Strategies for Discriminating and Comparing Unknown Unitary Transformations

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Strategies for Discriminating and Comparing
Unknown Unitary Transformations

by

Guy F. Okoko

A dissertation submitted to the Graduate Faculty in Physics toward the fulfillment of the requirements for the degree of Doctor of Philosophy

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Abstract

Strategies for Discriminating and Comparing

Unknown Unitary Transformations

By

Guy F. Okoko

Advisor: Professor János A. Bergou

How to discriminate or compare two unitary transformations that are completely unknown?

We first examine the unambiguous discrimination of two unknown unitary transformations; we show that the results are the same as those found for the programmable discrimination of two unknown quantum states.

Next we consider the minimum-error comparison of two unknown unitary transformations; the results are obtained in the general case where the prior probabilities are different.

Last we study the unambiguous discrimination of two unknown unitary transformations in the case where multiple copies of data are available.
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Introduction

A. Quantum State Discrimination

Given two non-orthogonal quantum states $|\Psi_A\rangle$ and $|\Psi_B\rangle$, we consider a set of quantum systems, each of which is either in state $|\Psi_A\rangle$ or in state $|\Psi_B\rangle$. A system randomly selected from this set has some probability of being in state $|\Psi_A\rangle$, which we denote by $\eta_A$, and some probability of being in state $|\Psi_B\rangle$, which we denote by $\eta_B$. In general, $\eta_A$ and $\eta_B$ are not equal; however, all of the systems in the set have the same value of $\eta_A$ and the same value of $\eta_B$. We call these probabilities the a priori probabilities of the states $|\Psi_A\rangle$ and $|\Psi_B\rangle$ respectively. Since each system is necessarily either in state $|\Psi_A\rangle$ or in state $|\Psi_B\rangle$, conservation of probability requires that

$$\eta_A + \eta_B = 1$$

Now consider a system randomly selected from the set. We are interested in finding out the state ($|\Psi_A\rangle$ or $|\Psi_B\rangle$) in which the system is. The task of finding out, or, trying to find out, which of $|\Psi_A\rangle$ and $|\Psi_B\rangle$ is the state of the system is not an easy task. This task is referred to as quantum state discrimination (the discrimination of, or, between, the quantum states $|\Psi_A\rangle$ and $|\Psi_B\rangle$).

Quantum state discrimination can involve more than two states, which can be mixed states (represented by density operators).

The discrimination of unknown states has already been studied [1,2].
B. Generalized Quantum Measurements and the P.O.V.M. concept

We (attempt to) determine the state of a system by performing a measurement on this system. In the present case, where only two states are possible for the system, one might think of a measurement with two possible outcomes: one outcome to tell us that the system is in state $|\Psi_A\rangle$, and one outcome to tell us that the system is in state $|\Psi_B\rangle$. This would work if the discrimination of non-orthogonal states could be carried out perfectly. It has been proven that, in general, one cannot perfectly discriminate between non-orthogonal states. The impossibility to perfectly discriminate between $|\Psi_A\rangle$ and $|\Psi_B\rangle$ means the following:

1. The conclusion that we draw (the guess that we make), based on the result of the measurement, can be false. In other words, while the measurement result points to the state $|\Psi_A\rangle$, the system might actually be in the state $|\Psi_B\rangle$ (and vice versa).

2. The measurement can have an inconclusive result. That is, a result that does not point to any of the two states. In other words, it can happen that, after the measurement, we are not able to draw any conclusion (to make any guess), as to the state of the system.

Given the impossibility to perfectly discriminate, in general, between $|\Psi_A\rangle$ and $|\Psi_B\rangle$, we design the measurement to have three possible results, which we denote by $r_a$, $r_b$, and $r_f$. These results determine our conclusions (our guesses) as follows.

1. If the measurement yields the result $r_a$, we conclude that the system is in the state $|\Psi_A\rangle$.

2. If the measurement yields the result $r_b$, we conclude that the system is in the state $|\Psi_B\rangle$. 
3. If the measurement yields the result \( r \), we do not draw any conclusion; that is, we admit that we failed to indicate the state of the system.

If \( \ket{\Psi_A} \) and \( \ket{\Psi_B} \) were orthogonal, we would be able to discriminate between them, perfectly, using a Von Neumann measurement; that is, a standard quantum measurement. We would simply measure, on the system, an observable that has these states as eigenstates corresponding to different eigenvalues. Thus, the measurement’s result would unambiguously indicate the state of the system.

