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A Wavelet-based Method for Overcoming the Gibbs Phenomenon

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Abstract: The Gibbs phenomenon refers to the lack of uniform convergence which occurs in many orthogonal basis approximations to piecewise smooth functions. This lack of uniform convergence manifests itself in spurious oscillations near the points of discontinuity and a low order of convergence away from the discontinuities. Here we describe a numerical procedure for overcoming the Gibbs phenomenon called the inverse wavelet reconstruction method. The method takes the Fourier coefficients of an oscillatory partial sum and uses them to construct the wavelet coefficients of a non-oscillatory wavelet series.

Key–Words: Gibbs phenomenon, wavelets, Gegenbauer reconstruction, inverse polynomial reconstruction.

1 Introduction

Fourier and orthogonal polynomial series are known for their highly accurate expansions for smooth functions. In fact it is known that the more derivatives a function has, the faster the approximation will converge. However, when a function possesses jump-discontinuities the approximation will fail to converge uniformly. In addition, spurious oscillations will cause a loss of accuracy throughout the entire domain. This lack of uniform convergence is known as the Gibbs phenomenon. Methods for post-processing approximations which suffer from the Gibbs phenomenon include the Gegenbauer reconstruction method of Gottlieb and Shu [7,9,10], the method of Pade approximants due to Driscoll and Fornberg [2], the method of spectral mollifiers due to Gottlieb and Tadmor [8] and Tadmor and Tanner [17, 18], the inverse polynomial reconstruction method of Shizgal and Jung [13,14,15,16], and the Freund polynomial reconstruction method of Gelb and Tanner [6]. These reconstruction methods can be combined with an effective method for edge-detection developed by Gelb and Tadmor [3,4,5,6], to yield an exponentially accurate reconstruction of the original function. In this paper we describe a new numerical method for overcoming the Gibbs phenomenon following the work of Shizgal and Jung, called the inverse wavelet reconstruction method.

We begin with a brief review of the essential definitions of wavelets which we will need. Recall that a wavelet is a function $\psi \in L^2(\mathbf{R})$ satisfying:

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0 \tag{1}$$

and

$$\int_{-\infty}^{\infty} \frac{\left|\widehat{\psi}(\xi)\right|}{|\xi|} d\xi < \infty, \tag{2}$$

where $\widehat{\psi}$ here is the Fourier transform of ψ. The function ψ is also known as an analyzing wavelet or a mother wavelet since any function $f \in L^2(\mathbf{R})$ can be expressed as a continuous sum of translations and dilations involving ψ according to the continuous wavelet transform. The continuous wavelet transform is given by

$$(W_\psi f)(b,a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(t)\, \overline{\psi\left(\frac{t-b}{a}\right)} dt$$

and the inverse continuous wavelet transform is given by

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[(W_\psi f)(b,a)\right] \frac{\psi_{b,a}(t)}{a^2} da db \tag{3}$$

where

$$\psi_{b,a}(t) = |a|^{-1/2} \overline{\psi\left(\frac{t-b}{a}\right)}$$

and

$$C_\psi = \int_{-\infty}^{\infty} \frac{\left|\widehat{\psi}(\xi)\right|}{|\xi|} d\xi.$$

A discrete wavelet series is given by

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{j,k} \psi_{j,k}(x) \tag{4}$$

where the discrete wavelet series coefficients are given by

$$c_{j,k} = (W_\psi f)\left(\frac{k}{2^j}, \frac{1}{2^j}\right)$$

and

$$\psi_{j,k}(x) = 2^{j/2} \psi\left(2^j x - k\right).$$

For more information see, for example, Chui [1]. We will make use of wavelet series for our reconstruction technique further on.

2 The Inverse Wavelet Reconstruction Method

2.1 Inverse Polynomial Reconstruction

One numerical technique for dealing with the problem of the Gibbs phenomenon is the inverse polynomial reconstruction method of Shizgal and Jung [13, 14, 15, 16]. Their inverse method is itself an alternative approach to the original direct Gegenbauer reconstruction method of Gottlieb and Shu [7, 9, 10]. In the inverse method, one solves a system of linear equations for the Gegenbauer polynomial reconstruction coefficients in terms original Fourier coefficients, whose expansion suffers from the Gibbs phenomenon. The inverse polynomial reconstruction method begins with a Fourier partial sum

