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ALGEBRAIC FORMULATION OF HADRONIC SUPERSYMMETRY BASED ON OCTONIONS: NEW MASS FORMULAS AND FURTHER APPLICATIONS

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Abstract.

A special treatment based on the highest division algebra, that of octonions and their split algebraic formulation is developed for the description of diquark states made up of two quark pairs. We describe symmetry properties of mesons and baryons through such formulation and derive mass formulae relating π , ρ , N and Δ trajectories showing an incredible agreement with experiments. We also comment on formation of diquark-antidiquark as well as pentaquark states and point the way toward applications into multiquark formulations expected to be seen at upcoming CERN experiments. A discussion on relationship of our work to flux bag models, string pictures and to string-like configurations in hadrons based on spectrum generating algebras will be given.

1. Introduction

In mid sixties Miyazawa, in a series of papers^[1], extended the $SU(6)$ group to the supergroup $SU(6/21)$ that could be generated by constituent quarks and diquarks that could be transformed to each other. In particular, he found the following: (a) A general definition of $SU(m/n)$ superalgebras, expressing the symmetry between m bosons and n fermions, with Grassman-valued parameters. (b) A derivation of the super-Jacobi identity. (c) The relation of the baryon mass splitting to the meson mass splitting through the new mass formulae.

This work contained the first classification of superalgebras (later rediscovered by mathematicians in the seventies). Because of the field-theoretic prejudice against $SU(6)$, Miyazawa's work was generally ignored. Supersymmetry was, of course, rediscovered in the seventies within the dual resonance model by Ramond^[2], and Neveu and Schwarz^[3]. Golfand and Likhtman^[4], and independently Volkov and Akulov^[5], proposed the extension of the Poincaré group to the super-Poincaré group. Examples of supersymmetric field theories were given and the general method based on the super-Poincaré group was discovered by Wess and Zumino^[6]. The super-Poincaré group allowed transformations between fields



associated with different spins 0, $\frac{1}{2}$ and 1. The Coleman-Mandula theorem was amended in 1975 by Haag, Lopuszanski and Sohnius to allow super-Poincaré e group $\times G_{int}$ as the maximum symmetry of the S-matrix. Unfortunately, $SU(6)$ symmetry was still forbidden.

2. $SU(6)$ and Hadronic Supersymmetry

How do we interpret the symmetries of the QCD spectrum in this light? In the ultraviolet, the running coupling constant tends to zero and quarks behave like free point particles. Thus an approximate conformal symmetry exists, allowing spin to be conserved separately from orbital angular momentum. Thus spin behaves as an internal quantum number; this makes a $SU(6)$ symmetry possible, since the quarks are almost free Dirac particles. Single vector-gluon exchange breaks this symmetry; thus, as shown by Glashow, Georgi and deRujula^[7], the mass-degeneracy of hadrons of different spins is lifted by a hyperfine-interaction term.

Here is the main point. In the infrared we expect confinement to set in. The quark-antiquark potential becomes proportional to the distance. Careful studies of quarkonium spectra and lattice-gauge calculations show that at large separation the quark forces become spin-independent. QCD is also flavor independent. We therefore find approximate spin- and flavor-independent quark binding forces; these are completely consistent with $SU(6)$ symmetry. This is not an exact symmetry, but is a good starting point, before spin and flavor effects are included.

There is good phenomenological evidence that in a rotationally excited baryon a quark-diquark ($q - D$) structure is favored over a three-quark (qqq) structure^{[8],[9],[10]}. Eguchi^[11] had shown that it is energetically favorable for the three quarks in a baryon to form a linear structure with a quark on one end and bilocal structure qq at the other end. Similarly, Bars and Hanson^[12], and independently Johnson and Thorn^[13] had shown that the string-like hadrons may be pictured as vortices of color flux lines which terminate on concentration of color at the end points. A baryon with three valence quarks would be arranged as a linear chain of molecule where the largest angular momentum for a state of a given mass is expected when two quarks are at one end, and the third is at the other: At large spin, two of the quarks form a diquark at one end of the string, the remaining quark being at the other. Regge trajectories for mesons and baryons are closely parallel; both have a slope of about $0.9(GeV)^{-2}$. If the quarks are light, the underlying quark-diquark symmetry leads to a Miyazawa symmetry between mesons and baryons. Thus we studied QCD with a weakly broken supergroup $SU(6/21)$. Note that the fundamental theory is not supersymmetric. For quarks, the generators of the Poincaré group and those of the color group $SU(3)^c$ commute. It is only the effective Hamiltonian which exhibits an approximate supersymmetry among the bound states $q\bar{q}$ and qD .

Under the color group $SU(3)^c$, meson $q\bar{q}$ and diquark ($D = qq$) states transform as^{[10],[14]}

$$qq : \mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6} ; \quad q\bar{q} : \mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8} \tag{1}$$

and under the spin flavor $SU(6)$ they transform as

$$qq : \mathbf{6} \times \mathbf{6} = \mathbf{15} + \mathbf{21} ; \quad q\bar{q} : \mathbf{6} \times \bar{\mathbf{6}} = \mathbf{1} + \mathbf{35} \tag{2}$$

Dimensions of internal degrees of quarks and diquarks are shown in the following table:

	$SU_f(3)$	$SU_s(2)$	$dim.$
q	\square	$s = 1/2$	$3 \times 2 = 6$
D	$\square \square$	$s = 1$	$6 \times 3 = 18$
	\square	$s = 0$	$3 \times 1 = 3$
	\square		

If one writes qqq as qD , then the quantum numbers of D are $\bar{\mathbf{3}}$ for color since when combined with q must give a color singlet, and $\mathbf{21}$ for spin-flavor since combined with color must give antisymmetric wavefunctions. The quantum numbers for \bar{q} are for color, $\bar{\mathbf{3}}$, and for spin-flavor, $\bar{\mathbf{6}}$. Thus \bar{q} and D have the same quantum numbers (color forces can not distinguish between \bar{q} and D). Therefore there is a dynamic supersymmetry in hadrons with supersymmetric partners

$$\psi = \begin{pmatrix} \bar{q} \\ D \end{pmatrix}, \quad \bar{\psi} = (q \quad \bar{D}) \tag{3}$$

We can obtain all hadrons by combining ψ and $\bar{\psi}$: mesons are $q\bar{q}$, baryons are qD , and exotics are $D\bar{D}$ states. Inside rotationally excited baryons, QCD leads to the formation of diquarks well separated from the remaining quark. At this separation the scalar, spin-independent, confining part of the effective QCD potential is dominant. Since QCD forces are also flavor-independent, the force between the quark q and the diquark D inside an excited baryon is essentially the same as the one between q and the antiquark \bar{q} inside an excited meson. Thus the approximate spin-flavor independence of hadronic physics expressed by $SU(6)$ symmetry is extended to $SU(6/21)$ supersymmetry^{[10],[14]} through a symmetry between \bar{q} and D , resulting in parallelism of mesonic and baryonic Regge trajectories.