Since our states are not orthogonal, a Von Neumann measurement is inappropriate. This is because non-orthogonal states cannot form an orthonormal basis in the state space of the system. So, such states cannot be eigenstates of the same observable. To carry out our task we need a different kind of measurement, known as **generalized measurement**. This kind of measurement corresponds to a set of positive semi-definite operators \( \Pi_i \), which add up to the identity operator, \( \I \). That is,

\[
\sum_i \Pi_i = \I
\]

We call such an expansion of the identity operator, in terms of positive semi-definite operators, a P.O.V.M. (positive operator valued measure). We call the \( \Pi_i \)'s the **elements** of the P.O.V.M. The correspondence between a P.O.V.M. and a generalized measurement is as follows.

1. The number of elements of the P.O.V.M. is equal to the number of possible results of the measurement.

2. Each possible result of the measurement is associated with a different element of the P.O.V.M.

Let \( r \) be the possible result associated with the P.O.V.M. element \( \Pi_i \). This association means the following:
3. If the system is in a pure state $|\Phi\rangle$, the probability that the measurement will yield the result $r_i$ is equal to $\langle \Phi | \Pi_i | \Phi \rangle$.

4. If the system is in a mixed state represented by the density operator $\rho$, the probability that the measurement will yield the result $r_i$ is equal to $\text{Tr}(\rho \Pi_i)$; that is, the trace of the operator product in parentheses.

Since our measurement has three possible results, it corresponds to a three-element P.O.V.M., which we write as follows:

$$\Pi_a + \Pi_b + \Pi_f = I$$

The elements of this P.O.V.M., in the order in which they are written, are associated with the measurement results $r_a$, $r_b$, and $r_f$, respectively.

C. Unambiguous discrimination and minimum-error discrimination

It is possible to design the measurement process in such a way that errors cannot occur. In this case $P_e = 0$, so that we have

$$P_s + P_f = 1$$

State discrimination that is carried out under such conditions is referred to as *unambiguous discrimination*. 

Unambiguous discrimination was started by Ivanovic [3]. He studied the following problem. A group of quantum systems is prepared as follows: (1) each system is definitely in one or the other of two known states $|\Psi_A\rangle$ and $|\Psi_B\rangle$, with equal probabilities of being in these states; and (2) the states $|\Psi_A\rangle$ and $|\Psi_B\rangle$ are not orthogonal. Then the systems are given, one by one, to an observer whose task is to determine, for each system, which one of the two states has been prepared. The observer performs, on each system, a single measurement or a series of measurements expected to indicate the state that has been prepared. Ivanovic came to this conclusion: if inconclusive measurement results are allowed to occur, then in the remaining cases the observer can correctly determine the state of the system. This task can be accomplished using a von Neumann measurement, in a basis formed by $|\Psi_A\rangle$ and a state orthogonal to it, or a basis formed by $|\Psi_B\rangle$ and a state orthogonal to it. Ivanovic observed that a sequence of measurements can do better, sometimes, than a single von Neumann measurement. Then Dieks [4] found that such a sequence of measurements can be realized with a single POVM. Later on, Peres [5] showed that this POVM is optimal; that is, it has the minimum probability of inconclusive outcomes. This probability is $Q_{IDP} = \left|\langle \Psi_A | \Psi_B \rangle\right|$, and the success probability, known as Ivanovic-Dieks-Peres (IDP) limit is $P_{IDP} = 1 - Q_{IDP}$. This result, which corresponds to the case of equal preparation probabilities of the two states, was later generalized to the case of arbitrary preparation probabilities by Jaeger and Shimony [6]. With the preparation probabilities denoted by $\eta_A$ and $\eta_B$, the general expression for the optimal probability of inconclusive results is

$$Q_{POVM} = 2 \sqrt{\eta_A \eta_B |\langle \Psi_A | \Psi_B \rangle|}.$$
There have been studies on the discrimination of more than two states, but only a few general results have been obtained. Explicit solutions exist only in some special cases. Two general results are known for unambiguous discrimination. The first one, obtained by Chefles [7], is that only linearly independent states can be unambiguously discriminated. The second general result states that there are upper and lower bounds on the success probability. An upper bound is found in [8]. Using work by Duan and Guo [9], X. Sun, et al. derived a lower bound [10].

The discrimination of three nonorthogonal states was initially considered by Peres and Terno [11]. This problem was also considered, with a different method, in [9] and [12]. Chefles and Barnett studied the unambiguous state discrimination of N symmetric states [13]. For the case of equal prior probabilities, they found an analytical expression for the optimal success probabilities.

In [14], the unambiguous discrimination between sets of states was introduced. The case where there are only two sets is the simplest. In this case, if one set consists of only one state then the problem is referred to as quantum state filtering. Unambiguous discrimination between multiple sets of pure states was also investigated in [15]. These problems, of discriminating between sets of states, can be recast in the form of discriminating between mixed states. The filtering of a mixed state out of many was studied in [16].

