States			
	Potential Demand Level (000's)			
Alternatives:	750	500	300	100
Don't Produce	0	0	0	0
Small Scale	1,100	600	200	-200
Large Scale	1,625	750	50	-650
	.3	.2	.2	.3
	Probability Distribution			

To calculate the payoff values (in thousands of dollars) in the table we use the formula

$$P = (r - v) * x - F$$

from the *Cost-Volume-Profit* model. Here x is the demand level, (which we assume is equal to the number of units sold), the unit revenue r has the value of 10, and the fixed cost F and variable cost v depend on which alternative we are considering. For instance, if Alpha decides to go into small scale production, then the profit function is

$$P_{small}(x) = (10 - 8) * x - 400{,}00 = 2x - 400{,}000.$$

So if demand is for x = 300,000 units, then the profit for small scale production is given by

$$P_{small}(300{,}000) = 2 * 300{,}000 - 400{,}000 = \$200{,}000.$$

If Alpha decides to go into large scale production, then the profit function is

$$P_{large}(x) = (10 - 6.50) * x - 1{,}000{,}00 = 3.50x - 1{,}000{,}000.$$

Now if demand is for x = 300,000 units, then the profit for large scale production is

$$P_{large}(300{,}000) = 3.50*(300{,}000) - 1{,}000{,}000 = 50{,}000.$$

The table shows that the "Large Scale" decision alternative has the highest profit potential, especially if the demand level is high, but it is also the riskiest option because there is a reasonable likelihood that the decision will result in a considerable loss. The option of not producing is, of course, the least risky. Since it is obvious that this decision will have no payoff, it is usually omitted from the payoff table model, though the "Do Nothing" option should always be considered.

Payoff tables are also useful for inventory stocking decisions, especially when an item has a limited shelf-life. For example, when ordering magazines, perishable foods, or certain electronic consumer products, a store manager must keep in mind that there is a limited time horizon during which the products can be sold. Unsold items lose most, if not all, of their value after this period.

Example 2: A bookstore sells Goodtimes magazine for $8 each and buys them for $5. Unsold copies are sold for recycling for $1. Over the past 100 weeks the store has experienced the following demand.

Demand	10	20	40	60
Number of Weeks	20	30	40	10

Assuming that one cannot sell more units than have been ordered, (i.e., second orders are not possible), develop a payoff table to model the decision of determining how many magazines to order.

Solution: The reasonable alternatives for the bookstore owner are to stock 10 magazines, 20 magazines, 40 magazines, or 60 magazines. Clearly, by stocking less than 10, she would forego certain profits, (which we call an "opportunity loss"); by stocking more than 60, she would be wasting her money on issues that will never be sold. By similar logic, one could see that it would never be appropriate for her to stock a number between say 10 and 20, or between 20 and 40 – such a value would guarantee one of those types of loss. The amount of profit she makes depends on how many magazines she decides to stock and how many customers come in to buy them. Past history shows that the only possibilities for any given week are for a demand of 10, 20, 40 or 60 magazines. It also shows that the probability that the demand will be 60 is 10/(20+30+40+10) or 10%. The payoff table for the situation looks like the following:

	States			
Alternatives	Demand is for 10	Demand is for 20	Demand is for 40	Demand is for 60
Stock 10	30	30	30	30
Stock 20	-10	60	60	60
Stock 40	-90	-20	120	120
Stock 60	-170	-100	40	180
Probability:	.2	.3	.4	.1

If the bookstore stocks 60 magazines and the demand is only for 40, then the bookstore incurs a cost of (60)($5) = $300, and makes (40)($8) = $320 revenue for the magazines sold to customers plus (20)($1) = $20 for the magazines recycled. So the profit listed in the corresponding cell of the chart is $40. If the bookstore stocks 40 magazines and the demand is for 60, then the bookstore could only sell 40 issues, so the corresponding total profit of (40)($3) = $120 is listed in the chart. In general, we can find payoffs in this table by using the formula:

Payoff = 8*(number sold) – 5*(number ordered) + 1*(number unsold).

For example, when the demand is 10 and 40 are stocked, 10 are sold, 40 are ordered and therefore 30 are unsold. Hence the payoff is 8*10 – 5*40 + 1*30 = –90.

The numbers in the bottom row represent the probability distribution for the possible states. Obviously, these numbers must be non-negative and sum to one. With the information laid out this way, it is easy to see the consequence of each decision; this will make it easier to make a well informed decision.

In some decision analysis cases, the list of potential states is not immediately obvious. In the next example, a decision-maker must select an investment allocation consisting of different proportions of two stocks. The payoff depends on how *each* of the stock performs. To simplify some of the calculations we will consider the case where the performance of any one stock does not affect the performance of any other. That is, the returns are independent or very close to being independent. Recall that when events A and B are independent, then the probability of both events A and B occurring simultaneously is given by Prob(A and B) = Prob(A) * Prob(B).

Example 3: Suppose an investment analyst is considering two stock allocations. Allocation I invests $3,000 in stock X and $7,000 in stock Y. Allocation II invests $5,000 in X and $5,000 in Y. The investment period is short term, say one day. Over the investment period stock X can either increase by 2%, increase by 1% or decrease by 1% with probabilities .2, .4, and .4 respectively. Stock Y will either increase by 5% or decrease by 3% with probability .6 and .4 respectively. Determine the set of potential states and find the probability distribution. Then set up the payoff table for the investment decision. Assume that the performance of stocks X and Y are independent

Solution: Since there are 3 possible outcomes for stock X and 2 possible outcomes for stock Y, there are (3)(2) = 6 states. Since we are assuming independence between Stocks X and Y, the probability of each state is obtained by multiplying the probabilities. The states and their probabilities are as follows:

A. Stock X increases by 2%, stock Y increases by 5%, (Prob. = (.2) (.6) = .12)
B. Stock X increases by 2%, stock Y decreases by 3%, (Prob. = (.2) (.4) = .08)
C. Stock X increases by 1%, stock Y increases by 5%, (Prob. = (.4) (.6) = .24)
D. Stock X increases by 1%, stock Y decreases by 3%, (Prob. = (.4) (.4) = .16)
E. Stock X decreases by 1%, stock Y increases by 5%, (Prob. = (.4) (.6) = .24)
F. Stock X decreases by 1%, stock Y decreases by 3%. (Prob. = (.4) (.4) = .16)

Next we show how to obtain the payoffs. Consider Portfolio I when state A occurs. Portfolio I invests $3,000 in stock X and $7,000 in stock Y, and when state A occurs X increases by 2% and Y increases by 5%. This gives the following.

Increase in stock X = 3,000(.02) = 60
Increase in stock Y = 7,000(.05) = 350
Payoff for Allocation I-State A = 60 + 350 = 410.

The payoffs for the other scenarios are obtained similarly and are given below.

	States					
Alternatives:	A	B	C	D	E	F
Allocation I	410	-150	380	-180	320	-240
Allocation II	350	-50	300	-100	200	-200
Probability	.12	.08	.24	.16	.24	.16

Exercises

1. Megley Cheese Company is a small manufacturer of several cheese products. One of the products is a cheese spread that is sold to retail outlets. Jason Megley must decide how many cases of cheese spread to manufacture each month. The probability that the demand will be six cases is .1, seven cases .3, eight cases .3, and nine cases .3. The cost of every case is $45, and the price that Jason gets for each case is $95. Cases not sold by the end of the month are of no value due to spoilage. Construct a payoff table for this situation. Obviously, the amount sold can not be larger than the amount manufactured in a given month.

2. A T-shirt company has developed a new T-shirt that they want to market. The company has determined that there is a 40% probability that the annual sales will be roughly 500,000 units, a 35% chance that the annual sales will be roughly 300,000 units, and a 25% chance that the annual sales will be roughly 50,000. If the company decides to subcontract to carry out production, its annual fixed costs will be $0 and it can produce for a variable cost of $6 per unit. If the company decides to produce itself, its annual fixed costs are $1,000,000 and it can produce for a variable cost of $3.00 per unit. The selling price is $14 per unit regardless of how the item is produced. Construct a payoff table to analyze the production decision.

3. A bookstore sells Hardtimes magazine for $11 each, buys them for $4 each, and receives a $2 credit for every unsold copy. Over the past 120 weeks it has experienced the following demand.

