2007

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AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **39** (2007), Pages 273–282

Towards the computation of the convex hull of a configuration from its corresponding separating matrix

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Abstract

In this paper, we cope with the following problem: compute the size of the convex hull of a configuration $\mathcal{C}$, where the given data is the number of separating lines between any two points of the configuration (where the lines are generated by pairs of other points of the configuration).

We give an algorithm for the case that the convex hull is of size 3, and a partial algorithm and some directions for the case that the convex hull is of size bigger than 3.

1 Introduction

A finite set $\mathcal{P} = \{P_1, \cdots, P_n\}$ of n points in the oriented affine plane $\mathbb{R}^2$ is a *configuration in general position* if three points in $\mathcal{P}$ are never collinear. Two configurations of n points in general position $\mathcal{P}^1$ and $\mathcal{P}^2$ are *isotopic* if they can be joined by a continuous path of configurations in general position.

A line $L \subset \mathbb{R}^2$ *separates* two points $P, Q \in \mathbb{R}^2 \setminus L$ if P and Q are in different connected components of $\mathbb{R}^2 \setminus L$. Given a configuration in general position $\mathcal{P}$, we denote by $n(P,Q)$ the number of separating lines defined by pairs of points in $\mathcal{P} \setminus \{P,Q\}$.

Given a configuration in general position $\mathcal{P}$, we define *the separating matrix of a configuration*, $S(\mathcal{P})$, to be the symmetric matrix of order n, defined by:

$$(S(\mathcal{P}))_{ij} = n(P_i, P_j)$$

The interesting question is which data we can retrieve from this separating matrix. In this paper, we introduce an algorithm that partially computes the convex hull from the separating matrix.

There are matrices associated to planar configurations of points which determine the configurations, for example the λ-matrix which was defined by Goodman and Pollack [5]. An interesting question is to study the connection between these two matrices.

The paper is organized as follows. In Section 2, we give some simple observations regarding the separating matrix, which yield some restriction on this matrix. In Section 3, we present an algorithm for computing the convex hull of a configuration, in case that its size is 3. In Section 4, we give a partial algorithm for computing the convex hull in case that its size is bigger than 3. We also give possible directions for solving this problem.

2 Simple observations about the separating matrices associated to configurations

In this section, we point out some properties of the separating matrix.

Lemma 2.1 *Given a separating matrix which represents a configuration of n points. Then the maximal value for an entry of this matrix is $\binom{n-2}{2}$.*

Proof. The separating lines are generated from pairs of the remaining points. There are $n-2$ such points, and hence there are at most $\binom{n-2}{2}$ such pairs. □

The next point is how many odd and even entries we have in this matrix. For this we recall the Orchard relation (see [2] and [3]). Given a planar configuration of n points, we say that two points P, Q are *equivalent* if $n(P,Q) \equiv (n-1) \pmod 2$. We have shown that this is an equivalence relation with at most two equivalence classes [2, 3].

Lemma 2.2 1. *If n is even, then there are $i(n-i)$ even entries in the upper triangular part (excluding the diagonal) of the separating matrix for some $0 \leq i \leq n$. The rest of the entries are odd.*

2. *If n is odd, then there are $i(n-i)$ odd entries in the upper triangular part (excluding the diagonal) of the separating matrix for some $0 \leq i \leq n$. The rest of the entries are even.*

Proof. Assume n is even. By the definition of the Orchard relation, two points in different equivalence classes have an even number of separating lines. If one class has i points, then the other has $n-i$ points, thus there are $i(n-i)$ pairs of points in diffferent classes, and therefore, $i(n-i)$ even entries. All the rest correspond to pairs of points from the same equivalence class, and hence have odd entries.

For the case that n is odd, the proof is identical. □

Example 2.3 *For $n = 6$, the upper triangular part (excluding the diagonal) has* 15 *entries. The options for the number of even entries are:* $6 \cdot (6-6) = 0, 5 \cdot (6-5) = 5, 4 \cdot (6-4) = 8, 3 \cdot (6-3) = 9$.