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{-i\pi n x}. \tag{5}$$

The function also has a Gegenbauer-λ expansion given by

$$f(x) = \sum_{m=0}^{\infty} \widehat{f}^\lambda(m) C_m^\lambda(x)$$

where

$$\widehat{f}^\lambda(m) = \frac{1}{h_m^\lambda} \int_{-1}^{1} f(x) C_m^\lambda(x) \left(1 - x^2\right)^{\lambda - 1/2} dx \tag{6}$$

and

$$h_m^\lambda = \pi^{1/2} \frac{\Gamma(m + 2\lambda)}{m! \Gamma(2\lambda)} \frac{\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(\lambda)(m + \lambda)}.$$

The Fourier coefficients $\widehat{f}(n)$ can be expressed in terms of Gegenbauer-λ coefficients as follows.

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2} \int_{-1}^{1} f(x) e^{-i\pi n x} dx \\
&= \frac{1}{2} \int_{-1}^{1} \sum_{m=0}^{\infty} \widehat{f}^\lambda(m) C_m^\lambda(x) e^{-i\pi n x} dx \\
&= \sum_{m=0}^{\infty} \widehat{f}^\lambda(m) \left(\frac{1}{2} \int_{-1}^{1} C_m^\lambda(x) e^{-i\pi n x} dx\right) \\
&= \sum_{m=0}^{\infty} \widehat{f}^\lambda(m) c_{m,n}^\lambda
\end{aligned} \tag{7}$$

The connection coefficients $c_{m,n}^\lambda$ are given by

$$c_{m,n}^\lambda = \frac{1}{2} \int_{-1}^{1} C_m^\lambda(x) e^{-i\pi n x} dx \tag{8}$$

and can be computed numerically. An explicit formula for $c_{m,n}^\lambda$ was derived by this author, Greene [11, 12], and is given for the cases $n \neq 0$ and $n = 0$ respectively by:

$$\begin{aligned}
c_{m,n}^\lambda &= \sum_{j=0}^{[m/2]} \frac{(-i)^{m-2j}}{\sqrt{2n}} J_{m-2j+\frac{1}{2}}(\pi n) \\
&\quad \times \frac{\left(\lambda - \frac{1}{2}\right)_j (\lambda)_{m-j} \left(\frac{3}{2}\right)_{m-2j}}{j! \left(\frac{3}{2}\right)_{m-j} \left(\frac{1}{2}\right)_{m-2j}}
\end{aligned} \tag{9}$$

$$c_{m,0}^\lambda = \begin{cases} \frac{\left(\lambda - \frac{1}{2}\right)_{m/2} (\lambda)_{m/2}}{j! \left(\frac{3}{2}\right)_{m/2}}, & m \text{ even} \\ 0, & m \text{ odd.} \end{cases} \tag{10}$$

The inverse polynomial reconstruction procedure is obtained by truncating the infinite series above and solving the system of equations

$$\widehat{f}(n) = \sum_{m=0}^{2N} \widehat{f}^\lambda(m) c_{m,n}^\lambda, \; n = -N \ldots N. \tag{11}$$

One then computes the Gegenbauer reconstruction approximation

$$S_{2N}^\lambda f(x) = \sum_{m=0}^{2N} \widehat{f}^\lambda(m) C_m^\lambda(x). \tag{12}$$

2.2 Inverse Wavelet Reconstruction

We explore here an analogous approach which seeks to reconstruct the original function in terms of wavelets. We begin with a Fourier partial sum as before and assume that the function $f(x)$ can also be expressed as a discrete wavelet series:

$$f(x) = \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{m,l} \psi_{m,l}(x)$$

where

$$c_{m,l} = \int_{-\infty}^{\infty} f(x) \overline{\psi_{m,l}(x)} dx$$

and

$$\psi_{m,l}(x) = 2^{-m/2} \psi(2^m x - l).$$

We now derive a formula expressing the Fourier coefficients in terms of wavelet coefficients.

$$\begin{aligned} \widehat{f}(n) &= \frac{1}{2} \int_{-1}^{1} f(x) e^{-i\pi n x} dx \\ &= \frac{1}{2} \int_{-1}^{1} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{m,l} \psi_{m,l}(x) e^{-i\pi n x} dx \\ &= \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{m,l} \left(\frac{1}{2} \int_{-1}^{1} \psi_{m,l}(x) e^{-i\pi n x} dx \right) \\ &= \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{m,l} \widehat{\psi}_{m,l}(n) \end{aligned} \tag{13}$$

The inverse wavelet reconstruction method is obtained by truncating the doubly infinite sum described above and solving for the wavelet coefficients. We suggest solving the following system of equations.

