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How Much Should You Pay For a Financial Derivative?

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Introduction: Investing in Google Stock

- On December 3, 2015, a share of Google stock had a closing price of $768.20.
- You had the belief that Google price would go up to around $800 within a month.
- You had spent most of your money to buy expensive textbooks and you had only $500 left.
- You really wanted to buy Google stock but you could not afford to buy even a single share.
- What could you do?

Betting on Google Stock with Options

- If you believe the stock price would go up then you should buy *call options on the stock*.
- Options give you *leverage* since their cost is only a fraction of the stock price.
- A call option with a strike of $770 expiring on 12/31/15 had a price of $15 on the CBOE.
- You could have bought 33 options for $495, hoping the price of Google stock will go above $770 on 12/31/15.
- How was the call option price of $15 computed and why is that a fair price to pay?

Option Mechanics

If the stock price reaches $800, then you would exercise your call options, which give you the right to buy Google stock for the strike price of $770, instead of their market price of $800 on 12/31/15, and make a profit of $30 per option, or $990 in total, doubling your investment, but if not then your options would expire worthless and you would lose the $495 paid for option premiums, ending up with nothing.

Options and Their Uses

- Speculation
- Insurance
- Risk Management

We'll explain some key mathematical ideas behind the pricing of financial derivatives, starting from a simple coin toss model and ending with the classical *Black-Scholes-Merton* formula for pricing a European call option. We also explain where the $15 option price comes from, using actual market data.

The modern history of option pricing begins in 1900 with Louis Bachelier's Theory of Speculation and fully enters the modern world of finance in the early 1970s. In 1997 the *Nobel Prize in Economics* was awarded to Robert Merton and Myron Scholes. Fischer Black died in his mid-fifties in 1995.

A bit of History . . .

"In the 1970s, Merton tackled a problem that had been partially solved by two other economists, Black and Scholes: deriving a formula for the correct price of a stock option. Then he graciously waited to publish until after his peers did; thus the formula would ever be known as the Black-Scholes model."

"Few people would have cared given that no active market for options existed. But coincidentally, a month before the formula appeared, the Chicago Board Options Exchange had begun to list stock options for trading. Soon, Texas Instruments was advertising in The Wall Street Journal, *Now you can find the Black-Scholes value using our calculator.* This was the true beginning of the derivatives revolution. Never before had professors made such an impact on Wall Street."

When Genius Failed by Roger Lowenstein.

Financial Derivatives: Forwards

Financial derivatives derive their values from some underlying asset, which may be a stock, bond, index, currencies, and so on. Derivatives are traded on stock exchanges and over the counter markets. *Forwards* and *Options* are examples of derivatives.

Definition: A *long forward* is an *obligation to buy* the asset at time T for a price F, called the *forward price*. A *short forward* is an *obligation to sell* for F.

Financial Derivatives: Options

Two types of European vanilla options: *call and put options*.

Definition: *European call options* give the holder of the option *the right but not the obligation to buy*, while *European put options* give the holder of the option *the right but not the obligation to sell* the underlying asset at the strike price K, specified in the contract, when the option expires at a future time T.

American options allow you to exercise the option at any time up to expiry, but we consider only European options on stocks that pay no dividends, since they are mathematically easier.

Note that we would not exercise the option if it is not in our interest to do so, but we pay a price for the rights we get.

Arbitrage

We assume that markets are *liquid* and *efficient*, and do not allow for *arbitrage*.

Definition: *Arbitrage* is a trading strategy set up with a zero net value, with zero probability of losing money, and a positive probability of making money.

Only mathematical models that admit no arbitrage are used for financial modeling. Otherwise, profits could be created from nothing without taking any risk, leading to financial paradoxes.

Arbitrage opportunities do exist in the real world, and some hedge funds exist to discover and exploit them, but generally they "quickly" disappear as the markets correct themselves.

Long and Short Positions

Options have *asymmetric sides*. We distinguish between a long position and a short position in an option.

Definition: We enter a *long option position* when we buy an option. We enter a *short option position* when we sell an option.

It costs money to enter a long option position, but by selling an option we receive money (the option premium) from the long side of the contract. Unlike the long side, the short side does not have any rights but only the obligation to fulfill the contract in case the long side decides to exercise their rights. The short side gets compensated with the option price for this obligation.