One can also design the measurement process in a way that makes inconclusive results impossible. In this case $P_f = 0$; then we have

$$P_s + P_e = 1$$

State discrimination that is done under such conditions is known as minimum-error discrimination. For the case of two states, either pure or mixed, the minimum error probability was derived in [17]. Analytical expressions for the minimum error probability have been determined only for some special
cases. For equiprobable and symmetric pure states, the solution of the minimum error problem was derived in [18]. An extension of this solution to the case of N equiprobable and symmetric mixed states was obtained in [19] and [20]. Other analytically solved cases include certain classes of linearly independent states [21]. The minimum-error strategy for multiple symmetric pure states was found by Barnett [22]. The case of three mirror symmetric pure states was solved by Andersson et al. [23].

When the given states are linearly independent, the minimum-error strategy for state discrimination is always a von Neumann measurement, as has been proved by Eldar [24].

The error minimizing strategy has also been studied under the condition that inconclusive results are allowed to occur, but with a fixed prescribed probability. This problem was first investigated for pure states by Chefles and Barnett [25] and by Zhang et al. [26]. The problem was later generalized to the case of mixed states in [27] and [28].

Early studies in quantum state discrimination are reviewed in [29]. Barnett and Rijs [30] provide a simple physical picture of an error-minimizing state discrimination measurement.

Apart from the unambiguous and minimum-error strategies, state discrimination measurements can also be optimized with respect to other criteria, like requiring the maximum of mutual information [31] or of the fidelity [32]. Recent reviews on quantum state discrimination are due to S. M. Barnett and S. Croke [45], and J. A. Bergou [46].
D. Discrimination of Unitary Transformations

The discrimination of unitary transformations in quantum mechanics is similar to that of quantum states. To discriminate between unitary transformations, we let these transformations act on a reference state, and then perform measurements on the results. That is, we discriminate between the output states. The discrimination of two known transformations has already been treated [33-36]. In this work we address the discrimination of two unknown unitary transformations. A more detailed analysis is given in [44] and [47].

E. Comparison of States and Unitary Transformations

We sometimes need to establish whether or not two quantum systems have been prepared in the same state. This problem, which consists of determining whether two unknown quantum states are the same or different, is known as quantum state comparison [37-39].

Comparison of unitary transformations is similar to comparison of states. The problem consists of determining whether two unknown unitary transformations are the same or different. The unambiguous comparison of unknown unitary transformations has been broadly discussed [40], and a linear optical implementation has been proposed [41].

In this work we compare unknown unitary transformations using the minimum-error strategy. A broader analysis is done in [44]. The case of equal prior probabilities has already been
treated [42]. With this strategy it is possible to obtain the result that the transformations are identical. This result cannot be obtained when we compare unknown transformations unambiguously. When doing unambiguous comparison, we can detect the transformations as being different but cannot detect them as being identical.
Part 1: Discrimination and Minimum-Error Comparison of Two Unknown Unitary Transformations

Chapter 1: Discrimination of two unknown unitary transformations

1.1. Equivalence with the discrimination of two unknown states

We consider, on one hand, two unknown unitary transformations which we denote by V and W, and on the other hand, a third unitary transformation which we denote by U. The transformation U is also unknown, but it is guaranteed to be the same as either V or W, with a probability \( \eta_1 \) of being the same as V, and a probability \( \eta_2 = 1 - \eta_1 \) of being the same as W. The transformations V and W are guaranteed to be different. Our task is to find out which of V and W is the same as U.

One way of approaching this problem is as follows. Consider the 8-dimensional Hilbert space of a system of three qubits, labeled A, B, and C. On a state \( |\psi_{ABC}\rangle \) of this Hilbert space we apply the transformation \( V^A U^B W^C \). By this notation we mean that the transformations V, U, and W act on the 2-dimensional Hilbert spaces of qubits A, B, and C respectively. If U is the same as V, then the above transformation can be written as \( V^A V^B W^C \). If, instead, U is the same as W, then the transformation can be written as \( V^A W^B W^C \). Let \( |\phi_1\rangle = V^A V^B W^C |\psi_{ABC}\rangle \) and \( |\phi_2\rangle = V^A W^B W^C |\psi_{ABC}\rangle \). With a convenient choice of the input state \( |\psi_{ABC}\rangle \), we can replace the discrimination of V and W by the discrimination of \( |\phi_1\rangle \) and \( |\phi_2\rangle \).
1.2 Parametrization of an unknown unitary transformation

Any 2x2 unitary matrix \( U \) can be expressed as \( U = \exp(-i\frac{\alpha}{2} \hat{r}\vec{\sigma}) \), where \( \hat{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) is the unit position vector, and \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) is the Pauli vector.