One more check, based on the Orchard relation, is the following:

Lemma 2.4 *Let $S(\mathcal{P})$ be the separating matrix of a configuration P.*

1. *If* $(S(\mathcal{P}))_{ij} \equiv (S(\mathcal{P}))_{jk} \pmod 2$, *then* $(S(\mathcal{P}))_{ik} \equiv (n-1) \pmod 2$.
2. *If* $(S(\mathcal{P}))_{ij} \not\equiv (S(\mathcal{P}))_{jk} \pmod 2$, *then* $(S(\mathcal{P}))_{ik} \equiv n \pmod 2$

Proof. (1) If $(S(\mathcal{P}))_{ij} \equiv (S(\mathcal{P}))_{jk} \equiv n \pmod 2$, then P_i and P_j are in different classes, and P_j and P_k are in different classes. This implies that P_i and P_k are in the same class.

On the other hand, if $(S(\mathcal{P}))_{ij} \equiv (S(\mathcal{P}))_{jk} \equiv (n-1) \pmod 2$, then P_i and P_j are in the same class, and P_j and P_k are in the same class. Therefore, P_i and P_k are in the same class too.

(2) If $(S(\mathcal{P}))_{ij} \not\equiv (S(\mathcal{P}))_{jk} \pmod 2$, then we have two cases:

(a) P_i and P_j are in the same class, while P_j and P_k are in different classes.

(b) P_i and P_j are in different classes, while P_j and P_k are in the same class.

In both cases, we can conclude that P_i and P_k are in different classes, and the result follows. □

Hence, we have the following corollary:

Corollary 2.5 *Necessary conditions for a matrix to be a separating matrix of a configuration are:*

1. *It should be symmetric with diagonal* 0.
2. *All entries should be smaller (or equal) than* $\binom{n-2}{2}$
3. *The matrix should satisfy the conditions of the last two lemmas.*

3 Convex hull of size 3

We start with the simplest case, where the size of the convex hull is 3. We present the algorithm for the case of convex hull of size 3.

Algorithm 3.1 *1. Choose i, j, k such that $1 \leq i < j < k \leq n$*

2. If $(S(\mathcal{P}))_{ij} + (S(\mathcal{P}))_{ik} + (S(\mathcal{P}))_{jk} = n^2 - 4n + 3$, then return: "Convex hull is of size 3 and it is P_i, P_j, P_k".

3. If for all $1 \leq i < j < k \leq n$, $(S(\mathcal{P}))_{ij} + (S(\mathcal{P}))_{ik} + (S(\mathcal{P}))_{jk} \neq n^2 - 4n + 3$, then return: "Convex hull of size larger than 3".

3.1 The correctness of the algorithm

The correctness of the algorithm is based on the following lemmas.

Let us assume we have a configuration with convex hull points P_i, P_j, P_k. In our matrix $(S(\mathcal{P}))_{ij} = n(P_i, P_j)$ is the number of lines separating P_i and P_j, for all i and j.

First, we will show that if the convex hull is of size 3, then the sum of separating lines on the convex hull is indeed $n^2 - 4n + 3$:

Lemma 3.2 *Let C be a configuration of n points. Assume that the points P_i, P_j, P_k form its convex hull. Then:*

$$(S(\mathcal{P}))_{ij} + (S(\mathcal{P}))_{ik} + (S(\mathcal{P}))_{jk} = n^2 - 4n + 3.$$

Proof. We have $n-3$ internal points (which are not on the convex hull). Each one of them contributes 3 separating lines on the convex hull (by the lines generated by the internal point and the three points of the convex hull). Moreover, each pair of them contributes 2 to the number of separating lines (by the line generated by this pair of points). Hence we have the following number of separating lines on the convex hull:

$$3(n-3) + 2 \cdot \frac{(n-3)(n-4)}{2} = (n-3)(n-1) = n^2 - 4n + 3$$

as needed. □

Now, we will show that for any triple of points P_i, P_j, P_k, which is not the convex hull, we have: $(S(\mathcal{P}))_{ij} + (S(\mathcal{P}))_{ik} + (S(\mathcal{P}))_{jk} < n^2 - 4n + 3$.