Long Forward Payoff

In the case of a long forward contract, we are obligated to buy the asset for F no matter what the actual market price S_T is, thus we have a linear payoff

$$V_T = S_T - F$$

at expiry T. The fact that we have only obligations means that it costs nothing to enter a forward. Clearly, nobody wants to pay a premium for getting only obligations.

Long Call Option Payoff

Since a long call comes with rights, we would choose not to exercise the option if we would get a negative payoff. This optionality simply removes the negative part of the payoff.

We can buy the underlying asset at maturity T for the strike price K instead of the prevailing market price S_T, and the long call option payoff is $C_T = S_T - K$, provided the difference is positive, but if it is negative, we will choose not to exercise the option because it would not be in our interest to do so, and our payoff at maturity would be 0, thus the *long call option payoff* is:

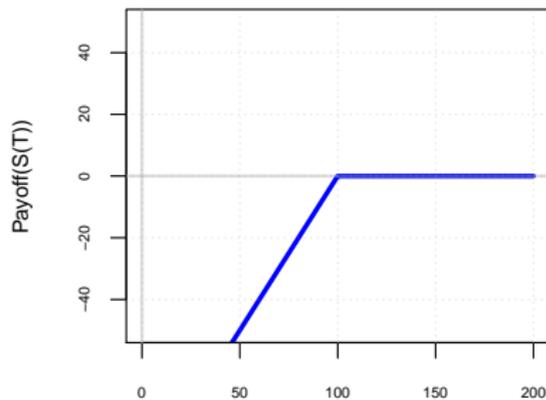$$C_T = (S_T - K)^+ = \max(0, S_T - K)$$

Payoff Diagrams

Here, we show the payoff diagrams at expiry for long call and short put options, with the same strike price $K = 100$, as a function of S_T, the asset price at expiry T.

Long Call Option Payoff
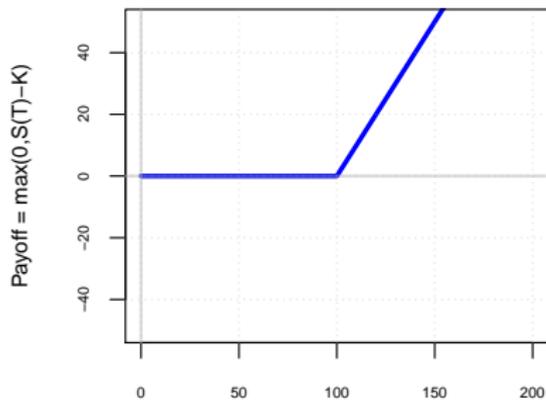
Short Put Option Payoff

Put-Call Parity

Finance Job Interview Question:
Express mathematically the payoff at expiry of a portfolio of a long call and a short put, written on the same asset, having the same expiry T and strike price K. Which derivative (already defined in this talk) has the same payoff at expiry?
Hint: use the payoff diagrams.
This observation is key in deriving the important **Put-Call Parity** (a common job interview question), which relates the prices of call and put options written on the same asset, with the same expiry and the same strike prices.

Put-Call Parity

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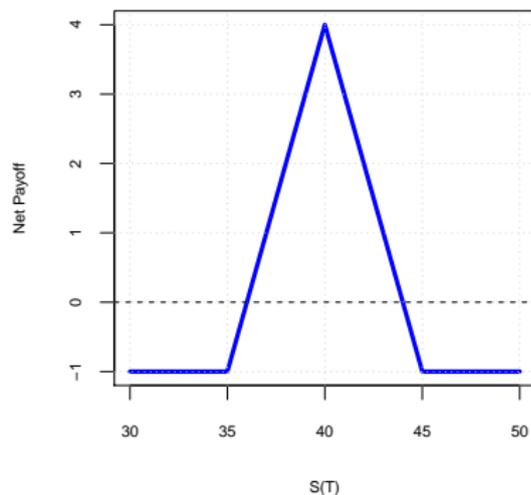
$$\boxed{C_0 - P_0 = S_0 - Ke^{-rT}}$$