$$\widehat{f}(n) = \sum_{m=-M}^{M-1} \sum_{l=-L}^{L-1} c_{m,l} \widehat{\psi}_{m,l}(n) \tag{14}$$

where

$$n = -N...N-1, \text{and } N = 2ML \tag{15}$$

for the wavelet coefficients $c_{m,l}$. One then computes the wavelet reconstruction approximation

$$S_{M,L} f(x) = \sum_{m=-M}^{M-1} \sum_{l=-L}^{L-1} c_{m,l} \psi_{m,l}(x). \tag{16}$$

The terms $\widehat{\psi}_{m,l}(n)$ are the nth Fourier coefficients of $\psi_{m,l}(x)$. Since we are solving a system of $2N$ equations in $(2M)(2L) = 4ML$ unknown wavelet coefficients, in order for the system to be invertible we must have $2N = 4ML$ or $N = 2ML$. It is this convention which we will use for our numerical implementation below, owing to the simplicity of the relation $N = 2ML$.

Alternatively, one may solve the system

$$\widehat{f}(n) = \sum_{m=-M}^{M} \sum_{l=-L}^{L} c_{m,l} \widehat{\psi}_{m,l}(n) \tag{17}$$

where

$$n = -N...N, \text{and } N = 2ML + M + L. \tag{18}$$

and then compute

$$S_{M,L} f(x) = \sum_{m=-M}^{M} \sum_{l=-L}^{L} c_{m,l} \psi_{m,l}(x). \tag{19}$$

The condition in (18) is due to the fact that we are solving a system of $2N + 1$ equations in $(2M + 1)(2L + 1)$ unknown wavelet coefficients. In order for the system to be invertible we must have $2N+1 = (2M + 1)(2L + 1)$ or $N = 2ML+M+L$. A detailed proof of convergence is the subject of concurrent work. However, we illustrate the apparent convergence for certain wavelets with some numerical examples.

3 Numerical Results

Numerical experiments show that the inverse wavelet reconstruction approach does yield accurate and uniform approximations for a variety of wavelet families. We illustrate this with three wavelet families: Poisson wavelets given by

$$\psi(x) = \frac{1}{\pi(1 + x^2)}, \tag{20}$$

Mexican hat wavelets given by

$$\psi(x) = \frac{2}{\sqrt{3}\pi^{1/4}} \left(1 - x^2\right) e^{-x^2/2}, \tag{21}$$

and Morlet wavelets given by

$$\psi(x) = \cos\left(\frac{1}{2}x\right) e^{-x^2/2}. \tag{22}$$

For each case we begin with a Fourier series of $2N = 16, 36, 64, 100$ expansion coefficients of an analytic non-periodic test function $f(x) = 4\tan^{-1}(x)$ whose partial sum suffers from the Gibbs phenomenon. We then compute the corresponding wavelet series reconstructions. We chose the number of Fourier coefficients such that $N = 2ML$ and $M = L$. The absolute error of these reconstructions, $|f(x) - S_{M,L} f(x)|$, are displayed in the figures provided. For the three wavelets shown the Morlets perform the best for a fixed number of Fourier coefficients, the Mexican hat wavelets perform comparably, though slightly less well, and the Poisson wavelets perform the least well.

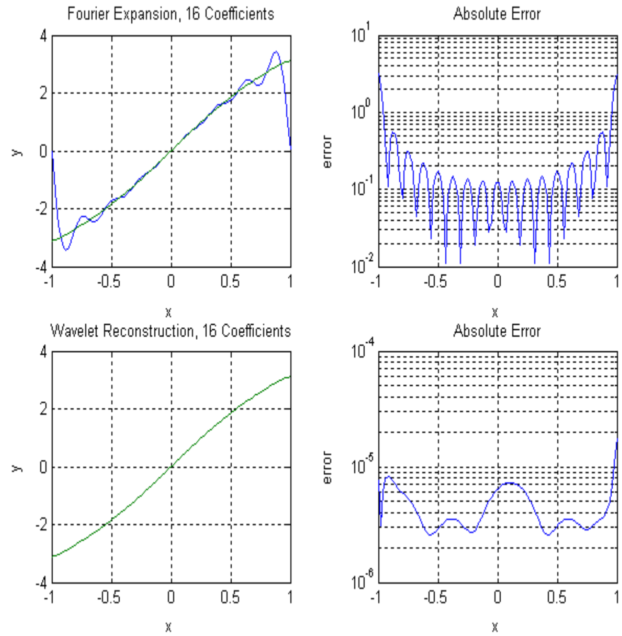

Figure 1: The graphs illustrate the Gibbs phenomenon for a 16 term Fourier partial sum. The 16 Fourier coefficients are used to reconstruct a 16 term Morlet wavelet series accurate to four decimal places.