3. Color Algebra and Octonions

We shall now give an algebraic justification to our remarks above. We will find all the answers in an algebra we build in terms of octonions and their split basis. The exact, unbroken color group $SU(3)^c$ is the backbone of the strong interaction. It is worthwhile to understand its role in the diquark picture more clearly.

In what follows we first give a simple description of octonion algebra (also known as Cayley algebra). Later we'll show how to build split octonion algebra that will close into a fermionic Heisenberg algebra. Split octonion algebra will then be shown to produce algebra of color forces in QCD in application to hadronic supersymmetry when the split units and their conjugates become associated with quark and antiquark fields, respectively.

An octonion x is a set of eight real numbers

$$x = (x_0, x_1, \dots, x_7) = x_0e_0 + x_1e_1 + \dots + x_7e_7 \tag{4}$$

that are added like vectors and multiplied according to the rules

$$e_0 = 1, \quad e_0e_i = e_ie_0 = e_i, \quad i = 0, 1, \dots, 7 \tag{5}$$

$$e_\alpha e_\beta = -\delta_{\alpha\beta} + \epsilon_{\alpha\beta\gamma} e_\gamma, \quad \alpha, \beta, \gamma = 1, 2, \dots, 7 \tag{6}$$

where e_0 is the multiplicative unit element and e_α 's are the imaginary octonion units. The structure constants $\epsilon_{\alpha\beta\gamma}$ are completely antisymmetric and take the value 1 for combinations

$$\epsilon_{\alpha\beta\gamma} = (165), (257), (312), (471), (543), (624), (736) \tag{7}$$

Note that summation convention is used for repeated indices.

The octonion algebra \mathcal{C} is an algebra defined over the field \mathbf{Q} of rational numbers, which as a vector space over \mathbf{Q} has dimension 8.

We shall now give reasons for incorporation of the octonion algebra for hadronic physics, showing only they through their split octonionic parts one can provide the correct description of the color algebra in hadrons. Later in another publication we shall show^[15] a previously unknown multiplication rules for octonions by producing a wheel that allows generalized multiplication rules for doublets and triplets of octonionic units.

First, the reasons: Two of the colored quarks in the baryon combine into an anti-triplet $\mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}} + (\mathbf{6})$, and in a nucleon $\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + (\mathbf{8})$. The $(\mathbf{6})$ partner of the diquark and the $(\mathbf{8})$ partner of the nucleon do not exist. In hadron dynamics the only color combinations to consider are $\mathbf{3} \times \mathbf{3} \rightarrow \bar{\mathbf{3}}$ and $\bar{\mathbf{3}} \times \mathbf{3} \rightarrow \mathbf{1}$. These relations imply the existence of split octonion units u_i defined below through a representation of the Grassmann algebra $\{u_i, u_j\} = 0, i = 1, 2, 3$. What is a bit strange is that operators u_i , unlike ordinary fermionic operators, are not associative. We also have $\frac{1}{2}[u_i, u_j] = \epsilon_{ijk} u_k^*$. The Jacobi identity does not hold since $[u_i, [u_j, u_k]] = -ie_7 \neq 0$, where e_7 , anticommute with u_i and u_i^* .

The behavior of various states under the color group are best seen if we use split octonion units defined by^[16]

$$u_0 = \frac{1}{2}(1 + ie_7), \quad u_0^* = \frac{1}{2}(1 - ie_7) \tag{8}$$

$$u_j = \frac{1}{2}(e_j + ie_{j+3}), \quad u_j^* = \frac{1}{2}(e_j - ie_{j+3}), \quad j = 1, 2, 3 \tag{9}$$

The automorphism group of the octonion algebra is the 14-parameter exceptional group G_2 . The imaginary octonion units $e_\alpha (\alpha = 1, \dots, 7)$ fall into its 7-dimensional representation.

Under the $SU(3)^c$ subgroup of G_2 that leaves e_7 invariant, u_0 and u_0^* are singlets, while u_j and u_j^* correspond, respectively, to the representations $\mathbf{3}$ and $\bar{\mathbf{3}}$. The multiplication table can now be written in a manifestly $SU(3)^c$ invariant manner (together with the complex conjugate equations):

$$u_0^2 = u_0, \quad u_0 u_0^* = 0 \tag{10}$$

$$u_0 u_j = u_j u_0^* = u_j, \quad u_0^* u_j = u_j u_0 = 0 \tag{11}$$

$$u_i u_j = -u_j u_i = \epsilon_{ijk} u_k^* \tag{12}$$

$$u_i u_j^* = -\delta_{ij} u_0 \tag{13}$$

where ϵ_{ijk} is completely antisymmetric with $\epsilon_{ijk} = 1$ for $ijk = 123, 246, 435, 651, 572, 714, 367$; and zero otherwise. Here, one sees the virtue of octonion multiplication. If we consider the direct products

$$C : \quad \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8} \tag{14}$$

$$G : \quad \mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6} \tag{15}$$

for $SU(3)^c$, then these equations show that octonion multiplication gets rid of $\mathbf{8}$ in $\mathbf{3} \otimes \bar{\mathbf{3}}$, while it gets rid of $\mathbf{6}$ in $\mathbf{3} \otimes \mathbf{3}$. Combining Eq.(12) and Eq.(13) we find

$$(u_i u_j) u_k = -\epsilon_{ijk} u_0^* \tag{16}$$

Thus the octonion product leaves only the color part in $\mathbf{3} \otimes \bar{\mathbf{3}}$ and $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$, so that it is a natural algebra for colored quarks.

