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Nataniel Greene

CUNY Kingsborough Community College

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Fourier Series of Orthogonal Polynomials

NATANIEL GREENE

Department of Mathematics and Computer Science
 Kingsborough Community College, CUNY
 2001 Oriental Boulevard, Brooklyn, NY 11235
 UNITED STATES

Abstract: Explicit formulas for the Fourier coefficients of the Legendre polynomials can be found in the Bateman Manuscript Project. However, similar formulas for more general classes of orthogonal polynomials do not appear to have been worked out. Here we derive explicit formulas for the Fourier series of Gegenbauer, Jacobi, Laguerre and Hermite polynomials.

Key-Words: Fourier series, orthogonal polynomials, Gegenbauer polynomials, Jacobi polynomials, Laguerre polynomials, Hermite polynomials.

1 Introduction

Explicit formulas for the Fourier coefficients of Legendre polynomials can be found in the Bateman Manuscript Project [3, 4, 5, 6]. For Legendre polynomials we find:

$$P_n(x) = \sum_{k=-\infty}^{\infty} a_k^{n, \frac{1}{2}} e^{ik\pi x} \quad (1)$$

where

$$\begin{aligned} a_k^{n, \frac{1}{2}} &= \frac{1}{2} \int_{-1}^1 P_n(x) e^{-ik\pi x} dx \\ &= \begin{cases} \frac{(-i)^n}{\sqrt{2k}} J_{n+\frac{1}{2}}(\pi k), & k \neq 0 \\ \delta_{0,n} & k = 0 \end{cases} \end{aligned}$$

It follows from Bateman [4] page 213 after setting $\lambda = \frac{1}{2}$. It can also be found with slight modification in Bateman [5] page 122.

However we are not aware of any reference where explicit formulas for the Fourier coefficients for Gegenbauer, Jacobi, Laguerre and Hermite polynomials can be found. In this article we use known formulas for the connection coefficients relating an arbitrary orthogonal polynomial to the Legendre polynomials to derive explicit formulas. Although we detail the formulas for the classical orthogonal polynomials, the method can be used to write explicit Fourier coefficients for any class of polynomials.

The formulas were developed by this author, Greene [11], in studying the Gegenbauer reconstruction method of Gottlieb and Shu [7, 8, 9, 10], which

is a technique for overcoming the spurious oscillations known as the Gibbs phenomenon which occur in Fourier and orthogonal polynomial approximations to piecewise smooth functions.

2 Gegenbauer and Jacobi Polynomials

The general connection formula relating a Gegenbauer- μ polynomial to a Gegenbauer- λ polynomial is found in Askey [2] page 77:

$$C_n^\mu(x) = \sum_{j=0}^{[n/2]} c_\lambda^\mu(j, n) C_{n-2j}^\lambda(x) \quad (2)$$

where

$$c_\lambda^\mu(j, n) = \frac{(\mu - \lambda)_j (\mu)_{n-j} (\lambda + 1)_{n-2j}}{j! (\lambda + 1)_{n-j} (\lambda)_{n-2j}}$$

The more general formula relating a Jacobi- (σ, τ) polynomial to a Jacobi- (α, β) polynomial is given by

$$P_n^{\sigma, \tau}(x) = \sum_{k=0}^n c_{\alpha, \beta}^{\sigma, \tau}(k, n) P_k^{\alpha, \beta}(x) \quad (3)$$

where

$$\begin{aligned} c_{\alpha, \beta}^{\sigma, \tau}(k, n) &= \frac{\Gamma(k + \alpha + \beta + 1) \Gamma(n + k + \sigma + \tau + 1)}{\Gamma(n + \sigma + \tau + 1) \Gamma(k + \sigma + 1)} \\ &\quad \times \frac{\Gamma(n + \sigma + 1)}{\Gamma(2k + \alpha + \beta + 1) (n - k)!} \end{aligned}$$

$$\times {}_3F_2 \left(\begin{matrix} -n+k, n+k+\sigma+\tau+1, k+\alpha+1 \\ k+\sigma+1, 2k+\alpha+\beta+2 \end{matrix} \middle| 1 \right)$$