Lemma 3.3 *Let C be a configuration of n points with a convex hull of size 3. Let P_i, P_j, P_k be a triple of points which is not the convex hull. Then: $(S(\mathcal{P}))_{ij} + (S(\mathcal{P}))_{ik} + (S(\mathcal{P}))_{jk} < n^2 - 4n + 3$.*

Proof. First, notice that from the previous lemma we can derive that $n^2 - 4n + 3$ is the highest possible amount of separating lines for a boundary of any triangle. Now, consider a triple of points P_i, P_j, P_k which are not the convex hull of the configuration. Hence, there is a point P which is outside the triangle generated by P_i, P_j, P_k. A simple observation shows that at least one of the lines generated by P and the one of the points P_i, P_j, P_k does not cross the boundary of the triangle, and hence the total number of separating lines is strictly smaller than $n^2 - 4n + 3$. □

By a similar argument to that of the previous lemma, we have:

Lemma 3.4 *Let C be a configuration of n points with a convex hull of size larger than 3. Let P_i, P_j, P_k be a triple of points. Then:*

$$(S(\mathcal{P}))_{ij} + (S(\mathcal{P}))_{ik} + (S(\mathcal{P}))_{jk} < n^2 - 4n + 3.$$

□

Hence, for determining whether a given matrix corresponds to a configuration whose size of its convex hull is 3, we have to do the following: For each triple of indices i, j, k, compute $(S(\mathcal{P}))_{ij} + (S(\mathcal{P}))_{ik} + (S(\mathcal{P}))_{jk}$. If for a triple i, j, k, we get the maximal value $n^2 - 4n + 3$, then the configuration has a convex hull of size 3, which consists of the points P_i, P_j, and P_k. If for all triples of indices i, j, k, we get

$$(S(\mathcal{P}))_{ij} + (S(\mathcal{P}))_{ik} + (S(\mathcal{P}))_{jk} < n^2 - 4n + 3,$$

then the configuration has a convex hull of size bigger than 3.

3.2 Complexity

It is easy to see that the complexity is $\mathrm{O}(n^3)$, since we have to check $\binom{n}{3}$ triples of points.

4 The general case

For the general case, we have some partial results.

We start with the expected number of separating lines on the convex hull of size k.

Lemma 4.1 *Let C be a configuration of n points. Assume that the points $P_{i_1}, P_{i_2}, \cdots, P_{i_k}$ form its convex hull in this order. Then:*

$$(S(\mathcal{P}))_{i_1 i_2} + (S(\mathcal{P}))_{i_2 i_3} + \cdots + (S(\mathcal{P}))_{i_{k-1} i_k} + (S(\mathcal{P}))_{i_k i_1} = n^2 - (k+1)n + k.$$

Proof. We have $n - k$ internal points (which are not on the convex hull). Each one of them contributes k separating lines on the convex hull (by the lines generated by the internal point and the k points of the convex hull, see Figure 1, Line (1)).

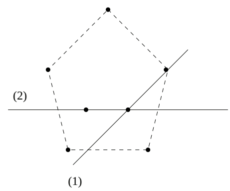

Figure 1: Examples for lines intersecting the convex hull

Moreover, Each pair of them contributes 2 to the number of separating lines (by the line generated by this pair of points, see Figure 1, Line (2)).

Hence we have the following total number of separating lines on the convex hull:

$$k(n-k) + 2 \cdot \frac{(n-k)(n-k-1)}{2} = (n-k)(n-1) = n^2 - (k-1)n + k$$

as needed. □

Lemma 4.2 *Let C be a configuration of n points. Assume that the points $P_{i_1}, P_{i_2}, \cdots, P_{i_k}$ form its convex hull in this order. Let σ be a permutation in S_k (the symmetric group on k elements). Then:*

$$\min_{\sigma \in S_k} \left(S(\mathcal{P}))_{i_{\sigma(1)} i_{\sigma(2)}} + \cdots + (S(\mathcal{P}))_{i_{\sigma(k-1)} i_{\sigma(k)}} + (S(\mathcal{P}))_{i_{\sigma(k)} i_{\sigma(1)}} \right) = n^2 - (k+1)n + k$$

Proof. We will compare the number of hits on paths going through these k points exactly once. We will show that the number of hits on the convex hull is *strictly smaller* than the number of hits on a path going through these k points exactly once, which is not the convex hull.