For convenience we now produce the following multiplication table for the split octonion units:

	u_0	u_0^*	u_k	u_k^*
u_0	u_0	0	u_k	0
u_0^*	0	u_0^*	0	u_k^*
u_j	0	u_j	$\epsilon_{jki} u_i^*$	$-\delta_{jk} u_0$
u_j^*	u_j^*	0	$-\delta_{jk} u_0^*$	$\epsilon_{jki} u_i$

It is worth noting that u_i and u_j^* behave like fermionic annihilation and creation operators:

$$\{u_i, u_j\} = \{u_i^*, u_j^*\} = 0, \quad \{u_i, u_k^*\} = -\delta_{ik} \quad (17)$$

For more recent reviews on octonions and nonassociative algebras we refer to papers by Okubo^[17], Baez^[18] and Catto^[19].

The quarks, being in the triplet representation of the color group $SU(3)^c$, are represented by the local fields $q_\alpha^i(x)$, where $i = 1, 2, 3$ is the color index and α the combined spin-flavor index. Antiquarks at point y are color antitriplets $\bar{q}_\beta^j(y)$. Consider the two-body systems

$$C_{\alpha j}^{\beta i} = q_\alpha^i(x_1)\bar{q}_\beta^j(x_2) \quad (18)$$

$$G_{\alpha\beta}^{ij} = q_\alpha^i(x_1)q_\beta^j(x_2) \quad (19)$$

so that C is either a color singlet or color octet, while G is a color antitriplet or a color sextet. Now C contains meson states that are color singlets and hence observable. The octet $q-\bar{q}$ state is confined and not observed as a scattering state. In the case of two-body G states, the antitriplets are diquarks which, inside a hadron can be combined with another triplet quark to give observable, color singlet, three-quark baryon states. The color sextet part of G can only combine with a third quark to give unobservable color octet and color decuplet three-quark states. Hence the hadron dynamics is such that the **8** part of C and the **6** part of G are suppressed. This can best be achieved by the use of the above octonion algebra^[20]. The dynamical suppression of the octet and sextet states in Eq.(18) and Eq.(19) is, therefore, automatically achieved. The split octonion units can be contracted with color indices of triplet or antitriplet fields. For quarks and antiquarks we can define the "transverse" octonions (calling u_0 and u_0^* longitudinal units)

$$q_\alpha = u_i q_\alpha^i = \mathbf{u} \cdot \mathbf{q}_\alpha, \quad \bar{q}_\beta = u_i^\dagger \bar{q}_\beta^i = -\mathbf{u}^* \cdot \bar{\mathbf{q}}_\beta \quad (20)$$

We find

$$q_\alpha(1)\bar{q}_\beta(2) = u_0 \mathbf{q}_\alpha(1) \cdot \bar{\mathbf{q}}_\beta(2) \quad (21)$$

$$\bar{q}_\alpha(1)q_\beta(2) = u_0^* \bar{\mathbf{q}}_\alpha(1) \cdot \mathbf{q}_\beta(2) \quad (22)$$

$$G_{\alpha\beta}(12) = q_\alpha(1)q_\beta(2) = \mathbf{u}^* \cdot \mathbf{q}_\alpha(1) \times \mathbf{q}_\beta(2) \quad (23)$$

$$G_{\beta\alpha}(21) = q_\beta(2)q_\alpha(1) = \mathbf{u}^* \cdot \mathbf{q}_\beta(2) \times \mathbf{q}_\alpha(1) \quad (24)$$

Because of the anticommutativity of the quark fields, we have

$$G_{\alpha\beta}(12) = G_{\beta\alpha}(21) = \frac{1}{2}\{q_\alpha(1), q_\beta(2)\} \quad (25)$$

If the diquark forms a bound state represented by a field $D_{\alpha\beta}(x)$ at the center-of-mass location x

$$x = \frac{1}{2}(x_1 + x_2) \quad (26)$$

when x_2 tends to x_1 we can replace the argument by x , and we obtain

$$D_{\alpha\beta}(x) = D_{\beta\alpha}(x) \quad (27)$$

so that the local diquark field must be in a symmetric representation of the spin-flavor group. If the latter is taken to be $SU(6)$, then $D_{\alpha\beta}(x)$ is in the 21-dimensional symmetric representation, given by

$$(\mathbf{6} \otimes \mathbf{6})_s = \mathbf{21} \quad (28)$$

If we denote the antisymmetric 15 representation by $\Delta_{\alpha\beta}$, we see that the octonionic fields single out the 21 diquark representation at the expense of $\Delta_{\alpha\beta}$. We note that without this color algebra supersymmetry would give antisymmetric configurations as noted by Salam and Strathdee^[21] in their possible supersymmetric generalization of hadronic supersymmetry. Using the nonsingular part of the operator product expansion we can write

$$\tilde{G}_{\alpha\beta}(\mathbf{x}_1, \mathbf{x}_2) = D_{\alpha\beta}(x) + \mathbf{r} \cdot \mathbf{\Delta}_{\alpha\beta}(x) \quad (29)$$

The fields $\Delta_{\alpha\beta}$ have opposite parity to $D_{\alpha\beta}$; \mathbf{r} is the relative coordinate at time t if we take $t = t_1 = t_2$. They play no role in the excited baryon which becomes a bilocal system with the 21- dimensional diquark as one of its constituents.

Now consider a three-quark system at time t . The c.m. and relative coordinates are

$$\mathbf{R} = \frac{1}{\sqrt{3}}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3) \quad (30)$$

$$\vec{\rho} = \frac{1}{\sqrt{6}}(2\mathbf{r}_3 - \mathbf{r}_1 - \mathbf{r}_2) \quad (31)$$

$$\mathbf{r} = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2) \quad (32)$$

giving

$$\mathbf{r}_1 = \frac{1}{\sqrt{3}}\mathbf{R} - \frac{1}{\sqrt{6}}\vec{\rho} + \frac{1}{\sqrt{2}}\mathbf{r} \quad (33)$$

$$\mathbf{r}_2 = \frac{1}{\sqrt{3}}\mathbf{R} - \frac{1}{\sqrt{6}}\vec{\rho} - \frac{1}{\sqrt{2}}\mathbf{r} \quad (34)$$

$$\mathbf{r}_3 = \frac{1}{\sqrt{3}}\mathbf{R} + \frac{2}{\sqrt{6}}\vec{\rho} \quad (35)$$