Demand	100	200	350	500
Number of Weeks	25	30	40	25

   a) Construct a payoff table for the decision of determining how many magazines to order. Use the four alternatives of ordering 100, 200, 350 or 500.
   b) Estimate the probability distribution of the demand.
   c) Determine the payoffs when 300 magazines are ordered. Assume that the states are still 100, 200, 350 and 500.

4. Infinity Computer Company has developed a new computer product and has obtained the following forecast of sales.

Sales	1,000	2,500	5,000
Probability	.45	.35	.20

The company is trying to decide how to produce the product and has identified three alternatives: subcontract, build a small production line, or build a large production line. If they subcontract there is no fixed cost and variable cost is $80 per unit. If they build a small production line, the fixed cost is $35,000 and the variable cost is $40. If they build a large production line, the fixed cost is $65,000 and the variable cost is $25. The product will be sold for $120 regardless of how it is made.

Fill in the payoff table given below for the decision of determining how to produce the product.

Sales

Alternatives	**1,000**	**2,500**	**5,000**
Subcontract			
Build Small			
Build Large			
Probability			

Criteria for Making a Decision

In this section we discuss the some common selection criteria that can be used in conjunction with a payoff table to select an alternative. When there are no estimates available concerning the relative likelihood of the states, the selection of an alternative is based solely on the values of the payoffs. Because of the uncertainty about which state will occur, each alternative comes with some risk that an undesirable payoff will be realized. Each of the criteria differs by the way they treat the risk vs. reward tradeoff. In general, the alternative selection process is to calculate a value for each decision alternative based on the chosen criterion, and then to choose the alternative having the best value.

The "Worst Case" criterion. Determine the minimum payoff (or maximum loss) for each alternative. Choose the alternative corresponding to the maximum of these minimum payoffs. This criterion is considered to be *risk-averse* because it considers only the worst case possibility when evaluating an alternative. It would be used by very conservative or pessimistic managers.

The "Average Case" criterion. Determine the average payoff for each alternative. Choose the alternative corresponding to the maximum average payoff. This criterion is considered to be *risk-neutral* because it gives equal consideration to all outcomes.

The "Minimax Regret" criterion. Compute the *Regret Table* as follows. Assuming that columns are labeled with the potential states, subtract every element in each column from the largest element in that column. Using the regret table, identify the maximum regret for each alternative. Choose the alternative corresponding to the minimum of these maximum regrets. Since regret is used as a way to measure risk, this criterion is considered to be risk-averse. In other words, for each state, ask yourself how much would you regret if that state occurred? For example, if the largest payoff for a given state is 200 and the alternative you chose had a payoff of 50, you'd regret that you lost out on 150 more by choosing what you chose instead of the one that had the highest payoff.

For all of the above criteria, we "do" the second half of the criteria first, i.e. the max, the min, the mean and the max, respectively, for each alternative. Put these values in a new column. Then "do" the first half of the word on the values in this new column.

For example, consider the payoff table for Carter Coffee:

		Demand:		
		High	Moderate	Low
	Build	500	250	–150
Alternatives:	Expand	350	300	100
	Lease	300	300	200

For the Worst Case criterion, we find the "min" first, getting –150 for Build, 100 for Expand and 200 for Lease. Then we find the "maxi" part. The largest of these numbers is 200, so Lease is the alternative chosen using the Worst Case criterion. To a person who is pessimistic and places an emphasis solely on the worst case potential of an alternative, the most preferable of these options is to Lease.

For the Average Case criterion, we find the "mean" first, getting 200,000 for Build, 250,000 for Lease and 266,666.67 for Lease. Then we find the "maxi". The largest of these corresponds to

Lease, so that is the alternative chosen using the Average Case criterion.

The Regret values (in thousands) are in the following table:

	High	Moderate	Low
Build	0	50	350
Expand	150	0	100
Lease	200	0	0

Notice, by definition, that the entries in a regret table are always non-negative. The "Build" option has a maximum regret of $350,000, the "Expand" option has a maximum regret of $150,000, and the "Lease" option has a maximum regret of $200,000. The Expand option has the smallest of these maximum regret values.

The following table summarizes the results of these calculations:

	Worst Case	***Average Case***	***Minimax Regret***
Build	-150	200	350
Expand	100	250	150*
Lease	200*	267*	200

From this table we draw the following conclusions. Risk-averse decision-makers would select to lease a facility from a sub-contractor, while other decision-makers would take more comfort by expanding instead.

Probability-based Criteria

A reliable estimate for the relative likelihood of each of the state's occurring would be represented as a discrete probability distribution, p_1, p_2, ...p_n, over the set of *n* states. This type of data may be obtained from past data, a forecast, or perhaps a marketing survey. With this information, we can determine the expected payoff and use some of the risk measures introduced in Section 1.3, such as the standard deviation and the percentile profile. The additional probability information and risk measures will allow the decision maker to make better decisions in general.

If alternative A has payoff values x_1, ..., x_n, corresponding with the *n* states, then the *expected payoff from A*, denoted EP(A), is a weighted average of the payoffs, that uses the probabilities as weights, i.e., $EP(A) = x_1 p_1 + x_2 p_2 + \ldots + x_n p_n$. Given probability information, the expected payoff represents a better measurement than the mean payoff calculated earlier because it takes into consideration the relative likelihood of the states. This gives rise to the following selection criteria.

The "Expected Payoff" criterion Determine the expected payoff for each alternative. Choose the alternative corresponding to the maximum value of EP(). When no probability information is available, we assume each state is equally likely, in which case this criterion is equivalent to the "Average Case" criterion. Consequently, this criterion is also considered to be risk-neutral

Let us return to the Carter Coffee company situation. Suppose that the probability of having high, moderate, and low demand is .5, .3, and .2, respectively. Based on the expected payoff criterion, the "Build" option for Carter Coffee has a value of $295,000, the "Expand" option has a value of $285,000, and the "Lease" option has a value of $280,000. The most preferable of these options is to Build.

Probability information can also be used to effectively measure the relative risk inherent to decision alternatives. Since risk is somewhat qualitative, it is difficult to measure directly. However, we can indirectly measure it. A wide range of values between the best case and worst case payoffs indicates a high level of risk. Measurements that incorporate probabilities such as the inter-quartile range or the standard deviation, are often used. They can be incorporated into decision criteria to allow the decision maker to select an alternative that provides a favorable return at an acceptable level of risk. Such criteria should recognize that some decision makers are willing to accept risk for a higher level of return, while others are not.

We re-acquaint readers with formulas for the "Variance" and "Standard-Deviation" of a probability distribution:

$$\mathrm{Var}(A) = [(x_1)^2\, p_1 + (x_2)^2\, p_2 + \ldots + (x_n)^2\, p_n] - EP(A)^2$$

$$SD(A) = \sqrt{\mathrm{Var}(A)}$$

These measures are used to assess relative risk in the following sense. If two investments have standard different deviations, then we infer that the investment with the smaller standard deviation is more predictable and hence involves less risk. Reasonable investors do not accept risk for free. A rational decision-maker would not select an alternative having a high level of risk unless there is a commensurately high level of expected payoff. So financial-based markets usually include conservative investment types which would have yearly returns that are somewhat modest but are predictable, i.e. with low risk, as well as more aggressive (and risky) investments opportunities that provide very good returns in some years and very low returns (or even losses) in other years. Such alternatives would only be attractive and remain in the market so long as their overall average yearly returns are higher than those of the conservative alternatives.

Nowadays, using the internet, an investor can easily obtain values for the average return and risk measures such as the standard deviation and beta of an investment, based on its past performance. If one expects that these measures are reflective of future performance, he or she can evaluate the alternatives based on some risk/return trade-off. These alternatives are for decision-makers who do not want to automatically take the alternative with the highest expected payoff (since it may have too much risk) nor the lowest risk (since it may have a poor expected payoff). Some approaches may include:

The "Risk-constrained" Expected Payoff criterion For each alternative A, calculate the expected payoff, EP(A), and a measure of the risk R(A), (e.g., the standard deviation SD(A)). Choose some value R, representing the maximum acceptable risk. Select the alternative having the largest value of EP(A), considering only those with $R(A) < R$. (If no such alternative exists, then R must be increased).

