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Matthew Alpert
Harvard University

Jens-P. Bode
Technische Universitat Braunschweig

Elie Feder
CUNY Kingsborough Community College

Heiko Harborth
Technische Universitat Braunschweig

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The Minimum of the Maximum Rectilinear Crossing Numbers of Small Cubic Graphs

Matthew Alpert[1], Jens-P. Bode[2], Elie Feder[3], Heiko Harborth[2]

[1]Harvard University, Cambridge, MA, USA. mna851@aol.com

[2]Diskrete Mathematik, Technische Universität, Braunschweig, Germany. jp.bode@tu-bs.de, h.harborth@tu-bs.de

[3]Department of Mathematics and Computer Science, Kingsborough Community College-CUNY, Brooklyn, NY, USA. efeder@kbcc.cuny.edu

ABSTRACT. Here we consider the minimum of the maximum rectilinear crossing numbers for all d-regular graphs of order n. The case of connected graphs only is investigated also. For $d = 3$ exact values are determined for $n \leq 12$ and some estimations are given in general.

1. Introduction

A *drawing* of the graph G with vertex set $V(G)$ and edge set $E(G)$ is defined as a representation of G in a plane such that the elements of $V(G)$ correspond to points in the plane and the elements of $E(G)$ correspond to continuous arcs. A *crossing* is defined to be the intersection of exactly two edges not at a vertex. We assume that each arc connects two vertices and that any pair of arcs has at most one point in common, either a vertexpoint or a crossing. A *rectilinear drawing* is a drawing of a graph in which all edges are represented as straight line segments in the plane. The *crossing number* of a graph G, denoted $\mathrm{cr}(G)$, is defined as the minimum number of crossings over all drawings of G. The *rectilinear crossing number* of a graph G, denoted $\overline{\mathrm{cr}}(G)$, is defined as the minimum number of crossings over all rectilinear drawings of G. Analogously, the *maximum crossing number*, denoted by $\mathrm{CR}(G)$, is defined as the maximum number of crossings over all drawings of G. The *maximum rectilinear crossing number* of a graph G, $\overline{\mathrm{CR}}(G)$, is defined as the maximum number of crossings over all rectilinear drawings of G.

The maximum crossing number and maximum rectilinear crossing number have been studied for several classes of graphs (see [3–10, 12]). Relevant to this paper are studies of the maximum rectilinear crossing number of C_n and of the Petersen Graph P. It has been shown in [3] and [5] that $\overline{\mathrm{CR}}(P) = 49$ and

$$\overline{\mathrm{CR}}(C_n) = \begin{cases} \frac{1}{2}n(n-3) & \text{if } n \text{ is odd,} \\ \frac{1}{2}n(n-4)+1 & \text{if } n \text{ is even.} \end{cases}$$

As a generalization, it has been considered the class $R_{n,d}$ of all d-regular graphs of order n. Let $M(n,d)$ be the maximum of $\overline{\mathrm{CR}}(G)$ for all d-regular graphs G of order n, that is,

$$M(n,d) = \max_{G \in R_{n,d}} \overline{\mathrm{CR}}(G).$$

In [1] it is a major result that

$$M(n,d) = \frac{1}{24} nd(3nd - 2d^2 - 6d + 2) \quad \text{if} \quad n + d \equiv 1 \pmod 2.$$

Other results are proved and conjectured for the cases of n and d even.

Now let $m(n,d)$ denote the minimum of $\overline{\mathrm{CR}}(G)$ for all d-regular graphs G of order n, that is,

$$m(n,d) = \min_{G \in R_{n,d}} \overline{\mathrm{CR}}(G).$$

If neccessary, by $R^c_{n,d}$ we denote the class of connected d-regular graphs of order n and by $m^c(n,d)$ the corresponding extremal value.

Here we present results for $d = 3$ (see Table 1). The value in italics is

n	4	6	8	10	12	14
$m(n,3)$	1	9	22	46	63	101
$m^c(n,3)$	1	9	27	49	78	*112*

Table 1: Values of $m(n,3)$ and $m^c(n,3)$.

an estimation in so far that all convex drawings, that is, with vertices in convex positions, are checked by computer. It may be noted that all values determined exactly coincide with the values for convex drawings.