Option Trading Strategies

- Bull Spread as a directional bet
- Butterfly Spread as a volatility bet

Bull Spread: Long20Call+Short40Call
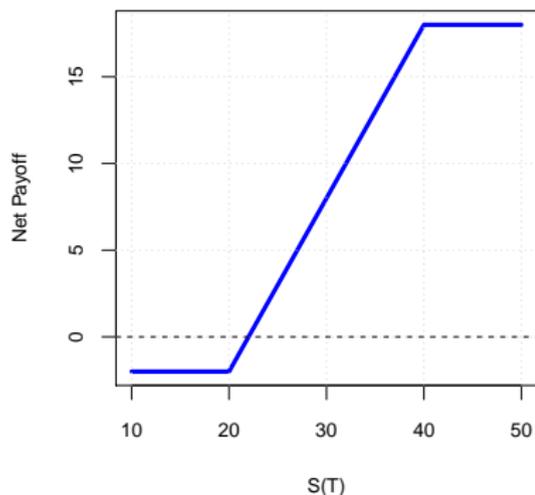
Butterfly Spread:Long45Call+Long35Put+Short40Call+Short40Put

Pricing Linear Derivative Payoffs

Any linear derivative payoff V_T at time T has the form:

$$V_T = a + bS_T$$

where S_T is the price of the asset at time T, *not known today*, and $a, b \in \mathbb{R}$ are numbers (with units) at time T, *known today*.

Problem: find the value of the derivative today $V_0 = F[V_T]$, where F is a *linear pricing operator* that satisfies the following conditions: $F[a] = ae^{-rT}$ (time value of \$) and $F[bS_T] = bS_0$:

$$V_0 = F[a + bS_T] = F[a] + F[bS_T] = ae^{-rT} + bS_0$$

where r is the *risk-free* annual interest rate with continuous compounding. This is one way to derive the Put-Call Parity.

Pricing Nonlinear Derivative Payoffs

A nonlinear derivative payoff at time T is some function f of S_T:

$$V_T = f(S_T)$$

Problem: find the value of the derivative today $V_0 = F[V_T]$. When viewed from today S_T is a random variable that has some distribution; to find V_0, we need a model for this asset price distribution, **not to predict the future but for the purpose of relative pricing** of derivatives with nonlinear payoffs.
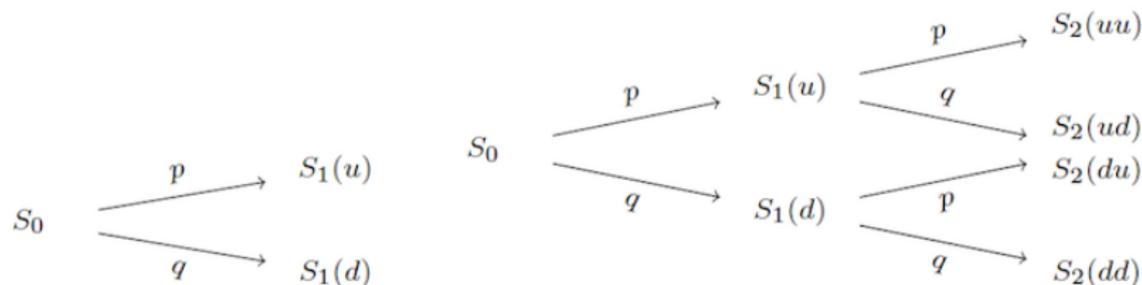
Coin-Toss Models for Relative Pricing

Figure 2. 1-step and 2-step coin-toss models for the asset price movements.

Imagine we toss a coin repeatedly and whenever we get heads the stock price moves up by the factor u, and whenever we get tails, the stock price moves down by the factor d, $0 < d < u$.

Pricing in a 1-Step Coin-Toss Model

We construct a static replicating portfolio, based on 1-step coin-toss model for the price of the asset, which consists of Δ units of the asset and b units of cash. The value of this portfolio today is given by V_0, and its value at expiry time $T = \Delta t$ is V_1:

$$V_0 = \Delta S_0 + b \rightarrow V_1 = \Delta S_1 + be^{r\Delta t}$$

A derivative on the asset with payoff at expiry $\Pi_1 = f(S_1)$ is a *Bernoulli* random variable:

$$\Pi_1 = f(S_1) = \begin{cases} \Pi_1(u) = f(S_1(u)), & p^* \\ \Pi_1(d) = f(S_1(d)), & q^* = 1 - p^* \end{cases}$$

Law of One Price and Replication

Any two financial securities having identical future payoffs, no matter how the future turns out, should have identical prices today. The law of one price is not a mathematical law but a reflection on human behavior for if the prices are not identical then one can profit without taking any risk and exploit an arbitrage by buying the cheaper security and selling the more expensive one.