The "Payoff-constrained" Minimum Risk criterion For each alternative A, calculate the expected payoff, EP(A), and a measure of the risk R(A), (e.g., the standard deviation SD(A)). Choose some value P, representing the minimum acceptable payoff. Select the alternative having the smallest value of R(A), considering only those with EP(A) > P. (If no such alternative exists, then P must be decreased).

The "Payoff-at-Risk" criterion For each alternative A, calculate the expected payoff, EP(A), and a measure of the risk R(A). Choose some appropriate non-negative value, α. The *payoff-at-risk* criterion selects the alternative having the largest value of EP(A) – α R(A).

These criteria select an alternative that is a compromise between risk and return. The value for α in the payoff-at-risk criterion is set to match how much emphasis the decision-maker wants to place on risk. Risk-accepting decision-makers use very low values for α; risk-averse decision-makers use larger values for α.

Let us return once again to the Carter Coffee Company decision involving expanding their operation. Suppose that now there are three options for how to build a new facility, namely, small, medium (the original build alternative) and large, plus the "expand" and "lease" alternatives. This gives a total of five alternatives. The results of the expected payoff and various risk calculations for each alternative are listed below.

	Demand:		
	High	Moderate	Low
Build Small	400	200	100
Build Medium	500	250	-150
Build Large	750	375	-400
Expand	350	300	100
Lease	300	300	200
Probability:	.5	.3	.2

Alternative	**Expected Payoff**	**Standard Deviation**
Build Small	280	125
Build Medium	295	247
Build Large	408	435
Expand	285	95
Lease	280	40

Suppose that one wanted to use the risk constrained expected payoff criterion and choose the alternative with the best (largest) expected payoff subject to a standard deviation of at most 250. With this criterion, the Build Large alternative becomes infeasible because its standard deviation is too large. This results in choosing the Build Medium alternative. Suppose that one wanted to use the payoff constrained minimum risk criterion and choose the alternative with the best (smallest) standard deviation subject to an expected payoff of at least 285. With this criterion the Build Small and Lease alternative become infeasible because their expected payoffs are too small. The best alternative is to Expand. Next we give a summary table with these criteria and the payoff at risk criteria as well.

	Risk constrained Expected Payoff R(A) < 250	Payoff Constrained Relative Risk (SD) EP(A) > 285	Payoff at Risk with $\alpha = .25$
Small	280	***	249
Medium	295*	247	233
Large	***	435	299*
Expand	285	95*	261
Lease	280	***	270
Best:	**Build Medium**	**Expand**	**Build Large**

Example 1 Consider the following payoff table:

Alternatives	**States** s_1	s_2	s_3
A_1	9	12	-1
A_2	3	15	−2
A_3	10	−3	−4
A_4	5	5	5
Probability	.6	.1	.3

Find the alternative that would be selected based on the Worst Case and Expected Case criteria. Find the alternative selected under the risk constrained expected payoff criterion, using standard deviation for relative risk and a maximum relative risk of 5.00. Find the alternative selected under the value constrained minimum risk criterion, with a minimum expected payoff of 4.00.

Solution A summary table is given below. Calculations for expected payoffs and SD are given below. A triple asterisk (***) is used when an alternative is not feasible due to the constraint.

	Worst Case	**Expected Payoff**	**Expected Payoff w/ R(A) < 5.00**	**Min Risk w/ EP(A) > 4.00**
A_1	-1	6.3*	***	5.19
A_2	-2	2.7	2.7	***
A_3	-4	4.5	***	6.74
A_4	5*	5.0	5.0*	0*

<u>Expected Payoff Calculations</u>

$EP(A_1) = 9(.6) + 12(.1) + (-1)(.3) = 6.3$
$EP(A_2) = 3(.6) + 15(.1) + (-2)(.3) = 2.7$
$EP(A_3) = 10(.6) + (-3)(.1) + (-4)(.3) = 4.5$
$EP(A_4) = 5(.6) + (5)(.1) + (5)(.3) = 5.0$

Standard Deviation Calculations

$Var(A_1) = (9)^2(.6) + (12)^2(.1) + (-1)^2(.3) – 6.3^2 = 26.91$
$Var(A_2) = (3)^2(.6) + (15)^2(.1) + (–2)^2(.3) – 2.7^2= 21.80$
$Var(A_3) = (10)^2(.6) + (–3)^2(.1) + (–4)^2(.3) – 4.5^2 = 45.43$
$Var(A_4) = (5)^2(.6) + (5)^2(.1) + (5)^2(.3) – 5^2 = 0.00$

$SD(A_1) = 5.19$
$SD(A_2) = 4.67$
$SD(A_3) = 6.74$
$SD(A_4) = 0.00$

Example 2 An investor has $10,000 to invest in two different asset classes. Her selection is between classes A and B. She feels that over the investment period there is a 60% chance of doubling the investment and a 40% chance of losing half of the investment, for each of the investments. Her alternatives are:

1. Invest the entire amount in Asset Class A.
2. Invest the entire amount in Asset Class B.
3. Invest $5,000 in each Asset Class.

a) Construct the payoff table. Which alternative has the highest expected payoff?
b) Which outcome has the lowest probability of losing money?
c) Determine the standard deviation for each alternative.
d) Which alternative is best using the SD-based payoff-at-risk criterion with $\alpha = .05$?

Solution The payoff table is as follows.

Change in A:	+100%	+100%	-50%	-50%
Change in B:	+100%	-50%	+100%	-50%
Invest $10,000 in A	10,000	10,000	-5,000	-5,000
Invest $10,000 in B	10,000	-5,000	10,000	-5,000
Invest $5,000 in each	10,000	2,500	2,500	-5,000
Probability:	0.36	0.24	0.24	0.16

Notice that the probability of a loss when investing all $10,000 in A is 0.24 + 0.16 = 40%. The same is true when investing all $10,000 in B. But when investing $5,000 in A and B, the probability of a loss is reduced to 16%. Any of the selection criteria can be applied by a decision-maker. The remaining, probability based, measurements are as follows.

	Expected Payoff	Standard Deviation	Payoff at Risk with $\alpha = .05$
A	4,000 *	7,348	3632.6
B	4,000 *	7,348	3632.6
A&B	4,000 *	5,196	3740*

From the table we conclude that the option having the least risk is the one that includes some investment in Asset Class A and some in Asset Class B.

Example 3 A mutual fund company offers a variety of products in which an investor can choose to invest. For one of the products, say product A, past history indicates that yearly returns are normally distributed with a mean of 6% and a standard deviation of 3%. Another product, say product B, has returns that are uniformly distributed between –2% and 20%, in other words, every value between –2% and 20% is equally likely to occur.

Which product is better using the Payoff-at-risk criterion with $\alpha = .5$?

Solution Here we apply the concept of expected payoff and payoff at risk to the return percentages. Applying the payoff-at-risk criterion, the first product has a payoff at risk of

Payoff at risk for A = Expected return rate – (α)*Standard deviation = 6% – (.5)(3%) = 4.5%.

To evaluate the payoff at risk of product B we first note that from probability theory it is known that a uniform distribution with lower bound a, and upper bound b, has a mean of (a + b)/2 and the standard deviation is (b – a)/$\sqrt{12}$. For product B, a = –2% and b = 20%. Thus product B yields an expected return of (–2% + 20%) = 9% with a standard deviation of

$$(20\% - (-2\%))/\sqrt{12} = 22\% \div \sqrt{12} \approx 6.4\%.$$

This gives us

Payoff at risk for B = Expected return rate – (α)*Standard deviation = 9% – (.5)(6.4%) = 5.8%,

So the second investment product would be selected since it has a largeer payoff at risk return percentage.

Multiple State Sets

The traditional payoff table model presented in the previous examples is built with the implicit assumption that the states and their relative likelihood are identical, regardless of which alternative is selected. When there is a different set of states associated with each alternative, the model must be modified. In most cases, though, the calculations associated with the desired selection criterion can be performed, and an alternative can be selected.

Example 4 An investment analyst is studying two different portfolios given below.

Portfolio 1		***Portfolio 2***	
Payoff	**Probability**	**Payoff**	**Probability**
10,000	.5	15,000	.3
5,000	.3	3,000	.5
–3,000	.2	–1,000	.1
		–5,000	.1

Which portfolio is preferable?

Solution A single payoff table cannot represent this situation because each decision alternative has its own unique set of consequences. However, most of the measurements that help a decision-maker select an alternative can still be calculated.