2. Exact values for $n \leq 12$

At first we prove two general lemmas.

Lemma 1: For $d < n$ there exist d-regular graphs of order n if and only if not $n \equiv d \equiv 1 \pmod 2$.

Proof: Since nd counts twice the number of edges $n \equiv d \equiv 1 \pmod 2$ is impossible. Conversely, if n is even then the complete graph K_n can be partitioned into $n-1$ linear factors [2] and d of them give a desired d-regular graph. If n is odd then K_n can be partitioned into 2-factors [2] which can be combined for d even. □

Lemma 2: If $d < n$ and not $n \equiv d \equiv 1 \pmod 2$ then disconnected d-regular graphs of order n exist if and only if $n \geq 2d + 2$.

Proof: Since every component has at least $d+1$ vertices we have $n \geq 2(d+1)$. Conversely, we use K_{d+1} and a d-regular graph of order $n-(d+1)$ as components. The existence of the second component follows from Lemma 1 since $n-(d+1)>d$ and even if d is odd. $\square$

We now consider the values in Table 1.

Theorem 1: $m^c(4,3)=m(4,3)=1$ and $m^c(6,3)=m(6,3)=9$.

Proof: By Lemma 2 we have $m=m^c$ in both cases. Moreover, $R_{4,3}=\{K_4\}$ implies $m(4,3)=\overline{\mathrm{CR}}(K_4)=1$. The set $R_{6,3}$ consists of the two graphs in Figure 1 only. For the first graph in Figure 1, $K_{3,3}$, we have

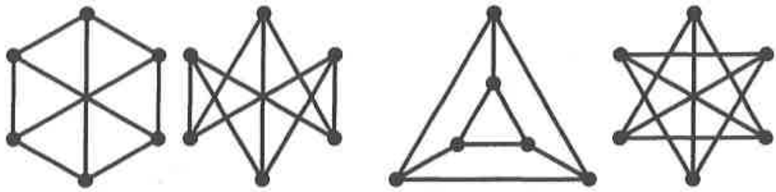

Figure 1: Drawings of $K_{3,3}$ and $K_3\times K_2$.

$\overline{\mathrm{CR}}(K_{3,3})=\binom{3}{2}\binom{3}{2}=9$. The second graph, $K_3\times K_2$, can be drawn with more than 9 crossings (see Figure 1) implying $m(6,3)=9$. $\square$

For $d=3$ and $n=8$ we obtain $m(8,3)<m^c(8,3)$. In the following the double triangle graph DT_n of order n, $n\equiv 0 \pmod 4$, is of interest consisting of $n/4$ double triangles $DT\cong K_4-e$ connected cyclically by edges (see Figure 2). The common edge of the 2 triangles of a DT will be

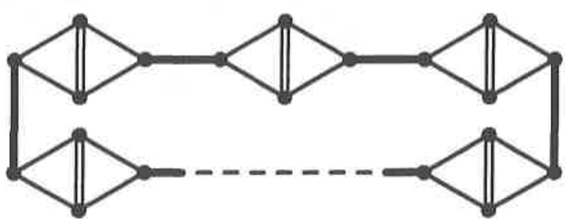

Figure 2: The graph DT_n with marked double edges and bold connecting edges.

called double edge of DT.

Lemma 3: The edges of one double triangle can have at most 17 crossings with the edges of another double triangle. This is possible as in Figure 3 only, where the two double edges intersect each other and the pairs of vertices of degree two of both DTs are on one side of the other DT.

Proof: Each edge of one double triangle (DT) has at most 4 crossings with the edges of the second double triangle (DT'), at most 2 with each of the 2 triangles of DT'. If such an edge has exactly 4 crossings then it does not intersect the double edge of DT'. If only one edge has 4 crossings

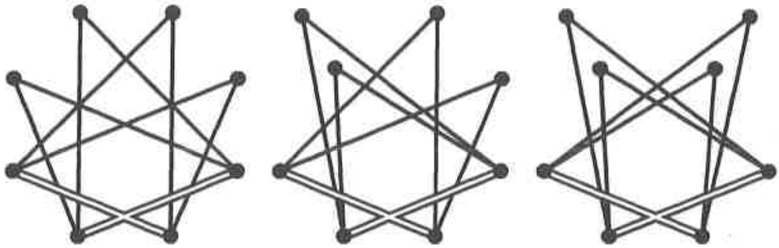

Figure 3: Drawings with 17 crossings between edges of two DTs.