The portfolio is *replicating* if the value of the portfolio at expiry T is the same as the payoff of the derivative:

$$\Pi_1 = V_1 = \Delta S_1 + be^{r\Delta t}$$

The Law of One Price

According to the law of one price, the price of the derivative Π_0 and the price of the replicating portfolio V_0 should be the same:

$$\Pi_0 = V_0 = \Delta S_0 + b$$

The asset price today S_0 is known, and we need to compute Δ and b to find the *fair, no-arbitrage price* of the derivative today.
The law of one price usually holds in liquid markets but there are always short- or even longer-term exceptions, and exploiting them forms the business model for some hedge funds and proprietary trading groups, whose very existence is a sign of market inefficiency. However, the markets self-correct such inefficiencies as more participants start exploiting them.

Replication in 1-Step Coin-Toss Model

$$\Pi_1 = \Delta S_1 + be^{r\Delta t}$$

Here, we can express the random variables as the vectors of possible values (cash has the same value in each state):

$$\begin{bmatrix} \Pi_1(u) \\ \Pi_1(d) \end{bmatrix} = \Delta \begin{bmatrix} S_1(u) \\ S_1(d) \end{bmatrix} + b \begin{bmatrix} e^{r\Delta t} \\ e^{r\Delta t} \end{bmatrix} = \begin{bmatrix} S_1(u) & e^{r\Delta t} \\ S_1(d) & e^{r\Delta t} \end{bmatrix} \begin{bmatrix} \Delta \\ b \end{bmatrix}$$

$$\begin{bmatrix} \Delta \\ b \end{bmatrix} = \begin{bmatrix} S_1(u) & e^{r\Delta t} \\ S_1(d) & e^{r\Delta t} \end{bmatrix}^{-1} \begin{bmatrix} \Pi_1(u) \\ \Pi_1(d) \end{bmatrix}$$

The Price of the Derivative

$$V_0 = \begin{bmatrix} S_0 & 1 \end{bmatrix} \begin{bmatrix} \Delta \\ b \end{bmatrix}$$

$$V_0 = \underbrace{\begin{bmatrix} S_0 & 1 \end{bmatrix} \begin{bmatrix} S_1(u) & e^{r\Delta t} \\ S_1(d) & e^{r\Delta t} \end{bmatrix}^{-1}}_{e^{-r\Delta t} \begin{bmatrix} p & q \end{bmatrix}} \begin{bmatrix} \Pi_1(u) \\ \Pi_1(d) \end{bmatrix}$$

$$V_0 = e^{-rT}(p\Pi_1(u) + q\Pi_1(d))$$

Probability Interpretation

$$e^{-r\Delta t}(pS_1(u) + qS_1(d)) = S_0, \quad p + q = 1$$

$$p = \frac{e^{r\Delta t} - d}{u - d}, \quad q = \frac{u - e^{r\Delta t}}{u - d}$$

We impose no-arbitrage conditions $d < e^{r\Delta t} < u$ that imply probability interpretation $0 < p, q < 1$ (later we set $\Delta t = T$). We can interpret p and q as probabilities, from some probability distribution $\mathcal{Q}$, for the two possible values of S_1.

$\mathcal{Q}$ is called *the risk-neutral probability distribution*.

The Risk-Neutral Interpretation

The numbers p and q, defining the risk-neutral distribution $\mathcal{Q}$, *represent a probabilistic interpretation of a linear algebra solution*, and not the actual physical probabilities of up and down moves. This observation implies that the actual physical probabilities p^* and q^* for the asset price distribution are irrelevant for computing the no-arbitrage price of any derivative on the asset.

The Risk-Neutral Distribution

The asset price today is the discounted expected value with respect to $\mathcal{Q}$ of asset price at expiry:

$$S_0 = e^{-rT}\mathbb{E}^{\mathcal{Q}}[S_1]$$

We can express the option price V_0 as the discounted, expected option payoff at expiry, with respect to $\mathcal{Q}$:

$$V_0 = e^{-rT}\mathbb{E}^{\mathcal{Q}}[V_1]$$

Multi-Step Coin-Toss Model

Imagine that we toss a coin repeatedly and whenever we get heads the stock price moves up by the factor u, and whenever we get tails, the price moves down by the factor d ($0 < d < u$).

- N time steps with a fixed time horizon T for the expiry of an option written on the asset.
- Divide the interval $[0, T]$ into N periods of size $\Delta t = T/N$.

We model the asset price evolution as a *discrete time stochastic process*, which is a collection of random variables indexed by time, $\{S_n\}_{n=0}^N$, where S_n is a random variable at time n, and S_0 is the asset price today.