From \( (\hat{r}\vec{\sigma})^2 = I \), it follows that \( \exp(-i\frac{\alpha}{2} \hat{r}\vec{\sigma}) = \cos \frac{\alpha}{2} I - i \sin \frac{\alpha}{2} (\hat{r}\vec{\sigma}) \). The above general expression of a 2x2 matrix can therefore be written as follows:

\[
U = \cos \frac{\alpha}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \sin \frac{\alpha}{2} \begin{pmatrix} \sin \theta \cos \phi & 0 \\ 0 & \sin \theta \cos \phi \end{pmatrix} + \sin \theta \sin \phi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos \theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

That is,

\[
U = \begin{pmatrix}
\frac{\cos \alpha}{2} - i \frac{\sin \alpha}{2} \cos \theta & - \sin \frac{\alpha}{2} \sin \theta (\sin \varphi + i \cos \varphi) \\
\sin \frac{\alpha}{2} \sin \theta (\sin \varphi - i \cos \varphi) & \frac{\cos \alpha}{2} + i \frac{\sin \alpha}{2} \cos \theta 
\end{pmatrix}
\] (1-1)

This matrix shows the parametrization of a unitary transformation in terms of three angular parameters \( \alpha \), \( \theta \), and \( \varphi \) whose ranges of variation are as follows: \( 0 \leq \alpha \leq 2\pi \), \( 0 \leq \theta \leq \pi \), and \( 0 \leq \varphi \leq 2\pi \).

1.3: Discrimination of \( |\phi_1\rangle = V^{A}V^{B}W^{C}|\psi\rangle_{ABC} \) and \( |\phi_2\rangle = V^{A}W^{B}W^{C}|\psi\rangle_{ABC} \)

Denoting by \( V_{ij} \) and \( W_{ij} \) the matrix elements of \( V \) and \( W \) respectively, and choosing the input state
\[ |\psi\rangle_{ABC} = \left( \frac{|0\rangle_A |0\rangle_B |0\rangle_C - |1\rangle_A |1\rangle_B |1\rangle_C}{\sqrt{2}} \right), \]

we can express the density matrices \(|\phi_1\rangle\langle \phi_1|\) and \(|\phi_2\rangle\langle \phi_2|\)
as linear combinations of outer products formed with the basis states \(|i\rangle_A |j\rangle_B |k\rangle_C\), \(i, j, k = 0, 1\). In these linear combinations, the coefficients of the outer products are functions of the matrix elements \(V_{ij}\) and \(W_{ij}\) and their complex conjugates. When we replace these matrix elements by the corresponding expressions given by the general matrix (1), the above mentioned coefficients become functions of the angular parameters \(\alpha, \theta, \) and \(\varphi\). We can then average each coefficient by using the Haar integral as follows:

\[
\langle C(\alpha, \theta, \varphi) \rangle = \frac{1}{4\pi^2} \int_0^{2\pi} d\alpha \int_0^\pi d\theta \int_0^{2\pi} d\varphi C(\alpha, \theta, \varphi) \sin^2 \frac{\alpha}{2} \sin \theta. \quad (1-2)
\]

We will denote the density operators with averaged coefficients by \(\rho_1 = \{ |\phi_1\rangle\langle \phi_1| \}_{av}\) and \(\rho_2 = \{ |\phi_2\rangle\langle \phi_2| \}_{av}\), and use the basis states \(|u_1\rangle_{AB} = |0\rangle_A |0\rangle_B\), \(|u_2\rangle_{AB} = \frac{|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B}{\sqrt{2}}\), \(|u_3\rangle_{AB} = |1\rangle_A |1\rangle_B\), and \(|u_4\rangle_{AB} = \frac{|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B}{\sqrt{2}}\). Similar basis states will be used for the qubit combination \(BC\). After averaging all coefficients using the Haar integral above, we find that

\[
\rho_1 = \frac{1}{6} P_{AB}^{sym} \otimes I_C \quad \text{and} \quad \rho_2 = \frac{1}{6} I_A \otimes P_{BC}^{sym}, \quad (1-3)
\]

where \(P_{AB}^{sym} = \sum_{i=1}^{3} |u_i\rangle_{AB} \langle u_i|\) and \(P_{BC}^{sym} = \sum_{i=1}^{3} |u_i\rangle_{BC} \langle u_i|\) are the projectors onto the symmetric subspaces of the Hilbert spaces corresponding to the systems \(AB\) and \(BC\) respectively.
Expressions (1-3) are the same as those found in Ref. [1] for the discrimination of two unknown states. Consequently, the results given in Ref. [1] for two unknown states are also valid in the case of two unknown unitary transformations.
Chapter 2: Minimum-Error comparison of two unknown unitary transformations

2.1. Problem formulation

Let \( V = e^{i\phi} \begin{pmatrix} a_v & -b_v \\ b_v^* & a_v^* \end{pmatrix} \) and \( W = e^{i\phi} \begin{pmatrix} a_w & -b_w \\ b_w^* & a_w^* \end{pmatrix} \) represent two unknown unitary transformations.