Two special cases are Askey [2] page 63

$$P_n^{\alpha,\tau}(x) = \sum_{k=0}^n c_{\alpha,\beta}^{\alpha,\tau}(k,n) P_k^{\alpha,\beta}(x) \quad (4)$$

where

$$c_{\alpha,\beta}^{\alpha,\tau}(k,n) = \frac{(\alpha+1)_n (-1)^{n-k} (\tau-\beta)_{n-k}}{(\alpha+\beta+2)_n (1)_{n-k}} \times \frac{(\alpha+\beta+1)_k (n+\alpha+\tau+1)_k}{(\alpha+1)_k (n+\alpha+\beta+2)_k} \times \frac{(\alpha+\beta+2)_{2k}}{(\alpha+\beta+1)_{2k}}$$

and

$$P_n^{\sigma,\beta}(x) = \sum_{k=0}^n c_{\alpha,\beta}^{\sigma,\beta}(k,n) P_k^{\alpha,\beta}(x) \quad (5)$$

where

$$c_{\alpha,\beta}^{\sigma,\beta}(k,n) = \frac{(\beta+1)_n (\sigma-\alpha)_{n-k} (\alpha+\beta+1)_k}{(\alpha+\beta+2)_n (1)_{n-k} (\beta+1)_k} \times \frac{(\alpha+\beta+2)_{2k} (n+\sigma+\beta+1)_k}{(\alpha+\beta+1)_{2k} (n+\alpha+\beta+2)_k}$$

The notation for the connection coefficients is our own and is meant to suggest cancellation of indices. We note that these two specific cases of connection coefficients can be combined to reproduce the general case using the relation

$$c_{\alpha,\beta}^{\sigma,\tau}(k,n) = \sum_{j=0}^{n-k} c_{\sigma,\beta}^{\sigma,\tau}(k+j,n) c_{\alpha,\beta}^{\sigma,\beta}(k,k+j). \quad (6)$$

Theorem 1 *The Fourier series of the Gegenbauer- μ polynomials is given by:*

$$C_n^\mu(x) = \sum_{k=-\infty}^{\infty} a_k^{n,\mu} e^{ik\pi x} \quad (7)$$

where the Fourier coefficients

$$a_k^{n,\mu} = \frac{1}{2} \int_{-1}^1 C_n^\mu(x) e^{-ik\pi x} dx \quad (8)$$

are given for $k \neq 0$ by

$$a_k^{n,\mu} = \sum_{j=0}^{[n/2]} \frac{(-i)^{n-2j}}{\sqrt{2k}} J_{n-2j+\frac{1}{2}}(\pi k) \times \frac{(\mu-\frac{1}{2})_j (\mu)_{n-j} (\frac{3}{2})_{n-2j}}{j! (\frac{3}{2})_{n-j} (\frac{1}{2})_{n-2j}} \quad (9)$$

and for $k = 0$ by

$$a_0^{n,\mu} = \begin{cases} \frac{(\mu-\frac{1}{2})_{n/2} (\mu)_{n/2}}{j! (\frac{3}{2})_{n/2}}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases} \quad (10)$$

Proof. Begin with the first lemma

$$P_n(x) = \sum_{k=-\infty}^{\infty} a_k^{n,\frac{1}{2}} e^{ik\pi x}$$

and write

$$P_{n-2j}(x) = \sum_{k=-\infty}^{\infty} a_k^{n-2j,\frac{1}{2}} e^{ik\pi x}.$$

Now multiply both sides by the connection coefficients for $\lambda = \frac{1}{2}$ and sum

$$\sum_{j=0}^{[n/2]} c_{\frac{1}{2}}^\mu(j,n) P_{n-2j}(x) = \sum_{j=0}^{[n/2]} c_{\frac{1}{2}}^\mu(j,n) \sum_{k=-\infty}^{\infty} a_k^{n-2j,\frac{1}{2}} e^{ik\pi x}$$

Use the second lemma for the left hand side and interchange the order of summation for the right hand side to get our result.

$$C_n^\mu(x) = \sum_{k=-\infty}^{\infty} \left(\sum_{j=0}^{[n/2]} c_{\frac{1}{2}}^\mu(j,n) a_k^{n-2j,\frac{1}{2}} \right) e^{ik\pi x}$$

and

$$a_k^{n,\mu} = \sum_{j=0}^{[n/2]} c_{\frac{1}{2}}^\mu(j,n) a_k^{n-2j,\frac{1}{2}}.$$