Multiplicative Binomial Model

$$S_{n+1} = S_n X_{n+1}$$

$$X_{n+1} = \begin{cases} u, & \text{with prob. } p \\ d, & \text{with prob. } q = 1 - p \end{cases} \quad \text{is indep. of } S_n$$

Simulated Binomial Tree with 500 Paths
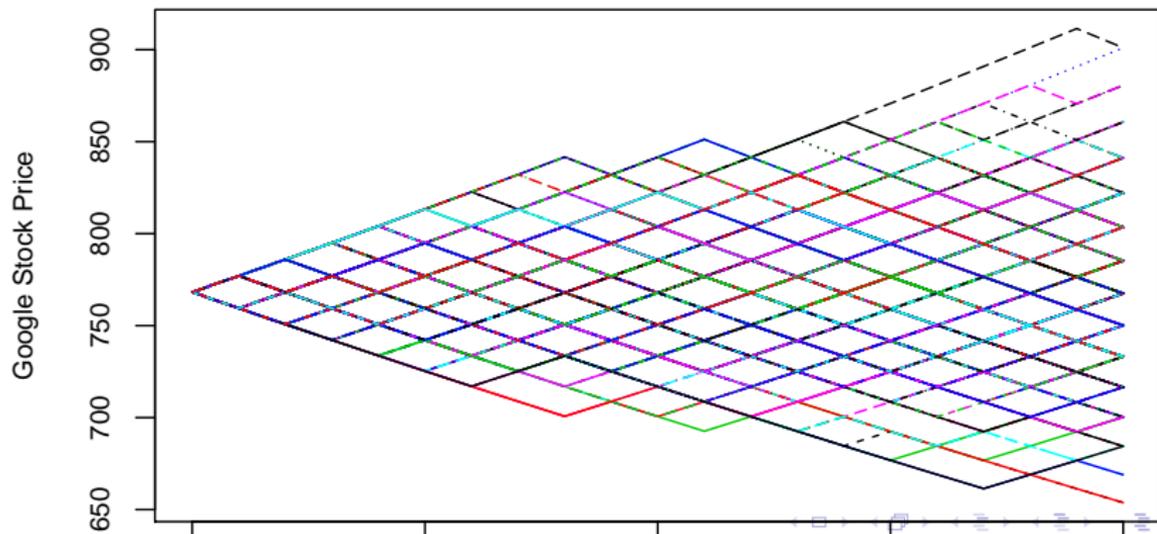
The Asset Value Martingale

The evolution of the asset price until time $n+1$ is governed by the vector $(X_1, \ldots, X_{n+1})$, which we can imagine to represent a sequence of $n+1$ independent coin tosses, resulting in the same up and down factors each time, with the same risk-neutral probabilities p and q, as in the one-step coin-toss model. The risk-neutral probabilities can be derived from the condition

$$e^{r\Delta t} = pu + qd = \mathbb{E}^{\mathcal{Q}}[X_1]$$

Discounted asset price process $\tilde{S}_n = e^{-rn\Delta t} S_n$ is a *martingale*:

$$\tilde{S}_n = \mathbb{E}_n^{\mathcal{Q}}[\tilde{S}_{n+1}]$$

for every n, where $\mathbb{E}_n^{\mathcal{Q}}$ is the expectation conditional on information available at time n.

The Option Value Martingale

Using a *dynamic, replicating and self-financing stochastic portfolio of asset and cash*, one can construct a stochastic process $\{V_n\}_{n=0}^{N}$ for the option value, such that $V_N = f(S_N)$ is the option payoff at expiry, where f is the option payoff function, and V_0 is the price of the option today.

The *key result* is that the discounted option value process $\tilde{V} = \{\tilde{V}_n\}_{n=0}^{N}$ is also a *martingale* under $\mathcal{Q}$. Martingales have constant expected value, so we can express the option price today as the discounted expected payoff at expiry:

$$V_0 = e^{-rT}\mathbb{E}^{\mathcal{Q}}[V_N]$$

A Geometric Random Walk Model

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}$$

where σ, known as *volatility*, is the standard deviation of the annual rate of log-return on the asset. The asset price dynamics is then given by the *geometric random walk*:

$$S_n = S_0 e^{\sigma\sqrt{\Delta t}\sum_{k=1}^{n} Y_k}, \ Y_k = \begin{cases} +1, & \text{with prob. } p \\ -1, & \text{with prob. } q = 1 - p \end{cases}$$
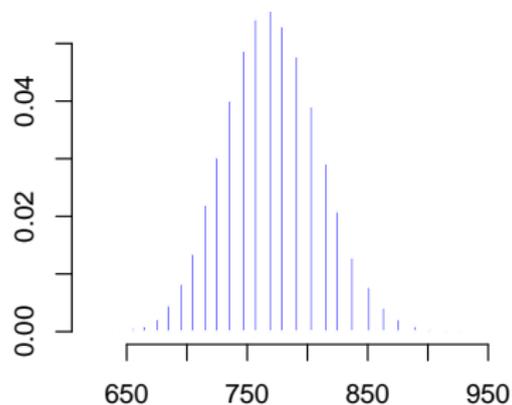
where $\{Y_k\}_{k=1}^{n}$ represent the independent coin tosses.

$$\mathbb{E}[Y_k] = 2p - 1, \quad \text{Var}(Y_k) = 1 - (2p - 1)^2$$

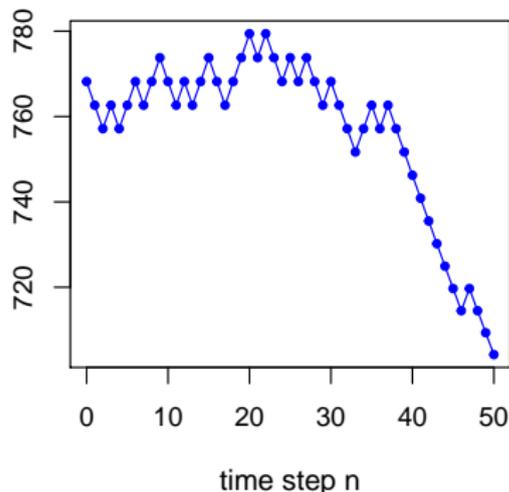# Random Walk of Google Stock Price

Simulated Google price distribution after 50 time steps, and one realization of the Google stock price path over 50 time steps.

Density Histogram of Google Stock Price

Google stock price after 50 time steps

A Random Walk of Google Stock Price

time step n

The Continuous Time Limit

The Taylor series expansion of $p(\Delta t)$:

$$p = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right) + O(\Delta t^{3/2})$$

where $\mu = r - \sigma^2/2$. Let $S_N = S_0 e^{Z_N}$ where:

$$Z_N = \sigma\sqrt{\Delta t}\sum_{k=1}^{N} Y_k$$

$$\mathbb{E}[Z_N] = (r - \sigma^2/2)T + O(T/N)$$

$$\mathrm{Var}(Z_N) = \sigma^2 T + O(T/N)$$

The *Central Limit Theorem* gives us the continuous time limits:

$$\lim_{N \to \infty} Z_N = Z_T \sim N((r - \sigma^2/2)T, \sigma^2 T)$$

$$\lim_{N \to \infty} S_N = S_T = S_0 e^{Z_T} = S_0 e^{(r - \sigma^2/2)T + \sigma\sqrt{T}Z}$$

where Z is the standard normal $Z \sim N(0, 1)$. The asset price S_T has a *continuous risk-neutral distribution*, which is *log-normal*, being the exponent of a normal distribution:

$$\log(S_T) \sim N(\log(S_0) + (r - \sigma^2/2)T, \sigma^2 T)$$

Check that $S_0 = e^{-rT}\mathbb{E}[S_T]$, using that $\mathbb{E}[e^X] = e^{\mathbb{E}[X] + \text{Var}(X)^2/2}$, for X normal. If we let T vary, we can construct this way a *geometric Brownian motion*.

The Pricing Formula

In the continuous-time limit, the discrete martingales $\tilde{S}$ and $\tilde{V}$ transform into continuous martingales, but to understand what happens, one needs *Stochastic Calculus*, which is powered by the differential of Brownian motion $dW \sim N(0, dt)$, for which $dW^2 = dt$, and using Taylor's expansions up to 2nd order to express the differential of $f(W, t)$ in terms of dt and dW.

In continuous time, we have the key pricing formula:

$$\boxed{V_0 = e^{-rT} \mathbb{E}^{\mathcal{Q}}[f(S_T)]}$$

$$S_T = S_0 e^{(r - \sigma^2/2)T + \sigma\sqrt{T}Z}$$

where f is the option payoff function and $Z \sim N(0, 1)$.