The baryon state must be a color singlet, symmetric in the three pairs (α, x_1) , (β, x_2) , (γ, x_3) . We find

$$(q_\alpha(1)q_\beta(2))q_\gamma(3) = -u_0^*F_{\alpha\beta\gamma}(123) \quad (36)$$

$$q_\gamma(3)(q_\alpha(1)q_\beta(2)) = -u_0F_{\alpha\beta\gamma}(123) \quad (37)$$

so that

$$-\frac{1}{2}\{\{q_\alpha(1), q_\beta(2)\}, q_\gamma(3)\} = F_{\alpha\beta\gamma}(123) \quad (38)$$

The operator $F_{\alpha\beta\gamma}(123)$ is a color singlet and is symmetrical in the three pairs of coordinates. We have

$$F_{\alpha\beta\gamma}(123) = B_{\alpha\beta\gamma}(\mathbf{R}) + \vec{\rho} \cdot \mathbf{B}'(\mathbf{R}) + \mathbf{r} \cdot \mathbf{B}''(\mathbf{R}) + C \quad (39)$$

where C is of order two and higher in $\vec{\rho}$ and \mathbf{r} . Because \mathbf{R} is symmetric in \mathbf{r}_1 , \mathbf{r}_2 and \mathbf{r}_3 , the operator $B_{\alpha\beta\gamma}$ that creates a baryon at \mathbf{R} is totally symmetrical in its flavor-spin indices. In the $SU(6)$ scheme it belongs to the (56) representation. In the bilocal $q - D$ approximation we have $\mathbf{r} = 0$ so that $F_{\alpha\beta\gamma}$

is a function only of \mathbf{R} and $\vec{\rho}$ which are both symmetrical in \mathbf{r}_1 and \mathbf{r}_2 . As before, \mathbf{B}' belongs to the orbitally excited 70^- representation of $SU(6)$. The totally antisymmetrical (20) is absent in the bilocal approximation. It would only appear in the trilocal treatment that would involve the 15-dimensional diquarks. Hence, if we use local fields, any product of two octonionic quark fields gives a (21) diquark

$$q_\alpha(\mathbf{R})q_\beta(\mathbf{R}) = D_{\alpha\beta}(\mathbf{R}) \quad (40)$$

and any nonassociative combination of three quarks, or a diquark and a quark at the same point give a baryon in the 56^+ representation:

$$(q_\alpha(\mathbf{R})q_\beta(\mathbf{R}))q_\gamma(\mathbf{R}) = -u_0^* B_{\alpha\beta\gamma}(\mathbf{R}) \quad (41)$$

$$q_\alpha(\mathbf{R})(q_\beta(\mathbf{R})q_\gamma(\mathbf{R})) = -u_0 B_{\alpha\beta\gamma}(\mathbf{R}) \quad (42)$$

$$q_\gamma(\mathbf{R})(q_\alpha(\mathbf{R})q_\beta(\mathbf{R})) = -u_0 B_{\alpha\beta\gamma}(\mathbf{R}) \quad (43)$$

$$(q_\gamma(\mathbf{R})q_\alpha(\mathbf{R}))q_\beta(\mathbf{R}) = -u_0^* B_{\alpha\beta\gamma}(\mathbf{R}) \quad (44)$$

The bilocal approximation gives the $(35 + 1)$ mesons and the 70^- baryons with $\ell = 1$ orbital excitation.

If we consider a (28×28) octonionic matrix

$$Z = \begin{pmatrix} u_0 M & u_0 B & \mathbf{u} \cdot \mathbf{Q} \\ u_0 B^\dagger & u_0 N & \mathbf{u} \cdot \mathbf{D}^* \\ \epsilon \mathbf{u}^* \cdot \mathbf{Q}^\dagger & \epsilon \mathbf{u}^* \cdot \mathbf{D}^\dagger & u_0^* L \end{pmatrix} \quad (45)$$

here ϵ can be 1, -1 or 0. M and N are respectively 6×6 and 21×21 hermitian matrices, B a regular 6×21 matrix, $\mathbf{u} \cdot \mathbf{Q}$ a 6×1 column matrix, $\mathbf{u} \cdot \mathbf{D}^*$ a 21×1 column matrix, and L a 1×1 scalar. Such matrices close under anticommutator operations for $\epsilon = 1$. Matrices iZ close under commutator operations. In either case, they don't satisfy the Jacobi identity. But for $\epsilon = 0$, when the algebra is no longer semi-simple, the Jacobi identity is satisfied and we obtain a hadronic superalgebra which is an extension of the algebra $SU(6/21)$. Its automorphism group includes $SU(6) \times SU(21) \times SU(3)^c$. Thus color is automatically incorporated.

4. QCD Justification of $U(6/21)$ Supersymmetry and Its Breaking

We shall first discuss the validity domain of $SU(6/21)$ supersymmetry. The diquark structure with spin $s = 0$ and $s = 1$ emerges in inelastic inclusive lepton-baryon collisions with high impact parameters that excite the baryon rotationally, resulting in inelastic structure functions based on point-like quarks and diquarks instead of three point-like quarks, In this case both mesons and baryons are bilocal with large separation of constituents.

Also, there is a symmetry between color antitriplet diquarks with $s = 0$ and $s = 1$ and color antitriplet antiquarks with $s = \frac{1}{2}$. This is only possible if the force between quark q and antiquark \bar{q} , and also between q and diquark D is mediated by a zero spin object that sees no difference between the spins of \bar{q} and D . The object can be in color states that are either singlet or octet since q and D are both triplets. Such an object is provided by scalar flux tubes of gluons that dominate over the one gluon exchange at large distances. Various strong coupling approximations to QCD, like lattice gauge theory^{[22],[23]}, 't Hooft's $\frac{1}{N}$ approximation^[24] when N , the number of colors, is very large, or the elongated bag model^[13] all give a linear potential between widely separated quarks and an effective string that approximates the gluon flux tube. In such a theory it is energetically favorable for the three quarks in a baryon to form a linear structure with a quark in the middle and two at the ends, or, for high rotational excitation, a

bilocal linear structure (diquark) at one end and a quark at the other end. In order to illustrate these points we start with the suggestion of Johnson and Thorn^[13] that the string-like hadrons may be pictured as vortices of color flux lines which terminate on concentration of color at the end points. The color flux connecting opposite ends is the same for mesons and baryons giving an explanation for the same slope of meson and baryon trajectories^[10].