We work out the case $k = 0$ explicitly:

$$a_0^{n,\mu} = \sum_{j=0}^{[n/2]} \delta_{0,n-2j} \frac{(\mu-\frac{1}{2})_j (\mu)_{n-j} (\frac{3}{2})_{n-2j}}{j! (\frac{3}{2})_{n-j} (\frac{1}{2})_{n-2j}} = \begin{cases} \frac{(\mu-\frac{1}{2})_{n/2} (\mu)_{n/2}}{j! (\frac{3}{2})_{n/2}}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

■

The formula for the Fourier series of Gegenbauer polynomials is used in [12].

Theorem 2 *The Fourier series of the Jacobi polynomial is given by:*

$$P_n^{\sigma,\tau}(x) = \sum_{k=-\infty}^{\infty} a_k^{n,\sigma,\tau} e^{ik\pi x} \quad (11)$$

where

$$a_k^{n,\sigma,\tau} = \frac{1}{2} \int_{-1}^1 P_n^{\sigma,\tau}(x) e^{-ik\pi x} dx \quad (12)$$

$$= \sum_{j=0}^n c_{0,0}^{\sigma,\tau}(j,n) \frac{(-i)^{n-2j}}{\sqrt{2k}} J_{n+\frac{1}{2}}(\pi k).$$

Proof. Begin with the formula for the Fourier series of a Legendre polynomial

$$P_n(x) = \sum_{k=-\infty}^{\infty} a_k^{n,\frac{1}{2}} e^{ik\pi x} \quad (13)$$

and write

$$P_{n-2j}(x) = \sum_{k=-\infty}^{\infty} a_k^{n-2j,\frac{1}{2}} e^{ik\pi x}. \quad (14)$$

Now multiply both sides by the connection coefficients for $\lambda = \frac{1}{2}$ and sum

$$\sum_{j=0}^n c_{0,0}^{\sigma,\tau}(j,n) P_{n-2j}(x)$$

$$= \sum_{j=0}^n c_{0,0}^{\sigma,\tau}(j,n) \sum_{k=-\infty}^{\infty} a_k^{n-2j,\frac{1}{2}} e^{ik\pi x} \quad (15)$$

Use the second lemma for the left hand side and interchange the order of summation for the right hand side to get our result.

$$P_n^{\sigma,\tau}(x) = \sum_{k=-\infty}^{\infty} \left(\sum_{j=0}^n c_{0,0}^{\sigma,\tau}(j,n) a_k^{n-2j,\frac{1}{2}} \right) e^{ik\pi x}. \quad (16)$$

■

3 Hermite Polynomials

Unlike Gegenbauer and Jacobi polynomials which share the orthogonality interval of $[-1, 1]$ with the complex exponentials $e^{ik\pi x}$, Hermite polynomials are orthogonal over $(-\infty, \infty)$ and Laguerre polynomials are orthogonal over the interval $[0, \infty)$. We offer here formulas for expressing a Hermite or Laguerre polynomial over a finite subinterval $[a, b]$ as a Fourier series. The choice of subinterval can depend on the application in question, however for Hermite and Laguerre polynomials we mention an interval which may be of particular interest for approximation purposes called the oscillatory region.

The following lemma found in Spanier and Oldham [14] page 217, describes the oscillatory region for Hermite polynomials.

Lemma 3 The n zeros, $\lfloor \frac{n}{2} \rfloor$ minima, and $\lceil \frac{n}{2} \rceil$ maxima of $H_n(x)$, for $n > 1$, are all located in the interval $[-\sqrt{2n}, \sqrt{2n}]$. Outside of this interval $|H_n(x)|$ increases monotonically, bounded globally by

$$|H_n(x)| < 1.09 \sqrt{2^n n! e^{x^2}} \quad (17)$$

Since the intervals described above are nested as n increases, we see that the resolving power of an $N + 1$ term Hermite partial sum lies within the oscillatory region $[-\sqrt{2N}, \sqrt{2N}]$.