Simulated Google Stock Price Evolution

Left plot: 10 simulated geometric random walks with $N = 500$ time
steps over $T = 28$ days. Right plot: the density function of the
log-normal distribution of S_T in dark blue, superimposed over the
density histogram of a large sample from S_N.

10 Geometric Random Walks of Google Stock Price
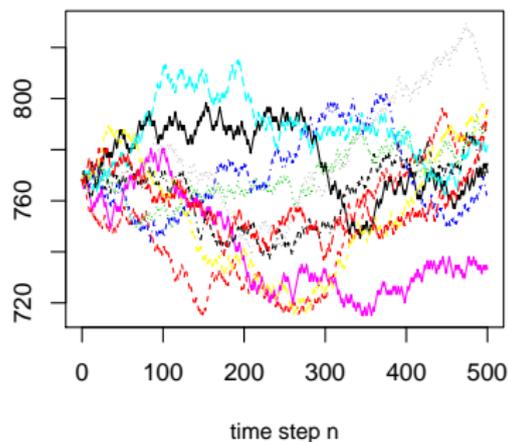
time step n

Stock Price Distribution from Geometric Random Walks
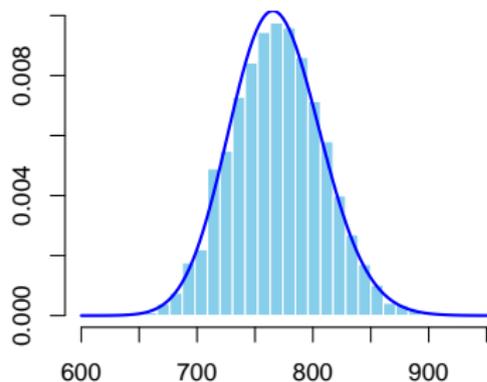
Google stock price after 28 days with 500 time steps

The Black-Scholes-Merton Formula

Consider now a European *call option* written on stock paying no dividends. The fair option price today, V_0, is given as the discounted expected value of the call option payoff at expiry $V_T = (S_T - K)^+$, where S_T is modeled by the log-normal r.v. $S_T = S_0 e^{(r - \sigma^2/2)T + \sigma\sqrt{T}Z}$ and the derivative price is given by:

$$V_0 = e^{-rT}\mathbb{E}[(S_T - K)^+]$$

We can compute the expected value by conditioning on the event $S_T > K$:

$$V_0 = e^{-rT}\mathbb{E}[(S_T - K)^+ | S_T > K]\mathbb{P}(S_T > K)$$

$$V_0 = e^{-rT}\mathbb{E}[S_T | S_T > K]\mathbb{P}(S_T > K) - Ke^{-rT}\mathbb{P}(S_T > K)$$

The Black-Scholes-Merton Formula

$$\mathbb{P}(S_T > K) = F(d_2)$$

where $F(x) = \mathbb{P}(Z \leq x)$ is the standard normal cdf and
$d_2 = \frac{1}{\sigma\sqrt{T}}(\log(S_0/K) + (r - \sigma^2/2)T)$.

$$\mathbb{E}[S_T | S_T > K] = S_0 e^{rT} \frac{F(d_1)}{F(d_2)}$$

where $d_1 = d_2 + \sigma\sqrt{T}$.

The celebrated *Black-Scholes-Merton* formula for the fair price of a European call option written on stock paying no dividends:

$$\boxed{V_0 = S_0 F(d_1) - K e^{-rT} F(d_2)}$$

The Price of the Google Call Option

Google stock market data from 12/03/2015:

$$S_0 = \$768.20, K = \$770, T = 28/365 \text{ years}$$

$$r = 0.5\%, \sigma = 18.5\%$$

The BSM formula gives us the call price on Google stock:

$$V_0 = \$14.98$$

at par with the price of $15 listed on the CBOE.

Conclusions

We emphasize that the BSM formula gives the "fair" option price only relative to our log-normal model but the "fair" price will change if we choose a different model for S_T.

The BSM model assumes a normal distribution for the log-returns on the asset price. However, practitioners have long realized that the BSM model does not quite reflect the observed *fat-tailed* distributions for the log-returns on asset prices.

Buying options costs only a fraction of the assets' prices, thus providing *big leverage for risk management, insurance, speculation* etc., and that is why options and other financial derivatives are so ubiquitous in finance.

Useful References

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