Using the minimum-error strategy, we want to determine whether \( V \) and \( W \) are the same or not.

2.2. Minimum probability of error

The minimum-error comparison of these unitary transformations is equivalent to the minimum-error discrimination of the states \( \psi_1 = VV^H \in \psi_1 \) and \( \psi_2 = WV^H \in \psi_2 \), where \( \psi_{in} \) is some input state.

Consider the corresponding density matrices \( \psi_1 \) and \( \psi_2 \), and let \( \eta_1 \) and \( \eta_2 \) be their respective prior probabilities. As our input state we choose

\[
|\psi_{in}\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle_v |0\rangle_w + |1\rangle_v |1\rangle_w \right).
\]

Using the above matrices of \( V \) and \( W \), and noting that

\[
|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

we have

\[
|\psi_2\rangle = \frac{1}{\sqrt{2}} e^{i(\phi_1 + \phi_2)} \left[ \begin{pmatrix} a_v \\ b_v^* \end{pmatrix} \begin{pmatrix} a_w \\ b_w^* \end{pmatrix} + \begin{pmatrix} -b_v \\ a_v^* \end{pmatrix} \begin{pmatrix} -b_w \\ a_w^* \end{pmatrix} \right].
\]

Setting \( \frac{1}{\sqrt{2}} e^{i(\phi_1 + \phi_2)} = C \), we have
Using the transformation relations
\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |00\rangle = \frac{1}{\sqrt{2}} \left( |\Phi^+\rangle + |\Phi^-\rangle \right),
\]
\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |11\rangle = \frac{1}{\sqrt{2}} \left( |\Psi^+\rangle - |\Psi^-\rangle \right),
\]
\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |01\rangle = \frac{1}{\sqrt{2}} \left( |\Psi^+\rangle + |\Psi^-\rangle \right) \quad \text{and}
\]
\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |10\rangle = \frac{1}{\sqrt{2}} \left( |\Psi^+\rangle - |\Psi^-\rangle \right),
\]
we can express \(|\psi_2\rangle\) in the Bell basis as follows:
\[
|\psi_2\rangle = \frac{C}{\sqrt{2}} \left\{ \begin{pmatrix} [a_v a_w + a_v^* a_w^*] + [b_v b_w + b_v^* b_w^*] \rangle \Phi^+ \rangle + [a_v a_w - a_v^* a_w^*] + [b_v b_w - b_v^* b_w^*] \rangle \Phi^- \rangle + \end{pmatrix} \right\}
\]

The expression of \(|\psi_1\rangle\) is found from the expression of \(|\psi_2\rangle\) by setting \(V = W\). With these expressions of \(|\psi_1\rangle\) and \(|\psi_2\rangle\), each of the density operators \(|\psi_1\rangle\langle\psi_1|\) and \(|\psi_2\rangle\langle\psi_2|\) can be expressed as a linear combination of 16 outer products formed with the Bell states. The coefficients of the former depend on \(a_v, a_v^*, b_v, b_v^*\); the coefficients of the latter depend on \(a_w, a_w^*, b_w, b_w^*\), as well as \(a_v, a_v^*, b_v, b_v^*\). In order to average these coefficients, we use the general matrix (1) by setting

\[
|\psi_2\rangle = \frac{C}{\sqrt{2}} \left\{ \begin{pmatrix} [a_v a_w + a_v^* a_w^*] + [b_v b_w + b_v^* b_w^*] \rangle \Phi^+ \rangle + [a_v a_w - a_v^* a_w^*] + [b_v b_w - b_v^* b_w^*] \rangle \Phi^- \rangle + \end{pmatrix} \right\}
\]
\[ a_v = \cos \frac{\theta_v}{2} - i \sin \frac{\theta_v}{2} \cos \phi_v \quad \text{and} \quad b_v = -i \sin \frac{\theta_v}{2} \sin \phi_v \left( \sin \phi_v + i \cos \phi_v \right), \]