To construct a solution which yields a maximal angular momentum for a fixed mass we consider a bag with elongated shape rotating about the center of mass with an angular frequency ω . Its ends have the maximal velocity allowed, which is the speed of light ($c = 1$). Thus, a given point inside the bag, at a distance r from the axis of rotation moves with a velocity

$$v = \vec{\omega} \cdot \vec{r} = \frac{2r}{L} \tag{46}$$

where L is the length of the string. In this picture the bag surface will be fixed by balancing the gluon field pressure against the confining vacuum pressure B , which (in analogy to electrodynamics) can be written in the form

$$\frac{1}{2} \sum_{\alpha=1}^8 (E_{\alpha}^2 - B_{\alpha}^2) = B \tag{47}$$

Using Gauss' law, the color electric field E through the flux tube connecting the color charges at the ends of the string is given by

$$\int \vec{E}_{\alpha} \cdot d\vec{S} = E_{\alpha} A = g \frac{1}{2} \lambda_{\alpha} \tag{48}$$

where $A(r)$ is the cross-section of the flux tube at distance r from the center and $g \frac{1}{2}$ is the color electric charge which is the source of E_{α} . By analogy with classical electrodynamics the color magnetic field $\vec{B}_{\alpha}(r)$ associated with the rotation of the color electric field is

$$\vec{B}_{\alpha}(r) = \vec{v}(r) \times \vec{E}_{\alpha}(r) \tag{49}$$

at a point moving with a velocity $\vec{v}(r)$. For the absolute values this implies

$$B_{\alpha} = v E_{\alpha} \tag{50}$$

because $\vec{v}(r)$ is perpendicular to $\vec{E}_{\alpha}(r)$. Using last three equations together with

$$\left\langle \sum_{\alpha=1}^8 \left(\frac{1}{2} \lambda_{\alpha}\right)^2 \right\rangle = \frac{4}{3} \tag{51}$$

for the $SU(3)^c$ triplet in Eq.(47) we obtain for the crosssection of the bag

$$A(r) = \sqrt{\frac{2}{3B}} g \sqrt{1 - v^2} \tag{52}$$

which shows the expected Lorentz contraction.

The total energy E of the bag

$$E = E_q + E_G + BV \tag{53}$$

is the sum of the quark energy E_q , the gluon field energy E_G and the volume energy of the bag, BV . Because the quarks at the ends move with the a speed close to speed of light, their energy is simply given by

$$E_q = 2p \tag{54}$$

where p is the momentum of a quark, a diquark or an antiquark, respectively. For the gluon energy, by analogy with electrodynamics, one obtains from Eqs.(48-50) the result

$$\begin{aligned} E_G &= \frac{1}{2} \int d^3x \sum_{\alpha=1}^8 (E_\alpha^2 + B_\alpha^2) \\ &= \sqrt{\frac{2}{3}} g \sqrt{BL} \int_0^1 dv \frac{1+v^2}{\sqrt{1-v^2}} = \sqrt{\frac{2}{3}} g \sqrt{BL} \frac{3\pi}{4} \end{aligned} \tag{55}$$

and for the volume energy

$$\begin{aligned} BV &= 2B \int_0^{\frac{L}{2}} A(r) dr = 2B \int_0^1 \sqrt{\frac{2}{3B}} g \sqrt{1-v^2} \frac{L}{2} dv \\ &= \sqrt{\frac{2}{3}} g \sqrt{BL} \frac{\pi}{4} = \frac{BA(0)L\pi}{4} \end{aligned} \tag{56}$$

It is obvious from Eq.(55) that the gluon field energy is proportional to the length L of the bag. The gluon field energy and the volume energy of the bag together correspond to a linear rising potential of the form

$$V(L) = E_G + BV = bL \tag{57}$$

where

$$b = \sqrt{\frac{2B}{3}} g \pi \tag{58}$$

The total angular momentum J of this classical bag is the sum of the angular momenta of the quarks at the two ends

$$J_q = pL \tag{59}$$

and the angular momentum J_G of the gluon field. From Eq.(49) we get

$$\vec{E}_\alpha \times \vec{B}_\alpha = \vec{v} E_\alpha^2 \tag{60}$$

for the momentum of the gluon field, and hence

$$\begin{aligned} J_G &= \left| \int_{bag} d^3\vec{r} \sum_{\alpha=1}^8 \vec{r} \times (\vec{E}_\alpha \times \vec{B}_\alpha) \right| = 2 \int_0^{\frac{L}{2}} dr A(r) r v E_\alpha^2 \\ &= \frac{16}{3L} g^2 \int_0^{\frac{L}{2}} \frac{r^2 dr}{A(r)} = \sqrt{\frac{2}{3}} g \sqrt{BL}^2 \frac{\pi}{4} \end{aligned} \tag{61}$$

where we have used Eq.(46) and Eq.(48) in the third step. We can now express the total energy of the bag in terms of angular momenta. Putting these results back into expressions for E_q and E_G , we arrive at

$$E_q = \frac{2J_q}{L}, \quad E_G = \frac{3J_G}{L} \quad (62)$$

so that the bag energy now becomes

$$\begin{aligned} E &= \frac{2J_q}{L} + \frac{3J_G}{L} + \sqrt{\frac{2B}{3}} L g \frac{\pi}{L} = \frac{2J_q + 4J_G}{L} \\ &= \frac{2(J + J_G)}{L} = \frac{1}{L} (2J + \sqrt{\frac{2}{3}} g \sqrt{B} L^2 \frac{\pi}{2}) \end{aligned} \quad (63)$$

Minimizing the total energy for a fixed angular momentum with respect to the length of the bag, $\frac{dE}{dt} = 0$ gives the relation

$$-\frac{2J}{L^2} + \sqrt{\frac{2}{3}} g \sqrt{B} \frac{\pi}{2} = 0 \quad (64)$$

so that

$$L^2 = \frac{4J}{g\pi} \sqrt{\frac{3}{2B}} \quad (65)$$

Re-inserting this into Eq.(63) we arrive at

$$E = 2\sqrt{Jg\pi} \left(\frac{2B}{3}\right)^{\frac{1}{4}} \quad (66)$$

or

$$J = \left(\sqrt{\frac{3}{2B}} \frac{1}{4\pi g}\right) E^2 = \left(\sqrt{\frac{3}{2B}} \frac{1}{8\pi^{\frac{3}{2}}} \frac{1}{\sqrt{\alpha_s}}\right) E^2 = \alpha'(0) M^2 \quad (67)$$

where $M = E$ and we used $\alpha_s = \frac{g^2}{4\pi}$, the unrationalized color gluon coupling constant. We can now let $\alpha'(0)$ defined by the last equation which is the slope of the Regge trajectory as

$$\alpha'(0) = \sqrt{\frac{3}{2B}} \frac{1}{8\pi^{\frac{3}{2}}} \frac{1}{\sqrt{\alpha_s}} = \frac{1}{4b} \quad (68)$$

where b was defined in Eq.(57).