	Maximax	**Worst Case**	**Expected Payoff**	**Standard Deviation**	**SD-based Payoff at Risk with $\alpha = .1$**
Portfolio 1	10,000	–3000*	5,900*	4948	5410*
Portfolio 2	15,000*	–5000	5,400	6741	4726

In this case, a risk-seeking individual would select Portfolio 2 because it has a larger potential payoff. Most other decision-makers would select Portfolio 1 because it has both a larger expected payoff and smaller risk.

Exercises

1. Suppose that a convenience store manager has determined the following distribution for the monthly demand of BigSki magazine.

Copies Demanded	Probability
20	.3
21	.4
22	.3

The magazine is purchased directly from the local distributor for a price of \$2.00 and sells for \$4.00 per copy. Unsold copies are returned at the end of the month for a \$.75 credit. Construct a payoff table for the decision of determining how many issues to order.

2. Construct the payoff table for the following investment decision:

(1) Invest \$10,000 in commercial property and \$5,000 in bonds.
(2) Invest \$8,000 in commercial property and \$7,000 in bonds.
(3) Invest \$4,000 in commercial property and \$11,000 in bonds.

Commercial property can increase by 10% with probability .6 or decrease by 5% with probability .4. Bonds can increase by 8% with probability .8 or decrease by 3% with probability .2. Assume that the performance of bonds is independent of the performance of commercial property are independent.

3. An investor has the following options. Invest \$10,000 in Stock A and \$20,000 in Stock B, or invest \$5,000 in Stock A and \$25,000 in Stock B. Stock A will either increase by 10% with probability .4, or decrease by 5% with probability .6. Stock B will either increase by 5% with probability .3, or decrease by 5% with probability .7. Assume the performance of Stock A and Stock B are independent of each other.

a) Construct a payoff table for this decision.
b) Give the probability distribution of the states.
c) Find the expected payoff for each alternative.

4. A payoff table associated with a decision is given below.

	States			
Alternatives	s_1	s_2	s_3	s_4
A_1	4	12	8	8
A_2	13	10	7	6
A_3	10	23	1	–6

a) Find the best alternative using the Worst Case, and Average Case criteria.
b) Construct the regret table and find the best alternative based on the minimax regret criterion.
c) Suppose that the probability distribution for the states is: $p_1 = .4$, $p_2 = .1$, $p_3 = .3$, and $p_4 = .2$. Find the best alternative under the expected payoff criterion.

d) Use the probabilities given in part (c) to calculate the standard deviation for each alternative and determine the alternative selected using the SD-based payoff-at-risk criterion with $\alpha = .5$.
e) Find the alternative selected under the risk constrained expected payoff criterion, using standard deviation for relative risk and a maximum relative risk of 5.00

5. A company is trying to decide how to increase its production capacity. The decision data has been organized in the following payoff table.

	States		
Alternatives	High Demand	Moderate Demand	Low Demand
Expand	35	80	20
Build New Facility	50	60	25
Subcontract	30	100	-20

a) Find the best decision alternative using each of the following criteria Worst Case, and Average Case.
b) Determine the regret table and find the best decision alternative using the minimax regret criterion.
c) Suppose that the probabilities of the states are .5, .4, and .1. Find the best alternative under the expected payoff criterion.
d) Calculate the standard deviation for each alternative and determine the best decision alternative using the SD-based payoff-at-risk criterion with $\alpha = .1$. Use the probabilities given in part (c).

6. A corporate committee is studying four different marketing alternatives A_1, A_2, A_3, A_4. The states associated with the alternatives are not the same. The following tables indicate the appropriate payoff values:

Table 1

	States		
Alternatives	s_1	s_2	s_3
A_1	10000	5000	–1000
A_2	8000	9000	–2000
Probability:	.1	.5	.4

Table 2

	States	
Alternatives	s_4	s_5
A_3	15000	–5000
A_4	12000	2000
Probability:	.3	.7

a) Which alternative would be selected using the expected payoff criterion?
b) Calculate the SD for each alternative. Which alternative would be selected using the Payoff-at-risk criterion with $\alpha = .5$?

7. A payoff table is given below.

	States			
Alternatives	s_1	s_2	s_3	s_4
A_1	10	15	2	8
A_2	9	20	6	3
A_3	15	10	1	4

a) Find the best decision alternative using the Worst Case, Average Case, and minimax regret criteria.

b) Suppose that the probabilities of the states are: $p_1 = .2$, $p_2 = .3$, $p_3 = .4$, and $p_4 = .1$. Find the best alternative under the expected payoff criterion.
c) Determine the mean absolute deviation associated with each alternative. Which alternative has the smallest risk?
d) Among all alternatives with expected payoff above 8, which one has the lowest risk?

8. An advertising agency is considering three alternative advertising media to promote a client's product: television, radio and newspaper. The payoffs will depend on the estimated size of the populations reached. The populations are assumed to be either 100,000, 200,000, or 300,000. For a population of 100,000 the payoffs are estimated to be 2,500 for television, 5,000 for radio, and 4,000 for newspaper. For a population of 200,000 the payoffs are estimated to be 3,500 for television, 1,500 for radio, and 5,500 for newspaper. For a population of 300,000 the payoffs are estimated to be 7,000 for television, 2,500 for radio, and 6,000 for newspaper.

   a) Construct the payoff table that gives the number of payoffs for each alternative.
   b) Which alternative is best using a Average Case decision criterion?
   c) Which alternative is best using a Worst Case decision criterion?

9. A manufacturer produces and sells chilled ready-to-eat pasta salad in lots of 50 serving units each. These items have a limited shelf life; therefore, if items are made but not sold, they have no value. Regular production runs are made on Friday of each week for sales the following week, however, if demand exceeds supply during the week, an extra production run can be made during the week. The cost per unit for a regular run is $5 per unit, whereas the cost of a production run during the week is $7 per unit. All items are sold for $10 per unit regardless of production cost. Historically, demand has been for 50, 100, or 150 units each week, so the company is trying to decide how many units should be made on Friday: 50, 100, or 150.

   a) Prepare a payoff table showing profits for each of the production lot sizes.
   b) If probability of demand for 50 units is .40, the probability of demand for 100 units is .50, and the probability of demand for 150 units is .10, what is the expected profit associated with each alternative lot size?
   c) Using the SD calculation, determine which alternative has the smallest risk. Which alternative is selected using the Payoff-at-risk criterion with $\alpha = .5$?

10. An investment analyst is studying three different portfolios given below.

Portfolio 1

Payoff	**Probability**
10,000	.6
–2,000	.4

Portfolio 2

Payoff	**Probability**
25,000	.2
10,000	.5
–2,000	.15
–10,000	.15

Portfolio 3

Payoff	**Probability**
6,000	.45
4,000	.55

a) Determine the expected payoff for each portfolio. Which portfolio would be selected if maximizing expected profit was the decision criterion?
b) Determine the risk as measured by the standard deviation for each portfolio. Which portfolio has the smallest risk as measured by the standard deviation?
c) Among all portfolios with expected payoff above 5,000, which one involves the least risk?

11. In the game of chuck-a-luck with three dice you pick a number from 1 to 6 and the operator rolls three dice. If the number you pick comes up on all three dice, he pays you $3; if it comes up on two of the three dice, he pays you $2; and if it comes up on just one of the three dice, he pays you $1. Only if the number you picked does not show up at all do you pay him exactly $1. Chuck-a-luck with four dice is played the same way with a chance to win either: $4, $3, $2, or $1, but in this game if the number picked does not show up at all the player must pay the operator $2.50. Use an expected criterion to determine which game is better for the player.