then there are at most $4+4\cdot 3=16$ crossings. If 2 nonadjacent edges, 2 edges of a triangle, or 3 edges have 4 crossings, then they have 2 adjacent endvertices on the same side of DT' so that this determined edge has no crossing with DT' and together there are at most $0+4\cdot 4=16$ crossings. It remains that exactly 2 adjacent edges have 4 crossings implying a maximum of $2\cdot 4+3\cdot 3=17$ crossings. The endvertices of these 2 adjacent edges are the 2 nonadjacent vertices of DT. For DT' and the 2 adjacent edges of DT there are the three possibilities as in Figure 4. To obtain 17 crossings, the

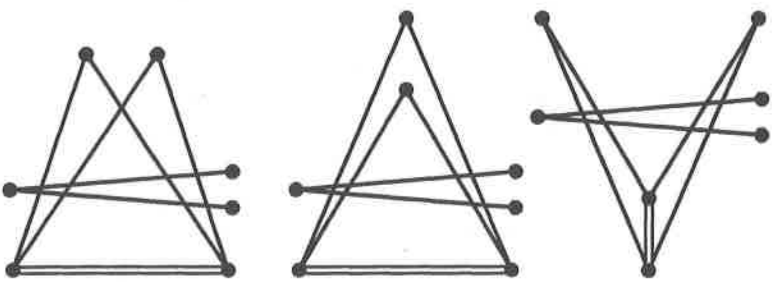

Figure 4: Two adjacent edges of DT intersecting 4 edges of DT' each.

3 edges incident to the remaining vertex of DT have to have 3 crossings each. Only for the first two cases of Figure 4 this is possible as in Figure 3. Note that due to the symmetry of DT and DT' both double edges have 3 crossings. Moreover, all 3 possibilities fulfill the asserted conditions. □

Theorem 2: $m^c(8,3)=\overline{\mathrm{CR}}(DT_8)=27$.

Proof: Figure 5 proves $\overline{\mathrm{CR}}(DT_8)\geq 27$.

A connecting edge intersects the edges of a double triangle at most twice and the other connecting edge at most once. Thus there are at most $1+2\cdot 4=9$ crossings involving connecting edges. To obtain a maximum of $9+2+17=28$ crossings, due to Lemma 3 the two DTs have to be as in the leftmost case of Figure 3. However, then one connecting edge has less than 4 crossings. This proves $\overline{\mathrm{CR}}(DT_8)\leq 27$.

There exist 4 further 3-regular connected graphs of order 8 (see Figure 6) all of which have a rectilinear drawing with more than 27 crossings. □

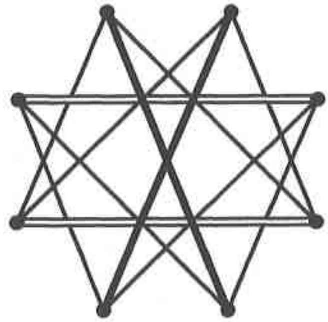

Figure 5: Drawing of DT_8 with 27 crossings.

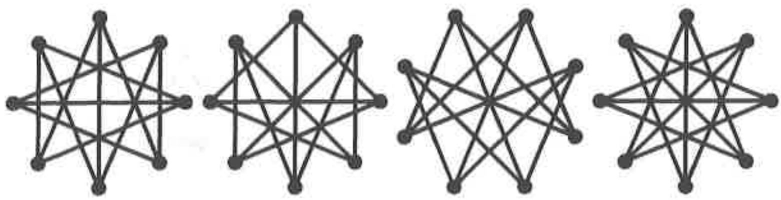

Figure 6: Remaining graphs of $R^c_{8,3}$ with more than 27 crossings.

Lemma 4: $\overline{\mathrm{CR}}(tK_4) = 20\binom{t}{2} + t = 10t^2 - 9t = \dfrac{5n^2 - 18n}{8}$ with $n = 4t$.