We have 3 classes of lines:

1. Lines determined by one internal point and one point on the convex hull (see Line (1) in Figure 2): Such a line intersects the cycle of the convex hull exactly once. For any other k-cycle, each of these lines hits at least once.

2. Lines determined by two internal points: For the hull, each of these lines intersects the hull exactly twice (see Line (2) in Figure 2). For any other k-cycle, each of these lines hits at least twice.

3. Lines determined by two points of the convex hull: These lines do not intersect the convex hull, since the convex hull is not self-intersecting. But any other cycle is self-intersecting and hence these lines will contribute (see the dotted lines in Figure 2). This contribution yields the “strictly smaller” part.

Hence we are done. □

Now we will show that if a configuration has a convex hull of size k, any other k points *in convex position* will have less separating lines over its convex hull.

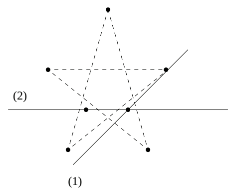

Figure 2: Examples for lines intersecting the hamiltonian cycle which is not convex

Lemma 4.3 *Let $\mathcal{P}$ be a configuration of n points whose convex hull is of size k. Let $P_{i_1}, \cdots, P_{i_k}$ be k points* **in convex position** *(in this order), which do not form the convex hull of P. Then:*

$$(S(\mathcal{P}))_{i_1 i_2} + \cdots + (S(\mathcal{P}))_{i_{k-1} i_k} + (S(\mathcal{P}))_{i_k i_1} < n^2 - (k+1)n + k.$$

The proof of this lemma uses the same argument as the proof of Lemma 3.3.

Similarly, it is easy to see the following:

Lemma 4.4 *Let $\mathcal{P}$ be a configuration of n points whose convex hull is of size k. Let $3 \leq m \leq n$, $m \neq k$. Let $P_{i_1}, \cdots, P_{i_m}$ be m points in convex position (in this order). Then:*

$$(S(\mathcal{P}))_{i_1 i_2} + \cdots + (S(\mathcal{P}))_{i_{m-1} i_m} + (S(\mathcal{P}))_{i_m i_1} < n^2 - (m+1)n + m.$$

Based on these lemmas, one can try to compute the convex hull from the separating matrix by using a similar algorithm to the case of convex hull of size 3:

Algorithm 4.5 *1. Set k:=3.*

2. Choose $i_1, i_2, \cdots, i_k$ such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$

3. If

$$\min_{\sigma \in S_k} \left((S(\mathcal{P}))_{i_{\sigma(1)} i_{\sigma(2)}} + \cdots + (S(\mathcal{P}))_{i_{\sigma(k-1)} i_{\sigma(k)}} + (S(\mathcal{P}))_{i_{\sigma(k)} i_{\sigma(1)}} \right) = n^2 - (k+1)n + k,$$

then return: "Convex hull is of size k and it is $P_{i_1}, P_{i_2}, \cdots, P_{i_k}$".

4. If for all $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, the condition in (3) is not satisfied, then $k := k+1$ and return to Step (2).

For covering all the subsets of k points out of the n points of the configuration, we have used the implementation of [6].

The problem of this algorithm is that it found "fake" convex hulls, i.e. one can have m points ($m \neq k$) NOT in convex position which still yield the correct minimal number of separating lines on its "convex hull".