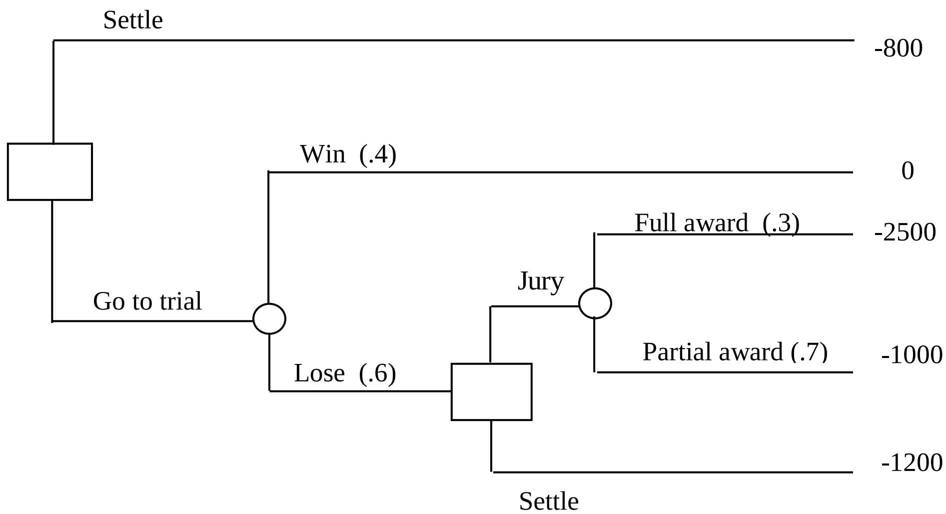
Decision Tree Models

Cookie Cutter Promotions

The owner of Cookie Cutter Promotions is being sued as a result of an injury suffered by a fan at a music festival in the park. The plaintiff was sitting on the grass when an enthusiastic crowd of people charged for the stage and trampled her. In her suit, the plaintiff accuses the promoter of being negligent and is asking for an award of $2.5 million in damages.

The promoter has been advised by his lawyers that there is a 60% chance he will be found negligent. However, even with a negative verdict, there is only a 30% chance that the jury will award the full $2.5 million. There is a 70% chance that the award will be for only $1 million. The lawyers also indicated that he could settle with the plaintiff for $800,000 before the trial. If the promoter goes to trial and is found negligent, he could still settle the suit for $1.2 million before the jury determines an award.

Suppose that a decision-maker is faced with a *sequence* of inter-related decisions. Each individual decision and the subsequent outcome have an impact on the next set of alternatives and their consequences. Since there are multiple interconnected decisions to evaluate, a payoff table cannot be used. A *decision tree* is a more general model that enumerates the consequences from sequences of decisions and lists their ultimate payoffs. It is a visual model that describes in hierarchical form the expanding set of possibilities that may occur over time. Each of the outcomes that may follow a decision alternative is branched off to enumerate all of the potential final states. The tree gives the payoff (in dollar or utility values) for each of these potential states. The decision tree also represents the probability distribution at each random event.



To represent the passage of time, the decision tree model is interpreted from left to right in chronological order. As the above decision tree show, the promoter must first decide whether to settle the suit right away or to try the case in court. (This decision is denoted by a box). If he decides to go to court, he may or may not win the case. (The possible states are describes as lines emanating from a circular node). The promoter has no control over which outcome will occur, but is assumed to have some information about the relative likelihood, (which is represented by the discrete probability values on the lines). If he loses the trial, he then has to decide whether to settle with the plaintiff before the jury determination, (over which, again, he has no control either).

As we shall see, the decision tree model representing the promoter's options will help him establish an appropriate course of action. The terminal points at the right end of the decision tree provide the ultimate payoff value (or cost if the value is negative) that represents the net financial consequence of a sequence of alternatives selected and outcomes occurring. In this situation the best case the promoter can hope for is to break even; in the worst case, he could lose as much as $2.5 million.

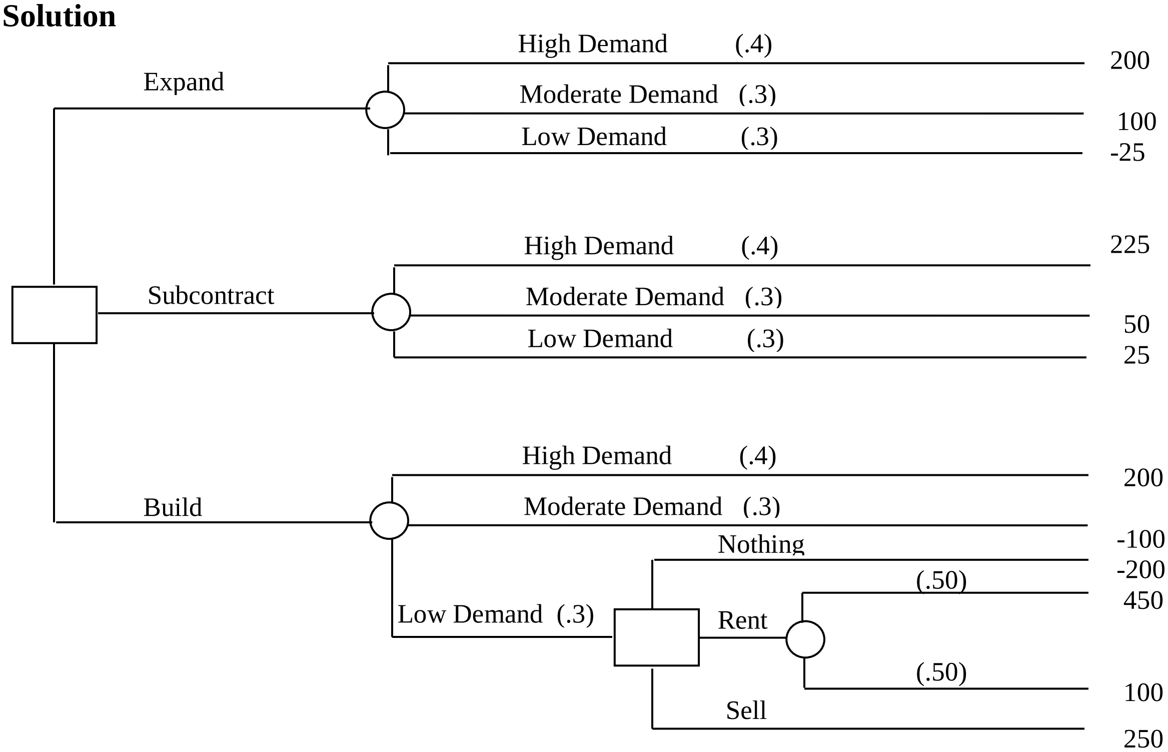
Example 1 A manager in a manufacturing company is trying to decide if they should expand their present production facility, or subcontract to increase production, or build a new production facility. Payoffs have been estimated and placed in the following table (in thousands).

		High Demand	Moderate Demand	Low Demand
	Expand	200	100	-25
Alternatives	Subcontract	225	50	25
	Build	200	−100	*
	Probability	.4	.3	.3

* If the manager of the company decides to Build and the demand turns out to be Low, then he has three options. He can do nothing which would result in a loss of $200,000. He can rent out the new facility which could create a profit of $450,000 with a 50% probability or a profit of $100,000 with a 50% probability. Or he can decide to sell the facility to earn a profit of $250,000.

Construct a decision tree which illustrates the decisions.

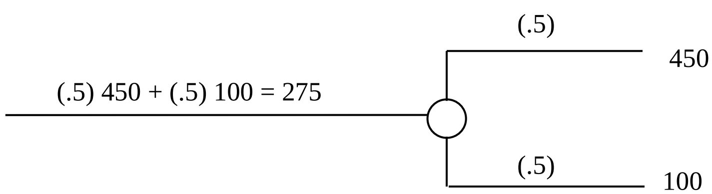
Solution



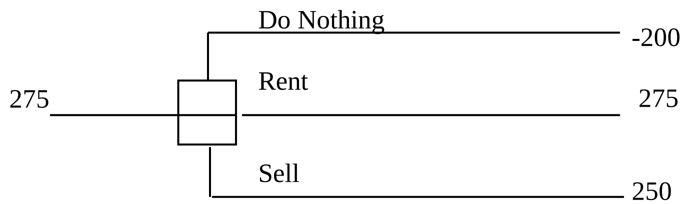
Analyzing a Decision Tree

After the decision tree is constructed, it is analyzed so that the sequence of decision alternatives that maximizes the expected payoff can be found. The following operations are performed, working from the right side of the tree back to the left side:

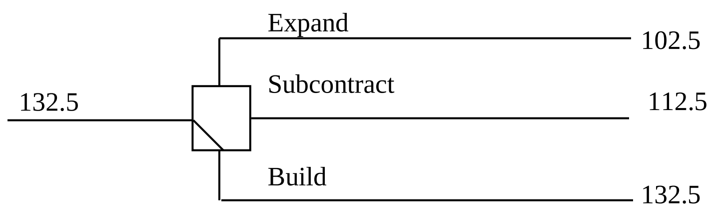
- For a circular node that represents a set of potential states with a probability distribution, calculate the *expected value* of the payoffs on the branches. For example, the branch of the tree in Example 1 that represents the possible states from renting the facility is evaluated as follows. The value associated with this alternative is $275,000.