The parameters B and α_s have been determined^{[25].[26]} using the experimental information from the low lying hadron states: $B^{\frac{1}{4}} = 0.146 \text{ GeV}$ and $\alpha_s = 0.55 \text{ GeV}$. If we use these values in Eq.(68) we find

$$\alpha'(0) = 0.88 (\text{GeV})^{-2} \quad (69)$$

in remarkable agreement with the slope determined from experimental data which is about $0.9 (\text{GeV})^{-2}$.

The total phenomenological non-relativistic potential then is the well known superposition of the Coulomb-like and confining potentials $V(r) = \frac{a}{r} + br$ where $r = |\vec{r}_1 - \vec{r}_2|$ is the distance between q and \bar{q} in a meson, or between q and D in a baryon with high angular momentum. This is verified in lattice QCD to a high degree of accuracy^[27] ($a = \frac{-c\alpha_c}{r}$, where c is the color factor and α_c the strong coupling strength).

It is interesting to know that all this is related very closely to the dual strings. Indeed we can show that the slope given in Eq.(68) is equivalent to the dual string model formula for the slope if we associate the "proper tension" in the string with the proper energy per unit length of the color flux tube and the volume.

By proper energy per unit length we mean the energy per unit length at a point in the bag evaluated in the rest system of that point. This will be

$$T_0 = \frac{1}{2} \sum_{\alpha} E_{\alpha}^2 A_0 + B A_0 \tag{70}$$

The fact that $\frac{1}{2} \sum_{\alpha} E_{\alpha}^2 = B$ in the rest system gives

$$T_0 = 2B A_0 \tag{71}$$

where A_0 is the cross-sectional area of the bag. If in Eq.(52) we let $A = A_0$ when $v = 0$, then using

$$A_0 = \sqrt{\frac{2}{3B}} g \tag{72}$$

we find

$$T_0 = 2\sqrt{\frac{2}{3}} g \sqrt{B} = 4\sqrt{\frac{2\pi}{3}} \sqrt{\alpha_s} \sqrt{B} \tag{73}$$

for the proper tension. In the dual string the slope and proper tension are related by the formula^[28]

$$T_0 = \frac{1}{2\pi\alpha'} \tag{74}$$

so the slope is

$$\alpha' = \frac{1}{8} \sqrt{\frac{3}{2}} \frac{1}{\pi^{\frac{3}{2}}} \frac{1}{\sqrt{\alpha_s}} \frac{1}{\sqrt{B}} \tag{75}$$

which is identical to the earlier formula we produced in Eq.(68).

It would appear from Eq.(73) that the ratio of volume to field energy would be one-to-one in one space dimension in contrast to the result one-to-three which holds for a three dimensional bag^[29]. However, the ratio one-to-one is true only in the rest system at a point in the bag, and each position along the x -axis is of course moving with a different velocity. Indeed we see from Eq.(55) and Eq.(56) that the ratio of the total volume energy to the total field energy is given by one-to three in conformity with the virial theorem^[29].

In the string model of hadrons we have $E^2 \sim J$ between the energy and the angular momentum of the rotating string. If we denote by $\rho(r)$ the mass density of the string, and by v and ω its linear and angular velocities, respectively, the energy and the angular momentum of the rotating string are given by

$$E = 2 \int \frac{\rho(r)}{\sqrt{1 - \omega^2 r^2}} dr = \frac{2}{\omega} \int_0^1 \frac{\rho(v)}{\sqrt{1 - v^2}} dv \tag{76}$$

and

$$J = 2 \int \frac{\rho(r)}{\sqrt{1 - \omega^2 r^2}} r^2 \omega dr = \frac{2}{\omega^2} \int_0^1 \frac{\rho(v)}{\sqrt{1 - v^2}} v^2 dv \tag{77}$$

and hence the relation

$$E^2 \propto J. \tag{78}$$

If the string is loaded with mass points at its ends, they no longer move with speed of light, however, the above relation still holds approximately for the total energy and angular momentum of the loaded string.

We now look at various ways of partitioning of the total angular momentum into two subsystems. Figures (a), (b) and (c) show the possible configurations of three quarks in a baryon.

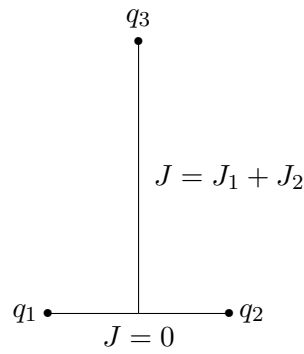


Fig.(a)

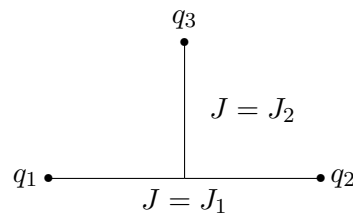


Fig.(b)

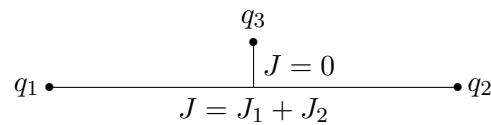


Fig.(c)

If we put the proportionality constant in Eq.(78) equal to unity, then the naive evaluation of energies yield

$$E^2 = J_1 + J_2 = E_1^2 + E_2^2 \leq (E_1 + E_2)^2 = E'^2 \quad (79)$$

where E and E' denote the energies corresponding to figures (a) or (c). In the case of figure (b), J_1 , J_2 are the angular momenta corresponding to the energies E_1 and E_2 of the subsystems. The equality

in Eq.(79) holds only if E_1 or E_2 is zero. Therefore for each fixed total angular momentum its most unfair partition into two subsystems gives us the lowest energy levels, and its more or less fair partition gives rise to energy levels on daughter trajectories. Hence on the leading baryonic trajectory we have a quark-diquark structure (Fig.(a)), or a linear molecule structure (Fig.(c)). On the other hand on low-lying trajectories we have more or less symmetric ($J_1 \sim J_2$) configuration of quarks.