and similarly for \( a_w \) and \( b_w \). The average density operators \( \rho_1 = \left\langle \psi_1 \right| \psi_1 \right\rangle_{av} \) and \( \rho_2 = \left\langle \psi_2 \right| \psi_2 \right\rangle_{av} \) result from averaging the coefficients of \( \left| \psi_1 \right\rangle \left\langle \psi_1 \right| \) and those of \( \left| \psi_2 \right\rangle \left\langle \psi_2 \right| \) by using the Haar integral (2). \( \rho_1 \) and \( \rho_2 \) are found to be as follows:

\[ \rho_1 = \frac{1}{3} S \quad \text{and} \quad \rho_2 = \frac{1}{4} I, \]

where \( S \) is the projector on the symmetric subspace, and \( I \) is the identity operator. Thus the eigenvalues of \( \Lambda = \eta_2 \rho_2 - \eta_1 \rho_1 \) are \( \lambda_1 = \lambda_2 = \lambda_3 = \frac{\eta_2}{4} - \frac{\eta_1}{3}, \) and \( \lambda_4 = \frac{\eta_2}{4} \).

The minimum probability of error is

\[ P_E = \frac{1}{2} \left( 1 - \sum_{k=1}^{4} |\lambda_k| \right). \]

That is,

\[ P_E = \frac{1}{2} \left( 1 - \sum_{k=1}^{4} \left| \frac{\eta_2}{4} - \frac{\eta_1}{3} - \frac{\eta_2}{4} \right| \right) \quad (2-1) \]

If \( \eta_1 = \eta_2 = \frac{1}{2} \), then \( P_E = \frac{3}{8} \).

Either one of the other symmetric Bell states, when used as input state, also leads to result (2-1). This probability will be denoted by \( P_{E}^{\text{sym}} \); we display it in Fig. 1 as a function of \( \eta_1 \).
Using the singlet state as our input state leads to $\rho_1 = A$ and $\rho_2 = \frac{1}{4} I$, where $A$ is the projector on the antisymmetric subspace, and $I$ is the identity operator. The resulting expression for the minimum probability of error is

$$P_E = \frac{1}{2} \left( 1 - 3 \frac{\eta_2}{4} - \left| \frac{\eta_2}{4} - \eta_1 \right| \right)$$

(2-2)

When the prior probabilities are equal, relation (5) yields $P_E = \frac{1}{8}$. The probability given by relation (2-2) will be denoted by $P_E^{\text{ant}}$; we display it in Fig. 2 as a function of $\eta_1$.
If, as our input state, we use a general 2-qubit state $\Psi = ad^++bc^+$, then, using relations (2-1) and (2-2), we find that

$$P_{E} = (1 - |d|^2)P_{E}^{\text{sym}} + |d|^2 P_{E}^{\text{ant}}$$

(2-3)

where $P_{E}^{\text{sym}}$ and $P_{E}^{\text{ant}}$ are the expressions given by relations (4) and (5) respectively. With equal prior probabilities, relation (2-3) yields $P_{E} = \frac{3}{8} \left(1 - |d|^2 \right) + \frac{1}{8} |d|^2$. Given that $0 \leq |d|^2 \leq 1$, the minimum value of this expression is $\frac{1}{8}$; it corresponds to $|d|^2 = 1$. Therefore using the singlet state is the optimal strategy.
The probability given by relation (2-3) is displayed in Fig. 3 as a function of $\eta_1$, for the case $|d|^2 = \frac{1}{2}$.

This corresponds to a “half symmetric – half antisymmetric” input state.

FIG. 3: Minimum Error Probability versus the prior probability with which the transformations are identical; case where the input state is a superposition state $\Psi = a\phi^+ + b\phi^- + c\psi^+ + d\psi^-$ such that $|d|^2 = \frac{1}{2}$. 

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Part 2: Unambiguous Discrimination of Two Unknown Unitary Transformations with Multiple Copies of Data

Chapter 3: Discrimination with two copies of data

3.1: Problem formulation

Let $X$ and $Y$ be the pair of unknown unitary transformations that we want to discriminate. Let $U$ be another (also unknown) unitary transformation, which is guaranteed to be identical with either $X$ or $Y$. If we have two copies of $U$, we can construct the transformation $XUUY$, which is an operator in the 16-dimensional Hilbert space of a 4-qubit system. Denoting the qubits by $A$, $B$, $C$, and $D$ we prepare the state $|\Psi^+\rangle_{AB} \otimes |\Psi^+\rangle_{CD}$ which we use as our input state, where $|\Psi^+\rangle_{AB}$ and $|\Psi^+\rangle_{CD}$ are the singlet Bell states in the 4-dimensional Hilbert spaces of sub-systems $(AB)$ and $(CD)$ respectively. The action of the transformation $XUUY$ on the above input state results in the output state

$$|\Omega\rangle = X^A U^B U^C Y^D |\Psi^+\rangle_{AB} \otimes |\Psi^+\rangle_{CD}. \quad (3-1)$$