Proof: Each triangle of one K_4 has at most 6 crossings with at most 2 triangles of a second K_4 and at most 4 crossings with the remaining 2 triangles of the second K_4. The justification of this is as follows: Assume triangle T_1 of the first K_4 has the maximum number of 6 crossings with each of the triangles T_2 and T_3 of the second K_4. Triangles T_2 and T_3 have an edge in common. The remaining edge of the second K_4 does not intersect triangle T_1 so that the remaining 2 triangles of the second K_4 have at most 4 crossings with T_1. Thus altogether there are at most $4\,(2 \cdot 6 + 2 \cdot 4) = 80$ crossings between the triangles of both K_4s. Each of the 2 edges of a crossing belongs to 2 triangles. Thus each crossing is counted $2 \cdot 2 = 4$ times and there are at most $80/4 = 20$ crossings between the edges of two K_4s. Since each K_4 has at most one crossing it follows $\overline{\mathrm{CR}}(tK_4) = 20\binom{t}{2} + t$.

A drawing of tK_4 with this number of crossings is depicted in Figure 7 which completes the proof. □

Theorem 3: $m(8,3) = \overline{\mathrm{CR}}(2K_4) = 22$.

Proof: By Lemma 4 we have $\overline{\mathrm{CR}}(2K_4) = 22$. Since $2K_4$ is the only 3-regular disconnected graph of order 8 and since $m^c(8,3) = 27$ by Theorem 2 the proof is complete. □

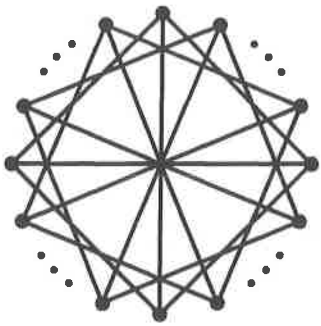

Figure 7: Extremal drawing of tK_4.

For $d = 3$ and $n = 10$ the extremal graphs are the Petersen graph P and $K_4 \cup K_{3,3}$.

Theorem 4: $m^c(10, 3) = \overline{\mathrm{CR}}(P) = 49$.

Proof: In [3] it is proved that $\overline{\mathrm{CR}}(P) = 49$. It can be checked that for the remaining 18 graphs of $R^c_{10,3}$ (see [11]) there exist drawings with more than 49 crossings. □

Theorem 5: $m(10, 3) = \overline{\mathrm{CR}}(K_4 \cup K_{3,3}) = 46$.

Proof: In $R_{10,3}$ there are only the two disconnected graphs in Figure 8, $K_4 \cup K_{3,3}$ with 46 crossings and $K_4 \cup K_3 \times K_2$ with 48 crossings. Together

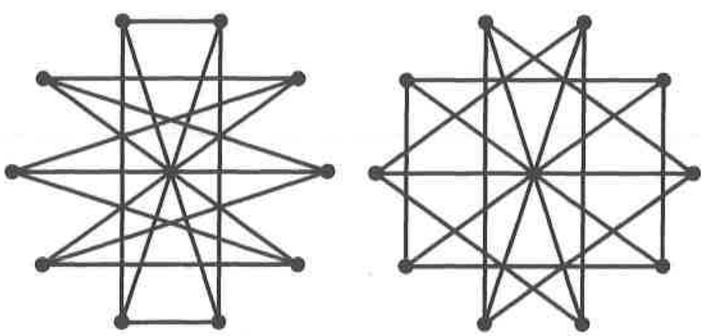

Figure 8: Extremal drawings of $K_4 \cup K_{3,3}$ and $K_4 \cup K_3 \times K_2$.

with $m^c(10, 3) = 49$ it follows $m(10, 3) \geq 46$.

Every edge of $K_{3,3}$ intersects at most 2 edges of each of the 4 triangles of K_4. Since every edge of K_4 belongs to exactly 2 triangles every edge of $K_{3,3}$ intersects at most $2 \cdot 4/2 = 4$ edges of K_4. Thus, see proof of Theorem 1, $\overline{\mathrm{CR}}(K_4 \cup K_{3,3}) \leq 9 \cdot 4 + \overline{\mathrm{CR}}(K_{3,3}) + \overline{\mathrm{CR}}(K_4) = 36 + 9 + 1 = 46$. □

Theorem 6: $m^c(12,3) = \overline{\mathrm{CR}}(DT_{12}) = 78$.