In Table 3, we give the flavor-spin content for three quark and quark-diquark baryons that correspond to Fig.(a) and Fig.(c) shown above.

Since high J hadronic states on leading Regge trajectories tend to be bilocal with large separation of their constituents, they fulfill all the conditions for supersymmetry between \bar{q} and D . Then the only difference between the energies of $(q\bar{q})$ mesons and (qD) baryons comes from the different masses of their constituents, namely $m_q = m_{\bar{q}} = m$, and $m_D \sim 2m$. For high J this is the main source of symmetry breaking which is spin independent. We will show how we can obtain sum rules from this breaking. The part of the mass operator that gives rise to this splitting is a diagonal element of $U(6/21)$ that commutes with $SU(6)$.

Let us now consider the spin dependent breaking of $U(6/21)$. For low J states the (qD) system becomes trilocal(qqq), the flux tube degenerates to a single gluon propagator that gives spin-dependent forces in addition to the Coulomb term $\frac{a}{r}$. In this case we have the regime studied by de Rujula, Georgi and Glashow, where the breaking is due to hyperfine splitting caused by the exchange of single gluons that have spin 1. These mass splittings give rise to different intercept of the Regge trajectories are given by

$$\Delta m_{12} = k \frac{\vec{S}_1 \cdot \vec{S}_2}{m_1 m_2}, \quad k = |\psi(0)|^2 \quad (80)$$

both for baryons and mesons at high energies. But at low energies the baryon becomes a trilocal object (with three quarks) and the mass splitting is given by

$$\Delta m_{123} = \frac{1}{2} k \left(\frac{\vec{S}_1 \cdot \vec{S}_2}{m_1 m_2} + \frac{\vec{S}_2 \cdot \vec{S}_3}{m_2 m_3} + \frac{\vec{S}_3 \cdot \vec{S}_1}{m_3 m_1} \right) \quad (81)$$

where m_1 , m_2 and m_3 are the masses of the three different quark constituents.

The element of $U(6/21)$ that give rise to such splittings is a diagonal element of its $U(21)$ subgroup and gives rise to $s(s+1)$ terms that behave like an element of the (405) representation of $SU(6)$ in the $SU(6)$ mass formulae. The splitting of isospin multiplets is due to a symmetry breaking element in the (35) representation of $SU(6)$. Hence all symmetry breaking terms are in the adjoint representation of $U(6/21)$. If we restrict ourselves to the non-strange sector of hadrons with approximate $SU(4)$ symmetry, effective supersymmetry will relate the splitting in m^2 between the Δ ($s = \frac{3}{2}, I = \frac{3}{2}$) and N ($s = \frac{1}{2}, I = \frac{1}{2}$) to the splitting between ω ($s = 1, I = 0$) and π ($s = 0, I = 1$) so that

$$m_\Delta^2 - m_N^2 \simeq m_\omega^2 - m_\pi^2 \quad (82)$$

which is satisfied to within 5%.

For classification of supergroups including $SU(m/n)$ we refer to a paper by Viktor Kac^[30].

5. Linear Mass Formulae

Based on the flavor $SU(3)$ and its breaking into its $SU(2) \times U(1)$ maximal subgroup of isospin and hypercharge, in 1962 the Gell-Mann-Okubo^{[31],[32]} mass formula illuminated the low lying hadronic spectrum. It led to the pseudoscalar mass formula. The mass formula for the vector mesons presented a more delicate problem since the isospin singlet members of the nine vector mesons, namely the physical ω and ϕ were mixtures of octet and singlet states, involving a mixing angle θ_V as a new parameter. A year later Okubo proposed^[33] a model for the determination of this mixing angle by requiring the nine vector mesons to fit into a 3×3 matrix. The group theoretic interpretation of ideal mixing followed

soon after^[34] with the enlargement of $SU(3)$ to $SU(3)_q \times SU(3)_{\bar{q}}$, one $SU(3)$ being associated with the quarks and the other $SU(3)$ with the antiquarks that are constituents of the vector mesons. Then the nonet corresponds to the representation $(3, 3)$ of this group. Since the u and d quarks are much lighter than the strange quark s , the $SU(2) \times SU(2)$ subgroup is not badly broken, so that we must decompose with respect to the subgroup $SU(2)_q \times U(1)_q \times SU(2)_{\bar{q}} \times U(1)_{\bar{q}}$ by using the (I, Y) labels for each $SU(2) \times U(1)$. They are shown in the table 1 below.

With respect to the diagonal $SU(3)$ subgroup the hypercharge Y is the sum of Y_q and $Y_{\bar{q}}$ while the isospin I is zero for ω and ϕ , one for ρ , and $\frac{1}{2}$ for K^* and \bar{K}^* . Now, Okubo's octet breaking hypothesis involves the octet-singlet mixture given by

$$K = I(I + 1) - \frac{Y^2}{4}. \tag{83}$$

For the nonet the energy breaking requires the combination

$$E = E_0 + a(K_q + K_{\bar{q}}) + bK. \tag{84}$$

We find the following assignments shown in the table below:

Particle	I_q	$I_{\bar{q}}$	I	Y_q	$Y_{\bar{q}}$	Y	K_q	$K_{\bar{q}}$	$K_q + K_{\bar{q}}$	K
ω	1/2	1/2	0	1/3	-1/3	0	13/18	13/18	13/9	0
ρ	1/2	1/2	1	1/3	-1/3	0	13/18	13/18	13/9	2
K^*	1/2	0	1/2	1/3	2/3	1	13/18	-1/9	11/18	1/2
\bar{K}^*	0	1/2	1/2	-2/3	-1/3	-1	-1/9	13/18	11/18	1/2
ϕ	0	0	0	-2/3	2/3	0	-1/9	-1/9	-2/9	0