If $U$ is identical with $X$, then $|\Omega\rangle$ is identical with

$$|\Omega_X\rangle = X^A X^B X^C Y^D |\Psi^+\rangle_{AB} \otimes |\Psi^+\rangle_{CD}. \quad (3-2)$$
Similarly, if $U$ is identical with $Y$, then $\Omega$ is identical with

$$\left| \Omega_Y \right\rangle = X^A Y^B Y^C Y^D \left| \Psi^- \right\rangle_{AB} \otimes \left| \Psi^- \right\rangle_{CD}. \tag{3-3}$$

The discrimination of the unitary transformations $X$ and $Y$ is equivalent to the discrimination of the states $\left| \Omega_X \right\rangle$ and $\left| \Omega_Y \right\rangle$. The prior probability $\eta_1$, with which $U$ is identical with $X$, can be thought of as the prior probability with which $\left| \Omega \right\rangle$ is identical with $\left| \Omega_X \right\rangle$. Similarly, the prior probability

$$\eta_2 = 1 - \eta_1,$$

with which $U$ is identical with $Y$, can be thought of as the prior probability with which $\left| \Omega \right\rangle$ is identical with $\left| \Omega_Y \right\rangle$.

### 3.2: Solution method

We design a measurement procedure with three possible results, $r_x$, $r_y$, and $r_i$ meaning the following:

- when we get the first result we infer that the transformation $U$ is the same as the transformation $X$;
- when we get the second result we infer that the transformation $U$ is the same as the transformation $Y$; the third result is inconclusive.

The measurement procedure consists of expanding the output state $\left| \Omega \right\rangle$ in the double Bell basis; the expansion has the form
\[ |\Omega\rangle = a_{11} |\Phi^+\rangle_{AB} \otimes |\Phi^+\rangle_{CD} + a_{12} |\Phi^+\rangle_{AB} \otimes |\Phi^-\rangle_{CD} + a_{13} |\Phi^+\rangle_{AB} \otimes |\Psi^+\rangle_{CD} + a_{14} |\Phi^+\rangle_{AB} \otimes |\Psi^-\rangle_{CD} + a_{21} |\Phi^-\rangle_{AB} \otimes |\Phi^+\rangle_{CD} + a_{22} |\Phi^-\rangle_{AB} \otimes |\Phi^-\rangle_{CD} + a_{23} |\Phi^-\rangle_{AB} \otimes |\Psi^+\rangle_{CD} + a_{24} |\Phi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD} + a_{31} |\Psi^+\rangle_{AB} \otimes |\Phi^+\rangle_{CD} + a_{32} |\Psi^+\rangle_{AB} \otimes |\Phi^-\rangle_{CD} + a_{33} |\Psi^+\rangle_{AB} \otimes |\Psi^+\rangle_{CD} + a_{34} |\Psi^+\rangle_{AB} \otimes |\Psi^-\rangle_{CD} + a_{41} |\Psi^-\rangle_{AB} \otimes |\Phi^+\rangle_{CD} + a_{42} |\Psi^-\rangle_{AB} \otimes |\Phi^-\rangle_{CD} + a_{43} |\Psi^-\rangle_{AB} \otimes |\Psi^+\rangle_{CD} + a_{44} |\Psi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD} \]

Conservation of probability requires that \( \sum_{i,j=1}^{4} |a_{ij}|^2 = 1 \).

The 16 basis vectors in this expansion represent elementary measurement outcomes. We can classify these basis vectors into three groups, so that one group will represent the measurement result \( r_X \), another group will represent the measurement result \( r_Y \), and the third group will represent the measurement result \( r_I \). What makes this classification possible is the following property of the singlet state: when identical operators are applied on its two parts, the singlet state is transformed into itself (see Appendix). The classification is as follows.

Group 1

The first group of basis vectors is composed of the three vectors \( |\Psi^-\rangle_{AB} \otimes |\Phi^+\rangle_{CD} \), \( |\Psi^-\rangle_{AB} \otimes |\Phi^-\rangle_{CD} \), and \( |\Psi^-\rangle_{AB} \otimes |\Psi^+\rangle_{CD} \). It is easy to realize that this group represents the measurement result \( r_X \).

In these three basis vectors, the state of subsystem (AB) has remained the same (the singlet state) as in the input state \( |\Psi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD} \), but the state of subsystem (CD) has changed.
Given relation (1), that is, $|\Omega\rangle = X^A U^B U^C Y^D |\Psi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD}$, the change in the state of subsystem (CD) indicates that the operators $U$ and $Y$ are different. Otherwise, the state of subsystem (CD) would have remained the same (the singlet state) as in the input state.