- When there is a rectangular node representing a decision point, choose the alternative with the *largest payoff*. For example, the branch of the tree in Example 9 where the decision maker must “Do Nothing” or “Rent” or “Sell” is evaluated as follows. The best alternative is to “Rent”.



Following this process through to the left side of the tree, the initial decision is evaluated as follows.



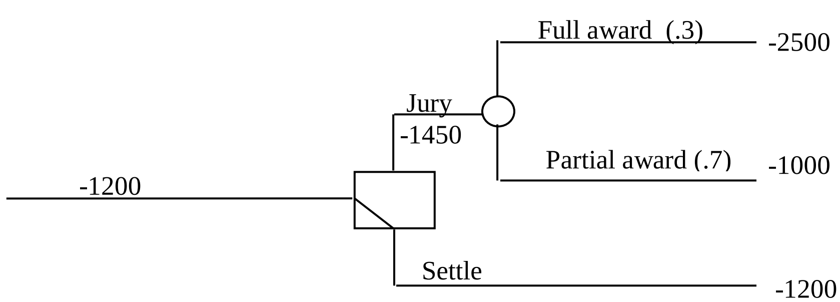
The course of action that has the maximum the total expected payoff ($132,500) is to build a new production facility; if the demand turns out to be low then rent out the facility.

In summary, to analyze a decision tree one works left to right and performs the following two steps.

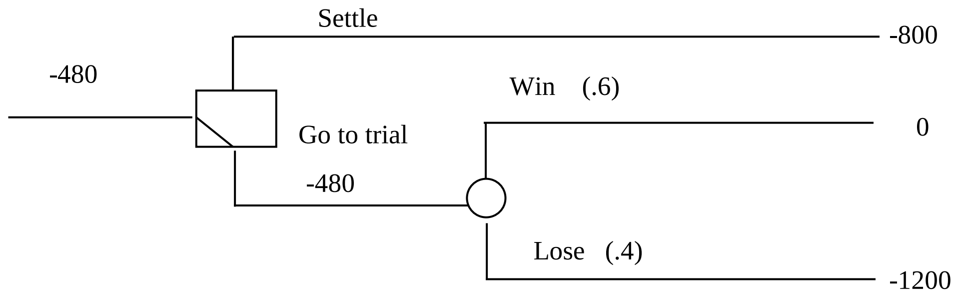
At every random event (indicated by a circle) perform an expected payoff or an expected cost calculation.

At every decision point (indicated by a rectangle) choose the best alternative and ignore the remaining alternatives. Use the payoff from the best alternative in any subsequent expected payoff (cost) calculations.

The evaluation process can be applied to the decision tree for the concert promoter, to help him find the best course of action. The decision about whether to let the jury determine the award if the court case is lost is evaluated as follows:



Now the decision about whether to go to trial can be evaluated as follows:



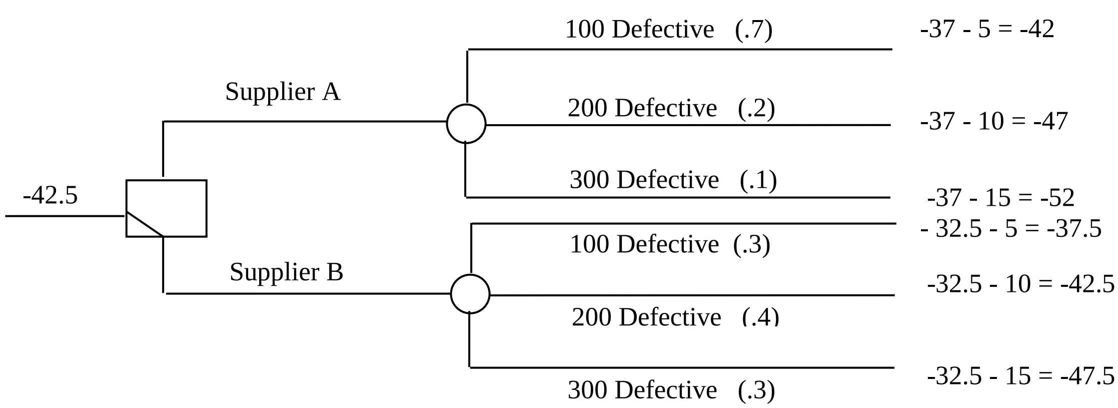
So the best course of action is for the promoter to go to trial, if he is found negligible then he should settle with the plaintiff before letting the jury determine the award. The expected cost of this strategy is $480,000.

Example 2 Quality Components needs to purchase on-off switches, which are available from two different suppliers. These switches are purchased in batches of size 1,000. The quality of any batch from the suppliers is described in the table below:

Number Defective	Prob. for Supplier A	Prob. for Supplier B
100	.70	.30
200	.20	.40
300	.10	.30

Defectives switches are repaired at a cost of $.05 each. The price of an order from Supplier A is $37 per batch; the price from Supplier B is $32.50. Build a decision tree to determine which supplier should be selected?

Solution



Since supplier B has a smaller expected cost per batch, it is selected as the vendor.

Example 3 Jerry Johnson, owner of Johnson Motors, is trying to decide what insurance policy to buy to cover hail damage on his inventory of more than 150 cars and trucks. The store is located in an area where storms occur frequently and they sometimes produce large balls of hail that can damage new vehicles. Jerry has obtained the following estimates on the potential damage from hail during a given year.

Hail Damage (in $1,000)	0	15	30	45	60	75	90	105
Probability	.35	.08	.10	.12	.15	.12	.05	.03

Jerry is considering one of the following three policy alternatives for managing his risk:

1. Buy an annual insurance policy for $50,000 covering 100% of any losses.
2. Buy an annual insurance policy for $25,000 that would cover all losses in excess of $35,000 (i.e. a $35,000 deductible).
3. He can self-insure and not purchase any insurance policy.

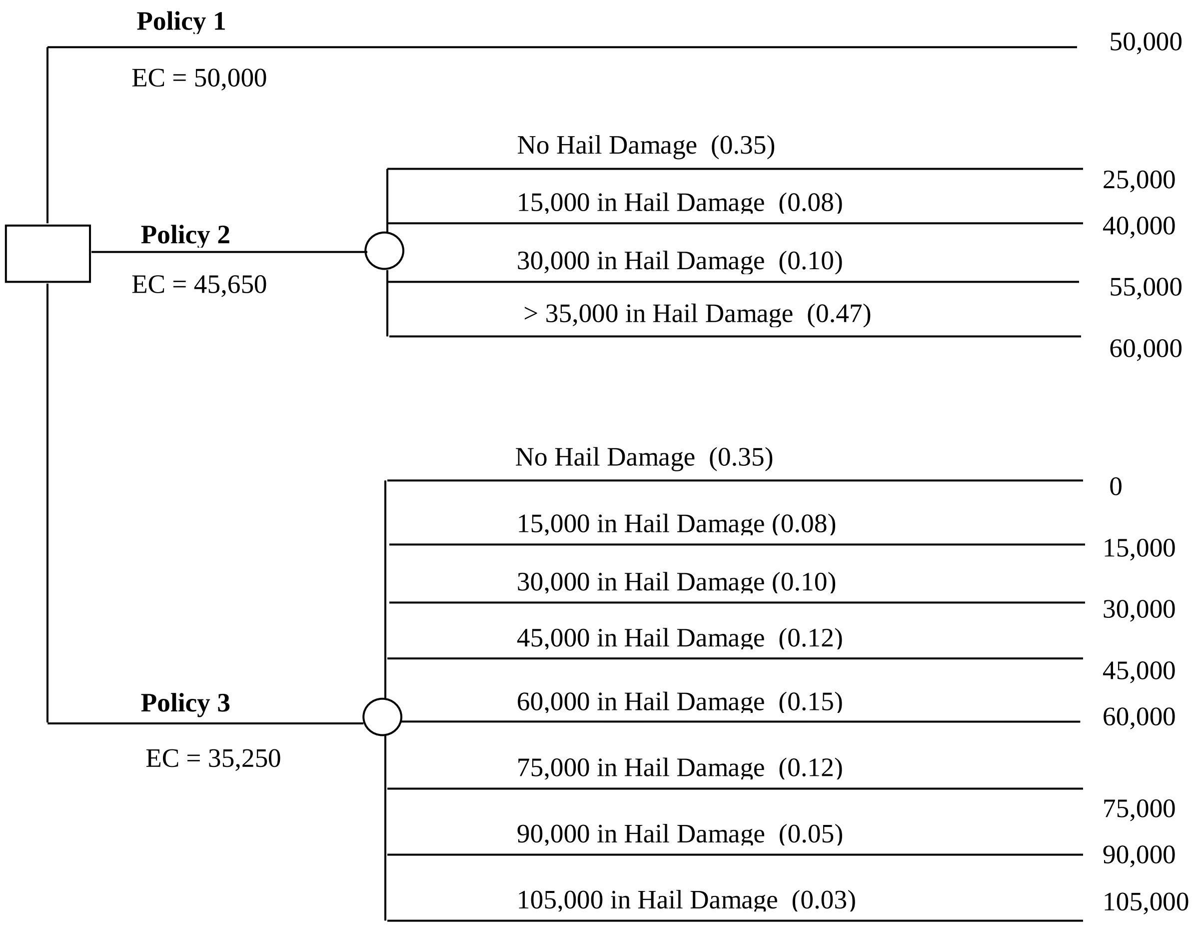
Construct a decision tree and determine which policy will minimize the expected cost.