Proof: The graph DT_{12} consists of three DTs and 3 connecting edges. Within a DT there is at most one crossing. The 3 connecting edges have at most 3 crossings. Each connecting edge has at most $2+2+4=8$ crossings with the edges of the three DTs. Each pair of DTs determines at most 17 crossings (Lemma 3). Together $\overline{\mathrm{CR}}(DT_{12}) \leq 3\cdot 1 + 3 + 3\cdot 8 + 3\cdot 17 = 81$.

If no pair of DTs has 17 crossings then there are at most $81-3=78$ crossings.

If all three pairs of DTs have 17 crossings then there exists one connecting edge of two DTs which remains without crossing with the third DT (see Lemma 3). Thus we have at most $81-4=77$ crossings.

If two pairs of DTs have 17 crossings, say, DT has 17 crossings with DT' and DT'', then by Lemma 3 the connecting edge of DT' and DT does not intersect the two edges of DT'' being incident to the double edge of DT'' and intersecting the double edge of DT. Thus this connecting edge has at most 3 crossings with DT'', that is, one less then 4. Due to symmetry of DT' and DT'' the corresponding connecting edge of DT'' and DT implies a further crossing less so that there are at most $81-1-2=78$ crossings.

If only one pair of DTs has 17 crossings then there are at most $81-2=79$ crossings. Let DT' and DT'' have 17 crossings as in the leftmost case of Figure 3 since otherwise there is at least one crossing less. The connecting edge of DT' and DT'' has to connect the rightmost vertex with the leftmost vertex such that this edge has 4 crossings. The remaining connecting edge of DT' has to have 4 crossings with DT'' and 2 crossings with DT'. The connecting edge of DT' and DT'' has to intersect 4 edges of DT so that the double edge of DT is not intersected. Then this double edge can intersect at most 2 edges of DT. Both edges of DT being adjacent to the connecting edge of DT' have at most 3 crossings each with the edges of DT'. To obtain 16 crossings between DT and DT' the remaining 2 edges have to have the maximum of 4 crossings each, however, one edge has only 2 crossings with DT' if the double edge of DT has 2 crossings. Thus $\overline{\mathrm{CR}}(DT_{12}) \leq 78$ is proved.

In the cases of two pairs of DTs with 17 crossings each and no pairs of DTs with 17 crossings each we obtain the drawings in Figure 9 both determining 78 crossings.

It was checked by computer that for the remaining 84 graphs of $R^c_{12,3}$ (see [11]) there exist convex drawings with more than 78 crossings. □

Theorem 7: $m(12,3) = \overline{\mathrm{CR}}(3K_4) = 63$.

Proof: Lemma 4 and Theorem 6 imply $\overline{\mathrm{CR}}(3K_4) = 63 < m^c(12,3)$. By computer it was checked that $\overline{\mathrm{CR}}(G) > 63$ for the remaining 8 disconnected graphs $G \in R_{12,3}$. □

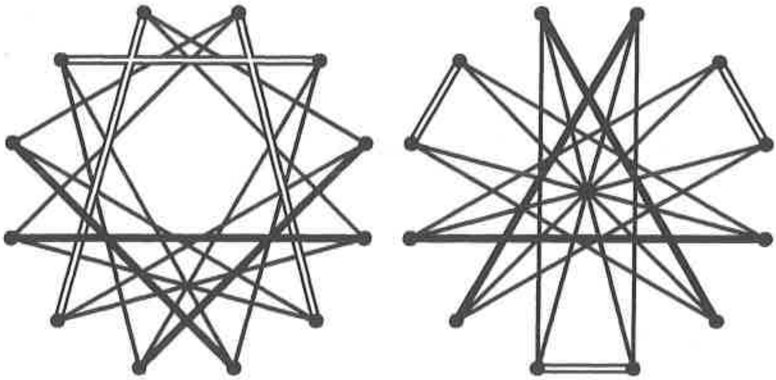

Figure 9: Drawings of DT_{12} with 78 crossings.

3. Further results

Now we determine some bounds for $\overline{\mathrm{CR}}(G)$ for cubic graphs G conjectured to be extremal.