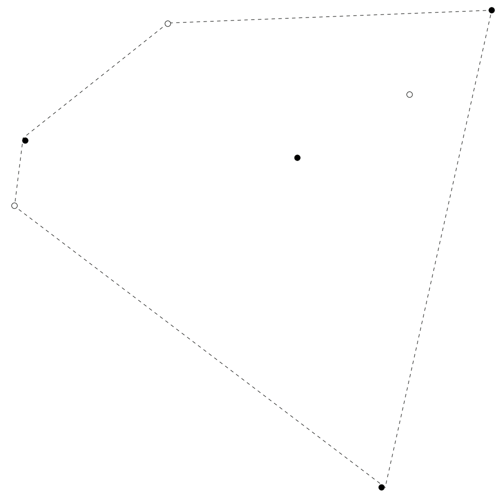

Figure 3: A configuration with a "fake" convex hull

For example, in Figure 3, the real convex hull of size 5 is dashed and the 4 black points are the "fake" convex hull (notice that they are not in convex position).

Out of 135 configurations of 7 points in general position, one gets 13 such "fake" convex hulls (for less than 7 points, there were no "fake" convex hulls). We have used the database of Aichholzer and Krasser (see [1]).

So, our next aim is to find how can we outrule the fake convex hulls and keep only the real ones.

One can try the following probabilistic way to rule out these "fakes". One can imagine that the sum of separating lines over the $n-1$ lines going out from a point on the convex hull (i.e. the sum of entries in the row corresponding to a point in the convex hull) will be higher than (or at least equal to) this corresponding sum for a point not on the convex hull (i.e. the sum of entries in the row corresponding to a point not in the convex hull).

For $n = 7$ and $n = 8$ points, this check indeed rules out the "fake" convex hulls. For $n = 9$ points, it also rules out the "fake" convex hulls, but it also rules out the correct convex hull in 28 cases (out of 158817 cases), since the corresponding sum of one of the points of the convex hull is strictly smaller than this sum for an internal point.

One direction for suggesting a different algorithm for this problem is the following: once we find out that a case of k points that satisfies the correct number of separating lines, but fails to satisfy the maximal row sum condition (see two paragraphs above), make an extra check: if in this case we have that the corresponding sums of the least m rows of the convex hull are equal (m can be equal to 1), then check if the corresponding rows of these k points are amongst the highest $k+m$ rows. If so, these

k points form the convex hull.

This check will fail if there will be a "fake" also here: k points which has the correct number of separating lines and satisfy the maximal row sum condition (such a "fake", if exists, has at least 10 points).

We now show that the cycle consisting of the points of the real convex hull (in order), combined with any number of internal points cannot be considered as a "fake" convex hull.

Lemma 4.6 *Let $\mathcal{P}$ be a configuration of n points whose convex hull is of size k. Let $P_{i_1}, \cdots, P_{i_k}$ be the k points of the convex hull (in this order) of P. Let P_{i_m} be an internal point. Consider the cycle $P_{i_1}, \cdots, P_{i_k}, P_{i_m}$. Then*

$$(S(\mathcal{P}))_{i_1 i_2} + (S(\mathcal{P}))_{i_2 i_3} + \cdots + (S(\mathcal{P}))_{i_{k-1} i_k} + (S(\mathcal{P}))_{i_k i_m} + (S(\mathcal{P}))_{i_m i_1} > n^2 - (k+2)n + (k+1).$$

Proof. We have $n - k - 1$ points which are not on this cycle. Each one of them contributes at least $k + 1$ separating lines to this cycle (by the lines generated by the points on this cycle with those not on this cycle). Moreover, each pair of them contributes at least 2 to the number of separating lines (by the line generated by this pair of points). Additionally, the line connecting P_{i_m} to P_{i_k}, and the line connecting P_{i_m} to P_{i_1} each contributes one separating line to the cycle.

Hence we have the following number of separating lines on the cycle:

$$\begin{aligned} (k+1)(n-k-1) + 2 \cdot \frac{(n-k-1)(n-k-2)}{2} + 2 &= (n-k-1)(n-1) + 2 \\ &= n^2 - (k+2)n + (k+3) \\ &> n^2 - (k+2)n + (k+1) \end{aligned}$$

as needed. $\square$

The same argument will apply if one adds any number of points to the cycle of the convex hull.

Acknowledgements

We thank Roland Bacher and Rom Pinchasi for fruitful discussions.

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(Received 30 Oct 2006)