Lemma 4 The following connection relation holds (Rainville [13] page 196):

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} c(k,n) P_{n-2k}(x) \quad (18)$$

where

$$c(k,n) = \frac{(-1)^k n! {}_1F_1(-k; \frac{3}{2} + n - 2k; 1) (2n - 4k + 1)}{k! (\frac{3}{2})_{n-2k}} \quad (19)$$

Lemma 5 Let

$$\varepsilon = \frac{b-a}{2}$$

and

$$\delta = \frac{b+a}{2}$$

so that $\xi \in [-1, 1]$ when $x \in [a, b]$. Then the following formula holds:

$$P_n(\varepsilon\xi + \delta) = \sum_{k=-\infty}^{\infty} a_k^{n,\frac{1}{2};\varepsilon,\delta} e^{ik\pi\xi} \quad (20)$$

where

$$a_k^{n,\frac{1}{2};\varepsilon,\delta} = \frac{1}{2} \int_{-1}^1 P_n(\varepsilon\xi + \delta) e^{-ik\pi\xi} d\xi \quad (21)$$

$$= e^{\pi i \delta \xi / \varepsilon} \frac{(-i)^n}{\sqrt{2k\varepsilon}} J_{n+\frac{1}{2}}\left(\pi \frac{k}{\varepsilon}\right).$$

Proof. Write the Fourier coefficient

$$a_k^{n,\frac{1}{2};\varepsilon,\delta} := \frac{1}{2} \int_{-1}^1 P_n(\varepsilon\xi + \delta) e^{-ik\pi\xi} d\xi \quad (22)$$

as the Fourier transform

$$\frac{1}{2} \int_{-\infty}^{\infty} \chi_{[-1,1]}(\xi) P_n(\varepsilon\xi + \delta) e^{-ik\pi\xi} d\xi. \quad (23)$$

and utilize the property

$$\phi(\widehat{\varepsilon\xi + \delta}) = \frac{1}{|\varepsilon|} e^{\pi i \delta k / \varepsilon} \widehat{\phi}\left(\frac{k}{\varepsilon}\right) \quad (24)$$

to obtain the result. ■

Theorem 6 *The Fourier expansion of the Hermite polynomial over the subinterval $[a, b]$ is given by:*

$$H_n(\varepsilon\xi + \delta) = \sum_{k=-\infty}^{\infty} a_k^{\varepsilon, \delta}(k, n) e^{ik\pi\xi} \quad (25)$$

where

$$\begin{aligned} a_k^{\varepsilon, \delta}(k, n) &= \frac{1}{2} \int_{-1}^1 H_n(\varepsilon\xi + \delta) e^{-ik\pi\xi} d\xi \quad (26) \\ &= \sum_{j=0}^{[n/2]} c(j, n) a_k^{n-2j, \frac{1}{2}; \varepsilon, \delta} \end{aligned}$$

and the terms in the summation are defined as in the previous lemmas.

Proof. Begin with the first lemma

$$P_n(\varepsilon\xi + \delta) = \sum_{k=-\infty}^{\infty} a_k^{n, \frac{1}{2}; \varepsilon, \delta} e^{ik\pi\xi} \quad (27)$$

and write

$$\begin{aligned} &\sum_{j=0}^{[n/2]} c(j, n) P_{n-2j}(\varepsilon\xi + \delta) \\ &= \sum_{j=0}^{[n/2]} c(j, n) \sum_{k=-\infty}^{\infty} a_k^{n-2j, \frac{1}{2}; \varepsilon, \delta} e^{ik\pi\xi} \quad (28) \end{aligned}$$

Use the second lemma for the left hand side and interchange the order of summation for the right hand side to get our result

$$H_n(\varepsilon\xi + \delta) = \sum_{k=-\infty}^{\infty} \left(\sum_{j=0}^{[n/2]} c(j, n) a_k^{n-2j, \frac{1}{2}; \varepsilon, \delta} \right) e^{ik\pi\xi} \quad (29)$$

■

4 Laguerre Polynomials

4.1 The Oscillatory Region of a Laguerre Series

The following lemma describes the oscillatory region for Laguerre polynomials.