Theorem 8: $\overline{\mathrm{CR}}(DT_n) \le \dfrac{13n^2-48n}{16}$ for $n \equiv 0 \pmod 4$,

$$\overline{\mathrm{CR}}(DT_n) \ge \begin{cases} \dfrac{25n^2-92n}{32} & \text{for } n \equiv 4 \pmod 8, \\[2mm] \dfrac{25n^2-104n+160}{32} & \text{for } n \equiv 0 \pmod 8. \end{cases}$$

Proof: If $n = 4t$ then there are at most t crossings within the t double triangles DT. Lemma 3 implies at most $17\binom{t}{2}$ crossings for pairs of DTs. The t connecting edges may intersect pairwise in $\binom{t}{2}$ crossings. A connecting edge has at most 2 crossings with each of the 2 neighboring DTs and at most 4 crossings with each of the remaining DTs. Altogether we have at most $t + 17\binom{t}{2} + \binom{t}{2} + t(4 + 4(t-2))$ crossings implying the upper bound with $t = n/4$.

For t odd, that is, $n \equiv 4 \pmod 8$, the rightmost drawing of Figure 9 can be generalized for a lower bound as follows. Consider n points on a circle. The double edges of any DT connect neighboring points and the 2 vertices of degree 2 of this DT are the opposite neighboring points. Then double edges alternate with pairs of vertices of degree 2. The connecting edges are drawn from one point to the neighbor of the opposite point. Then there are only $\binom{t}{2}$ crossings less than counted for the upper bound since each pair of DTs has only 16 crossings instead of 17.

For t even, that is, $n \equiv 0 \pmod 8$, a drawing as for $t = 6$ in Figure 10 can be used. Here the double edge of a DT connects points of distance 5 on the circle with n points and the 2 remaining vertices of this DT are the 2 neighboring opposite points. All further DTs are obtained using rotations

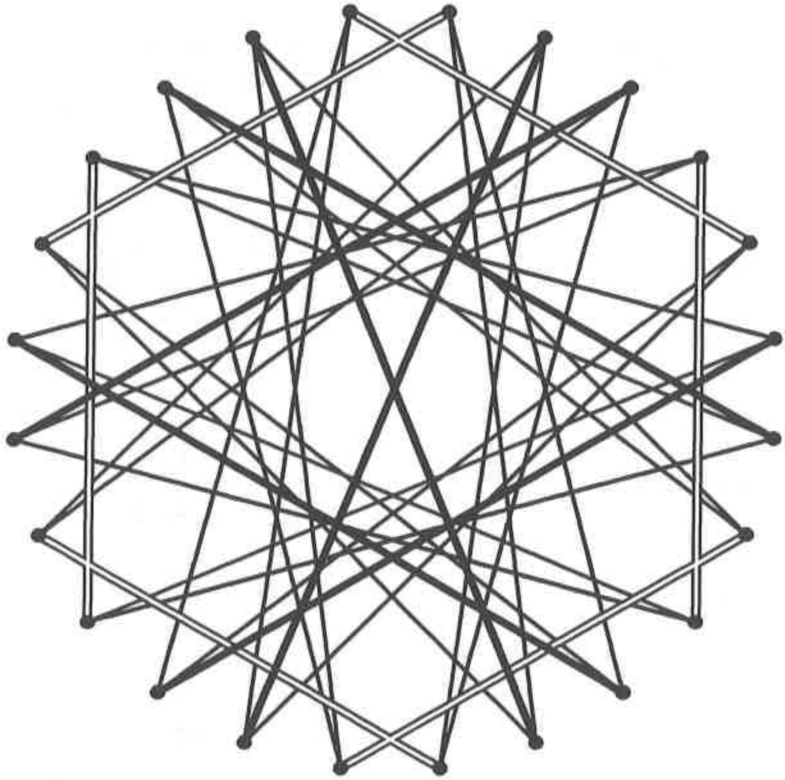

Figure 10: Lower bound for $\overline{\mathrm{CR}}(DT_{4t})$ for $t = 6$.

by 4 points. There are 2 connecting edges to opposite points starting from 2 points of distance 3. The endpoints of the remaining $t-2$ connecting edges have distances $n/3-2$. In this drawing the number of crossings counted for the upper bound is reduced as follows. There are $\binom{t}{2} - t$ crossings less due to pairs of DTs having 16 crossings only. There are $(t-2)/2$ crossings less due to pairs of nonintersecting connecting edges. Each of the $t-2$ short connecting edges intersects 2 DTs only 3 times so that there are $2(t-2)$ crossings less. Altogether $(t^2 + 2t - 10)/2$ has to be subtracted from the upper bound. □

The known exact values of $\overline{\mathrm{CR}}(DT_n)$, see Theorems 2 and 6, coincide with the lower bounds of Theorem 8.