Solution To construct a good decision tree it is helpful to group together some of the outcomes based on what the cost will be if alternative 2 is chosen. In particular, if hail damage is more than 35, the total cost to Jerry would be $25,000 for the policy plus $35,000 in damages. The insurance company would pay for any amount of damages over $35,000. So the cost is $55,000 in these situations. The expected cost for Policy 2 can then be calculated using the following cost table.

Hail Damage (in $1,000)	0	15	30	More than 35
Cost with Policy 2(in $1,000)	25	40	55	60
Probability	.35	.08	.10	.47

Expected cost = (25,000)(.35) + (40,000)(.08) + (55,000)(.10) + (60,000)(.47) = $45,650.

Using this distribution for Policy 2 and the expected cost of $45,650. The expected cost for Policy 3 is found using the given distribution for hail damages. The value is 35,250. The decision tree is given in figure below.



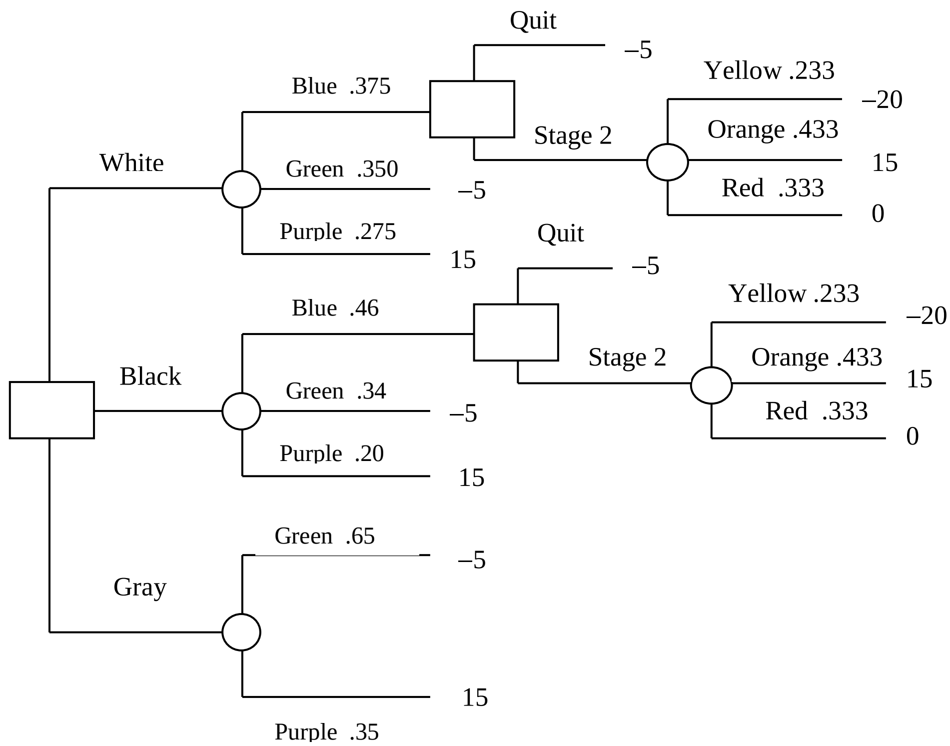
A Decision Tree for determining the best insurance policy

The final decision clearly depends on the objective. Policy 3 will minimize expected cost, but carries the most risk. Policy 1 is just the opposite, it carries the highest expected cost, but its SD is 0, and its worst case is a loss of $50,000 which is the cost of the premium. Policy 2 is between these two, with expected cost and risk level somewhere in between the extremes.

Example 4 A gambler has an opportunity to play the following two-stage game. At stage 1 he pays $5 and must choose between a white box, a black box, and a gray box. The white box contains 15 blue cards, 14 green cards, and 11 purple cards. The black box contains 23 blue, 17 green, and 10 purple cards. The gray box contains no blue, 26 green, and 14 purple cards. The cards are identical except for color. If a green card is drawn the player has lost and the game is over. If a purple card is drawn the house pays $20. If a blue card is drawn, the player may now quit or move on to stage 2 for an additional cost of $15. In stage 2, the player draws a card at random from a box that contains 7 yellow, 13 orange, and 10 red cards. If in stage 2 the player draws an orange card, the house pays $35. If he draws a yellow card, the house pays $0. If he draws a red card, the house pays $20.

Construct a decision tree and determine what strategy will maximize expected payoff.

Solution The decision tree is given below, followed by analysis of the tree.



Expected payoff calculations

Expected payoff of Stage 2 = (.233)(–20) + (.433)(15) + (.333)(0) = 1.83

Expected payoff of the White Box = (.375)(1.83) + (.350)(–5) + (.275)(15) = 3.06

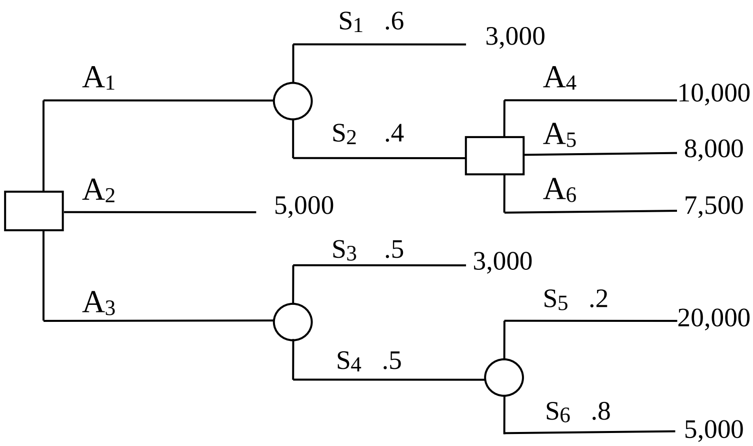
Expected payoff of the Black Box = (.46)(1.83) + (.34)(–5) + (.20)(15) = 2.14

Expected payoff of the Gray Box = (.65)(–5) + (.35)(15) = 2.00

At the decision points on the White Box and Black Box branch, choose to play Stage 2.

Conclusion Choose the White box to maximize expected payoff. If a blue card is chosen, then go on to Stage 2.

Example 5 A decision tree is given below. Determine the best course of action using an expected payoff criterion.



Solution To begin analyzing the decision tree we must choose between A_4, A_5 and A_6. We select A_4 since the maximum among the payoffs occurs with A_4. Next we must make three expected payoff calculations as follows.