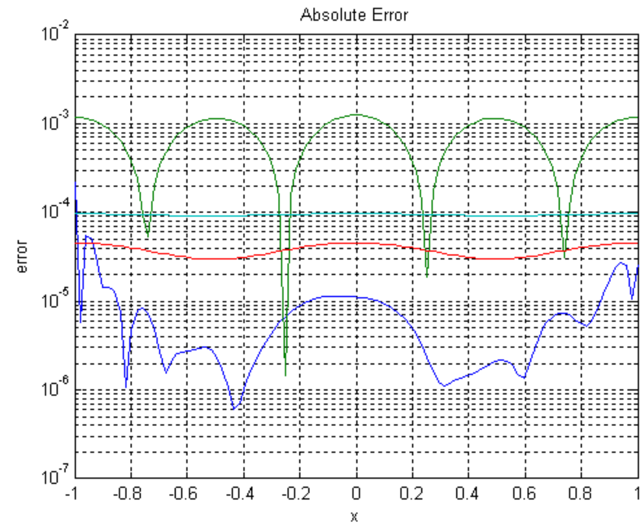

Figure 3: The absolute error of a Poisson wavelet reconstruction based on 16, 36, 64, and 100 Fourier coefficients.

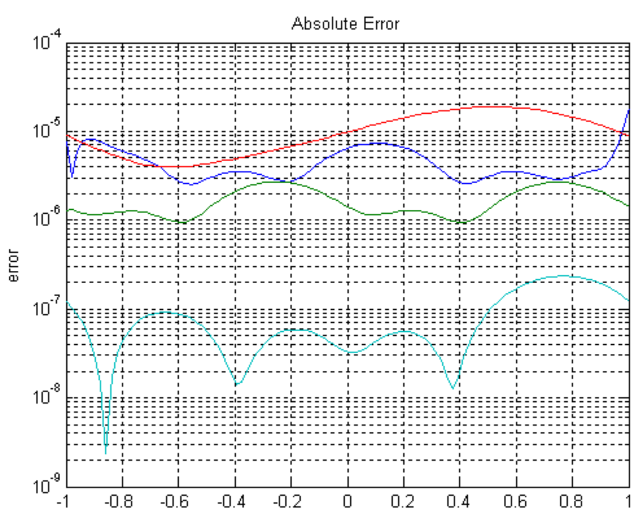

Figure 2: The absolute error of a Morlet wavelet reconstruction based on 16, 36, 64, and 100 Fourier coefficients.

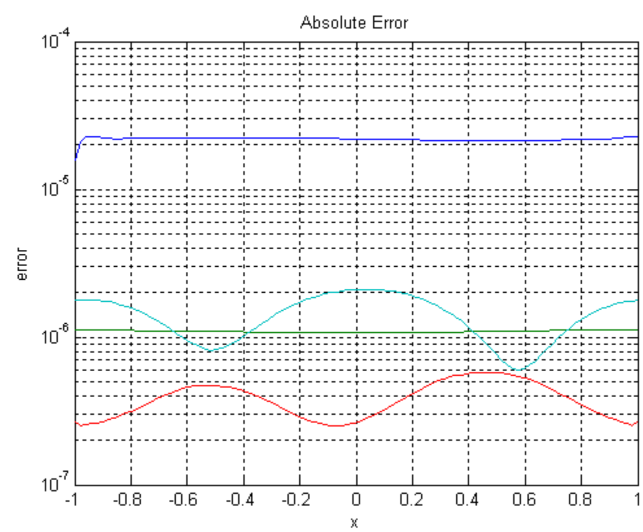

Figure 4: The absolute error of a Mexican hat wavelet reconstruction based on 16, 36, 64, and 100 Fourier coefficients.

4 Conclusions

The numerical results indicate that the inverse wavelet reconstruction method yields an accurate and uniformly converging reconstruction approximation for a variety of wavelets. Current work underway includes a study of the technique for a broader spectrum of wavelets, reconstruction from series other than Fourier series, such as orthogonal polynomials or wavelets, comparison with other methods, an analytic estimation of error and proof of convergence.

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