For $n = 14$ there exists a graph with at least 112 crossings (see Figure 11). By computer we have checked that there is no convex drawing with more than 112 crossings and that for all remaining 508 graphs of $R^c_{14,3}$ (see [11]) there are (convex) drawings with more than 112 crossings.

In the disconnected case, for $n \equiv 0 \pmod 4$ we have the exact value of

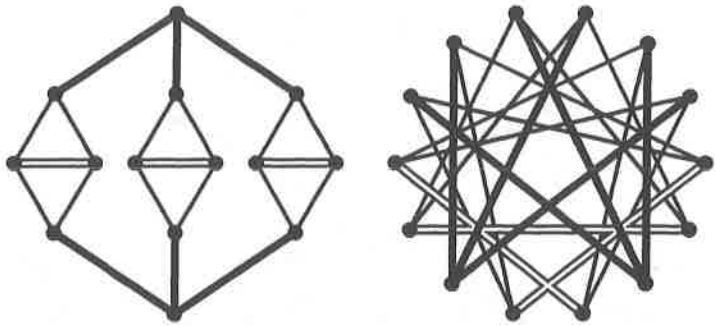

Figure 11: A graph of $R^c_{14,3}$ and a drawing of this graph with 112 crossings.

$\overline{\text{CR}}(tK_4)$, see Lemma 4, which coincides with $m(n,3)$ for $n = 8$ and $n = 12$ (see Theorems 3 and 7).

For $n \equiv 2 \pmod 4$, we consider the graph $tK_4 \cup K_3 \times K_2$.

Theorem 9: $\overline{\text{CR}}(tK_4 \cup K_3 \times K_2) = 20\binom{t}{2} + 33t + 15 = \frac{1}{8}(5n^2 - 14n + 24)$ with $n = 4t + 6$.

Proof: There are at most $20\binom{t}{2} + t$ crossings for the t copies of K_4, see Lemma 4. We have $\overline{\text{CR}}(K_3 \times K_2) \leq 15 = 6 + 3 \cdot 2 + 3$ since the two triangles have at most 6 crossings and each connecting edge has at most 2 crossings with the two triangles. Furthermore there are at most 3 crossings of the 3 connecting edges. Between $K_3 \times K_2$ and a K_4 there are at most $32 = 2 \cdot 10 + 3 \cdot 4$ crossings since each of the two triangles has at most 10 crossings with the K_4, see proof of Lemma 4, and a connecting edge has at most 4 crossings with a K_4. Altogether $\overline{\text{CR}}(tK_4 \cup K_3 \times K_2) \leq 20\binom{t}{2} + t + 15 + 32t$. A drawing with this number of crossings is depicted in Figure 12. □

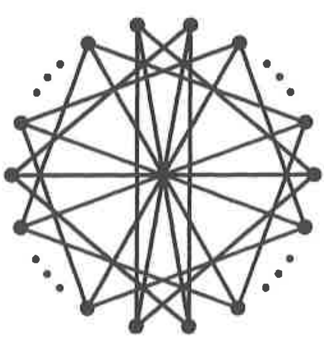

Figure 12: Extremal drawing of $tK_4 \cup K_3 \times K_2$.

Finally, we have a further exact value.

Theorem 10: $m(14,3) = \overline{\mathrm{CR}}(2K_4 \cup K_3 \times K_2) = 101$.

Proof: Theorem 9 implies $\overline{\mathrm{CR}}(2K_4 \cup K_3 \times K_2) = 101$. By computer it was checked that $\overline{\mathrm{CR}}(G) > 101$ for the remaining 539 graphs $G \in R_{14,3}$. $\square$

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