Lemma 7 *The n zeros, $[\frac{n}{2}]$ minima, and $[\frac{n}{2}]$ maxima of $L_n^\alpha(x)$, $n > 1$, where $\alpha > -1$, are all located in the interval $[0, \beta]$ where*

$$\begin{aligned} \beta &= 2n + \alpha + 1 + \sqrt{(2n + \alpha + 1)^2 + \frac{1}{4} - \alpha^2} \\ &\sim 4n + 2\alpha + 2 \end{aligned}$$

Outside of this interval $|L_n^\alpha(x)|$ increases monotonically, bounded globally by

$$|L_n^\alpha(x)| < \frac{(\alpha + 1)_n}{(1)_n} e^{x/2} \quad (30)$$

when $\alpha \geq 0$ and

$$|L_n^\alpha(x)| < \left(2 - \frac{(\alpha + 1)_n}{(1)_n} \right) e^{x/2} \quad (31)$$

when $-1 < \alpha < 0$.

Remark 8 *See Spanier and Oldham [14] page 209 for $\alpha = 0$ and Szego page [15] for the value of β when $\alpha \neq 0$ and Abramowitz and Stegun [1] page 786 for the bounds on $L_n^\alpha(x)$.*

Since the intervals described above are nested as n increases, we see that the resolving power of the Laguerre partial sum $S_N^\alpha f(x)$ lies within the oscillatory region $[0, \beta]$. This suggests that the subinterval $[0, \beta]$ may be a useful one to consider for approximation purposes.

Lemma 9 *The following connection formula holds (Rainville [13] page 216):*

$$L_n^\alpha(\xi) = \sum_{k=0}^n c^\alpha(k, n) P_k(\xi) \quad (32)$$

where

$$c^\alpha(k, n) = \frac{(-1)^k (1 + \alpha)_n (2k + 1)}{2^k (n - k)! \left(\frac{3}{2}\right)_n (1 + \alpha)_k}$$

$$\times {}_2F_3 \left(\begin{matrix} -\frac{1}{2}(n - k), -\frac{1}{2}(n - k - 1) \\ \frac{3}{2} + k, \frac{1}{2}(1 + \alpha + k), \frac{1}{2}(2 + \alpha + k) \end{matrix} \middle| \frac{1}{4} \right)$$

Theorem 10 *The Fourier expansion of the Laguerre polynomial over the subinterval $[a, b]$ is given by:*

$$L_n^\alpha(\varepsilon\xi + \delta) = \sum_{k=-\infty}^{\infty} a_k^{\varepsilon, \delta} e^{ik\pi\xi} \quad (33)$$

where

$$\begin{aligned} a_k^{\varepsilon, \delta} &= \frac{1}{2} \int_{-1}^1 L_n^\alpha(\varepsilon\xi + \delta) e^{-ik\pi\xi} d\xi \quad (34) \\ &= \sum_{j=0}^{[n/2]} c(j, n) a_k^{n-2j, \frac{1}{2}; \varepsilon, \delta} \end{aligned}$$

5 Orthogonal Polynomial Partial Sums and Concluding Remarks

Once formulas for the Fourier coefficients are known, the explicit Fourier series of the corresponding orthogonal polynomial partial sum can be written as in the following theorem.

Theorem 11 *Let*

$$\varepsilon = \frac{b - a}{2}$$

and

$$\delta = \frac{b + a}{2}$$

so that $\xi \in [-1, 1]$ when $x \in [a, b]$. Let $\{\phi_n(x)\}$ denote any class of orthogonal polynomials and let

$$S_N f(\varepsilon\xi + \delta) = \sum_{n=0}^N \widehat{f}^\phi(n) \phi(\varepsilon\xi + \delta) \quad (35)$$

be the partial sum of any arbitrary function f over the subinterval $[a, b]$. Then $S_N f(\varepsilon\xi + \delta)$ has the following Fourier series:

$$S_{M,N} f(\varepsilon\xi + \delta) = \sum_{k=-M}^M b_k^{\varepsilon,\delta} e^{ik\pi\xi} \quad (36)$$

where

$$b_k^{\varepsilon,\delta} = \sum_{n=0}^N \widehat{f}^\phi(n) a_\phi^{\varepsilon,\delta}(k, n) \quad (37)$$

and $a_\phi^{\varepsilon,\delta}(k, n)$ is the k th Fourier coefficient of the n th orthogonal polynomial ϕ_n over the subinterval $[a, b]$.

The technique of connection coefficients can be applied to obtain explicit Fourier coefficients for any class of polynomials. Some of the formulas described here were found to be useful in our own work and we hope that others may find the technique and the formulas themselves to be useful as well.

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