Table 1. Particle assignments

Note that the sum $K_q + K_{\bar{q}}$ of the two octet breakings gives equal spacing for the energy levels and degeneracy for ω and ρ . It was shown in Okubo's paper that the rest energy breaking formula leads to a quadratic mass formula when the energy differences are large with respect to the mean energy as in the case of the pseudoscalar mesons and to a linear mass formula when the ratio of the energy splittings to the mean energy is small as in the case of baryons. The vector mesons being nearer in mass to the baryons than the pseudoscalar mesons we can use the linear mass formula as also suggested by the value of the mixing angle being nearer perfect mixing in this case. Then we get

$$m_\omega = \mu + \frac{13}{9}a, \tag{85}$$

$$m_\rho = \mu + \frac{13}{9}a + 2b, \tag{86}$$

$$m_{K^*} = m_{\bar{K}^*} = \mu + \frac{11}{18}a + \frac{1}{2}b, \tag{87}$$

$$m_\phi = \mu - \frac{2}{9}a, \tag{88}$$

leading to the mass sum rule:

$$m_\omega + m_\rho = 2(2m_{K^*} - m_\phi). \tag{89}$$

With the choice

$$\mu = 987.9 \text{ MeV}, \quad a = -142.1 \text{ MeV}, \quad b = -3.6 \text{ MeV} \tag{90}$$

we find the following masses as compared to experimental values given in Particle Data Table:^[35]

Particle	our result	experiment
m_ω	782.74	782.65
m_ρ	775.54	775.49
$m_{K^*} = m_{\bar{K}^*}$	899.36	891.66
m_ϕ	1020	1019.46

Table 2. Calculated masses vs. experiment

and the mass formula Eq.(89) gives deviations of less than one percent with above choices. The Eq.(89) which we have derived here using purely the $SU(3) \times SU(3)$ group theoretical assignments shown in Table1 is usually written as the squared mass formula

$$4m_K^2 = 2m_\phi^2 + m_\omega^2 + m_\rho^2 \tag{91}$$

for which the deviations are much larger than the linear formula.

It is also important to note that the quantum numbers of Table 1 which forbid the the decay of the ϕ into pions, or more generally of the $s\bar{s}$ system into systems involving u and d quarks is consistent with the OZI rule. This rule must be violated in QCD through gluonic intermediate states and yet it is surprisingly well verified, reinforcing the symmetry breaking chain that gives Eq.(84).

The diquark structure in the baryon will cause the form factors in lepton-baryon scattering to deviate from the model with three point-quarks in the bag. Donachie et al.^[36] and Fredrickson et al.^[37] have shown that the inclusion of both spin one and spin zero diquark states in the nucleon can explain some of the deviations from the quark-parton model without including higher order gluon corrections. According to our analysis such diquark states should occur in deep inelastic scattering only at high impact parameter when the nucleon is thrown into a rotationally excited state.

The exotic meson states should occur whether one starts from a string model^[38], a relativistic oscillator model^[39], or a bag model^[40] within the mass range 1.2 - 2 GeV. Their experimental absence is a real puzzle. However recently some possible evidence for the formation of $s = 2, l = 2$ meson resonances that would fit nicely in the (405) representation of $SU(6)$ has been found in the analysis of $\gamma\gamma \rightarrow \rho\rho$ reactions by Li and Liu^[41]. More experimental confirmation must be forthcoming before the issue is settled.

We have also derived the Hamiltonian of the relativistic quark model and have found new exact solutions^[42] using confluent hypergeometric functions. In the process we have built generating functions for the hypergeometric series from the point of view of solutions of a differential equation and have obtained quadratic mass formulae in a remarkable agreement with experiments.

At this point we would also like to mention that Iachello and his collaborators ^[43] have obtained a similar mass formulae based on algebraic methods for the quadratic case. Iachello proposed that starting from a spectrum generating algebra G , one can write a chain of subalgebras

$$G \supset G' \supset G'' \supset \dots \tag{92}$$

and that the Hamiltonian can be expanded in terms of invariants of the chain of subalgebras

$$H = \alpha C(G) + \beta C(G') + \gamma C(G'') + \dots \tag{93}$$

where $C(G)$ denotes one of the invariants of G . They have also^{[44],[45]} successfully applied their analysis into the study of triatomic molecules. Connections and comparisons between their formalism and ours will be exploited in another publication^[46].

Table 3. Flavor-Spin Content for three quark and quark-diquark baryons

Configuration	$SU_c(3)$	$SU_f(3)$	$SU_s(2)$
q	$\square \equiv \underline{3}$	$\square \equiv \underline{3}$	$S = 1/2$
$D \equiv q^2$	$\begin{matrix} \square \\ \square \end{matrix} \equiv \underline{\bar{3}}$	$\left\{ \begin{array}{l} \square\square \equiv \underline{6} \\ \square \equiv \underline{\bar{3}} \end{array} \right.$	$\left\{ \begin{array}{l} S = 0 \text{ not allowed} \\ S = 1 \end{array} \right.$ $\left\{ \begin{array}{l} S = 0 \\ S = 1 \text{ not allowed} \end{array} \right.$
q^3	$\begin{matrix} \square \\ \square \\ \square \end{matrix} \equiv \underline{1}$	$\left\{ \begin{array}{l} \square\square\square \equiv \underline{10} \\ \square\square \equiv \underline{8} \\ \square \end{array} \right.$	$S = 3/2$ $S = 1/2$
$q - D$	$\begin{matrix} \square \\ \square \\ \square \end{matrix} \equiv \underline{1}$	$\left\{ \begin{array}{l} \underline{3} \times \underline{\bar{3}} \\ \underline{3} \times \underline{6} \end{array} \right. \left\{ \begin{array}{l} \begin{matrix} \square\square \equiv \underline{8} \\ \square \end{matrix} \\ \begin{matrix} \square \\ \square \\ \square \end{matrix} \equiv \underline{1} \end{array} \right.$	$S = 1/2$ $S = 1/2$
		$\left\{ \begin{array}{l} \square\square\square \equiv \underline{10} \\ \square\square\square \equiv \underline{10} \\ \begin{matrix} \square\square \equiv \underline{8} \\ \square \end{matrix} \\ \begin{matrix} \square\square \equiv \underline{8} \\ \square \end{matrix} \end{array} \right.$	$S = 1/2$ $S = 3/2$ $S = 1/2$ $S = 3/2$

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