Expected payoff on A_1 branch = (.6)(3,000) + (.4)(10,000) = 5,800

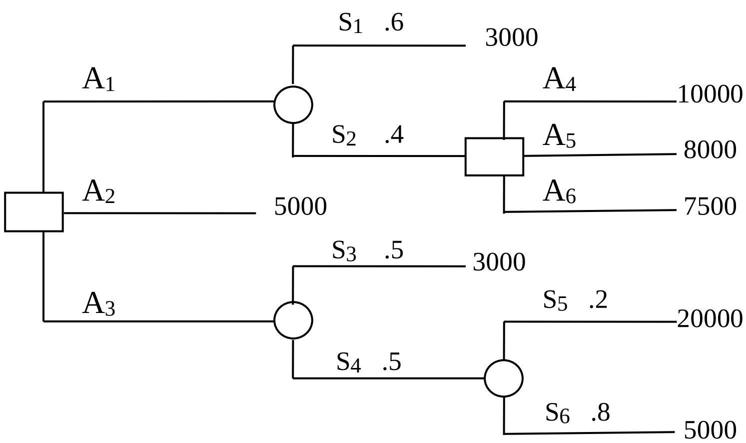
First expected payoff on A_3 branch = (.2)(20,000) + (.8)(5,000) = 8,000

Second expected payoff on A_3 branch = (.5)(3,000) + (.5)(8,000) = 5,500

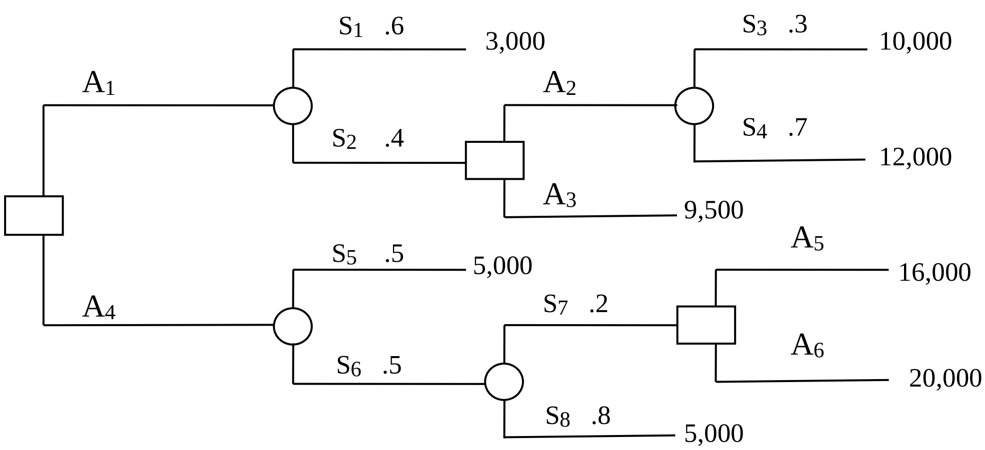
Conclusion Choose A_1, if S2 occurs, then select A_4.

Exercises

1. Determine the best course of action using an expected payoff criterion.



2. A decision tree is given below. Determine the best course of action using an expected payoff criterion.



3. A firm must decide whether to construct a small, medium, or large plant. A consultant's report indicates a 25% chance that demand will be low, and a 75% chance it will be high. If the firm builds a small facility and demand turns out to be low, the payoff will be $22 million. If demand turns out to be high, then the firm may subcontract and realize a profit of $45 million, or it could expand and obtain a profit of $48 million. A medium facility could be built as a hedge: if demand turns out to be low, its payoff will be $42 million; if demand is high the firm can do nothing and obtain a payoff of $46 million, or it could expand for a $50 million profit. If the firm builds a large facility and demand is low, the will be a loss of $20 million. Whereas high demand results in a $72 million profit.

   Illustrate the decision of selecting a plant size using a decision tree. Label the decision alternatives, the states of nature, the probabilities and payoffs appropriately. Be sure to distinguish between square nodes and round nodes where appropriate. Analyze the tree to determine which size is best.

4. A gas company can either buy its supply this year at a cost of $600,000 or spend $1,000,000 to drill for natural gas. If it hits a gusher (20 % chance) the company will bring in $2,700,000 of revenue, but if the deposit is moderate (30% chance) it will bring in $1,400,000. In either case, they will have enough supply. If the well comes up dry, then the company will have to buy its supply at new market prices. For the new market, there is a 20% chance the price will stay the same, a 10% chance it will drop by 5%, a 20% chance it will be 5% higher, a 25% chance it will be 10% higher, and otherwise it will be 15% over the current price. If the company buys its supply, then it believes that a revenue of $900,000 will be produced. Construct a decision tree for this situation and determine the best strategy

5. A company is trying to decide if it should expand its present production facility, subcontract to increase production, or build a new production facility. Payoffs have been estimated as follows.

		High	Moderate	Low
	Expand	300,000	200,000	100,000
Alternatives	Subcontract	300,000	300,000	50,000
	Build	500,000	250,000	-100,000

a) Construct a decision tree which illustrates this decision. Analyze the decision tree using the expected profit criterion where the probabilities for the states are .2, .5, and .3.

b) Suppose that in addition to the above data, if the company decided to build and demand is low, then it has two options. The company can either do nothing which results in a loss of $100,000 or it can rent out the new facility which would create a profit of $75,000. Construct a decision tree which illustrates the above decision including this new information. Analyze the decision tree using the expected profit criterion.

6. Electric sensors are bought from two different suppliers Supplier A and Supplier B. The quality of these sensors is given in the following tables:

Supplier A

Percent Defective	**Probability**
1	.30
2	.50
5	.20

Supplier B

Percent Defective	**Probability**
1	.7
3	.2
4	.1

Orders are always placed for a quantity of 1,000 sensors. Defectives from Supplier A are all repaired at a cost of $3.00 each. Defectives from Supplier B are all repaired at a cost of $5.00 each.

a) What is the expected cost of repair of defective sensors when buying from Supplier A?
b) What is the expected cost of repair of defective sensors when buying from Supplier B?
c) Illustrate the decision with a decision tree and determine what will minimize expected cost.

7. A gambler has an opportunity to play the following two-stage game. Initially the gambler must pay $5 and must choose between a white box and a black box. The white box contains 5 blue cards, 4 green cards, and 6 purple cards. The black box contains 3 blue cards, 5 green cards, and

12 purple cards. The cards are all identical except for color. If a green card is drawn, the player has lost and the game is over. If a purple card is drawn, the house pays \$15. If a blue card is drawn, the player may now quit, or move on to stage 2 for an additional cost of \$10. In stage 2 the player draws a card at random from a box that contains 3 yellow and 7 orange cards. If in stage 2 the player draws an orange card, the house pays \$35. If a yellow card is selected, the house pays \$0.

Construct a decision tree and determine the best strategy based on maximizing expected payoff.

8. An investor can invest his money in one of three different investment plans over an 18-month period. The return on his investment depends on the type of investment plan chosen and the future state of the economy. The three plans consist of buying convertible bonds (CB), purchasing government bonds (GB), or investing in money market funds (MMF). In particular, he can buy CB for \$10,000, invest \$8,000 in MMF, or buy \$15,000 worth of GB. The economy has been forecasted to be gloomy with a probability of .30, stable with a probability of .45, or rosy with a probability of .25. The total amount collected, including the initial investment, for the GB is \$16,000 for a rosy economy, \$15,900 for a stable economy, and \$14,500 for a gloomy economy. The amount collected for the MMF investment is \$9,000 for rosy, \$8,900 for stable economies. However, when the economy is gloomy, the investor can pay a fee of \$350 and sell his MMF prematurely in which case he collects \$8,900. Otherwise, he may wish to do nothing and collect \$8,700. The CB investment will result in collecting \$11,000 in a rosy economy. Under a stable economy, the investor can sell the CB prior to maturity for a fee of \$200 and collect \$11,100, or wait until the end of the 18 months and collect \$10,500. When the economy is gloomy, he can sell the CB prematurely and invest in real estate bonds at a cost of \$500 in which case he will collect \$10,500, or he can do nothing and collect \$9,800.

   a) Construct a decision tree that represents the investment plans.
   b) Determine the optimal investment plan which will maximize his expected profit.

9. A manager is considering three options concerning one of his production line machines. He can purchase a new one for \$400 and it will easily last three years. He is also considering purchasing a used machine or repairing the current machine. If he repairs the current one, he estimates a repair cost of \$150, but also believes that there is only a 30% chance that it will not last a full three years and he will end up purchasing a new one anyway. If he buys a used machine for \$200, he estimates a 60% chance it will last the three years. If it breaks down, he will have the option of repairing it for \$150 or buying a new one. Construct a decision tree for this situation and determine the best strategy.