Thus the above three basis vectors, as measurement results, indicate that the operators $U$ and $Y$ are different. Equivalently, these three elementary measurement outcomes indicate that the operators $U$ and $X$ are identical (because $U$ is guaranteed to be identical with either $X$ or $Y$). Therefore, the measurement result $r_X$ is represented by the above three elementary outcomes.

Group 2

The second group of basis vectors is composed of the vectors $|\Phi^+\rangle_{AB} \otimes |\Psi^-\rangle_{CD}$, $|\Phi^+\rangle_{AB} \otimes |\Psi^-\rangle_{CD}$, and $|\Psi^+\rangle_{AB} \otimes |\Psi^-\rangle_{CD}$. In these basis vectors, the state of subsystem (CD) has remained the same (the singlet state) as in the input state $|\Psi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD}$, but the state of subsystem (AB) has changed. Clearly, this group represents the measurement result $r_Y$ (the argument is similar to the one conducted above).

Group 3

The third group of basis vectors is composed of the remaining 10 basis vectors. This group represents the inconclusive result $r_I$. 
Using the coefficients in the above expansion of the output state $|\Omega\rangle$, we can express the probabilities of our conclusive measurement results as follows:

$$P(r_X) = |a_{41}|^2 + |a_{42}|^2 + |a_{43}|^2$$

(3-4)

and

$$P(r_Y) = |a_{14}|^2 + |a_{24}|^2 + |a_{34}|^2$$

(3-5)

From conservation of probability, it follows that the probability of the inconclusive result is

$$P(r_I) = 1 - P(r_X) - P(r_Y).$$

(3-6)

### 3.3: Use of the conclusive probabilities as conditional probabilities

The probabilities given by relations (3-4) and (3-5) represent conditional probabilities. The former is the conditional probability of finding result $r_X$ given that $U = X$, and the latter is the conditional probability of finding result $r_Y$ given that $U = Y$.

In fact, if it is given that $U = X$, that is, that $|\Omega\rangle$ is identical with

$$|\Omega_X\rangle = X^AX^BX^C|\Psi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD}.$$ 

then the above expansion of $|\Omega\rangle$ reduces to 4 terms.

This is because, in this case, the state of subsystem (AB) will necessarily remain the same (the singlet
state $|\Psi^-\rangle_{AB}$, because the operators acting on its two parts are identical. This means that, in the above expansion of $|\Omega\rangle$, every basis vector in which the first tensor factor is not $|\Psi^-\rangle_{AB}$ must have a zero coefficient. Consequently, the above expansion reduces to

$$|\Omega\rangle = \sum_{j=1}^{4} a_{4j} |\Psi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD} + a_{42} |\Psi^-\rangle_{AB} \otimes |\Phi^-\rangle_{CD} + a_{43} |\Psi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD} + a_{44} |\Psi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD}$$

where $\sum_{j=1}^{4} |a_{4j}|^2 = 1$.

With this reduced expansion the probability of finding result $r_X$ is still given by relation (3-4), but now it is a conditional probability: the conditional probability of finding result $r_X$ given that $U = X$.

Similarly, if it is given that $U = Y$, that is, that $|\Omega\rangle$ is identical with

$$|\Omega_Y\rangle = X^A Y^B Y^C Y^D |\Psi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD}$$

then the state of subsystem (CD) will necessarily remain the same (the singlet state $|\Psi^-\rangle_{CD}$), because the operators acting on its two parts are identical. Thus, in the expansion of $|\Omega\rangle$, every basis vector in which the second tensor factor is not $|\Psi^-\rangle_{CD}$ must have a zero coefficient. Consequently, in this case the expansion of $|\Omega\rangle$ reduces to

$$|\Omega\rangle = a_{14} |\Phi^+\rangle_{AB} \otimes |\Psi^-\rangle_{CD} + a_{24} |\Phi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD} + a_{34} |\Psi^+\rangle_{AB} \otimes |\Psi^-\rangle_{CD} + a_{44} |\Psi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD}$$

where $\sum_{i=1}^{4} |a_{4i}|^2 = 1$.

With this expansion the probability of finding result $r_Y$ is still given by relation (3-5), but it is now a conditional probability: the conditional probability of finding result $r_Y$ given that $U = Y$.  

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3.4: Average values of the conditional probabilities

Consider the above reduced expansion of \( |\Omega\rangle \) corresponding to the condition that \( U = X \); that is,

\[
|\Omega\rangle = |\Omega_X\rangle = a_{41}|\Psi^-\rangle_{AB} \otimes |\Phi^+\rangle_{CD} + a_{42}|\Psi^-\rangle_{CD} \otimes |\Phi^+\rangle_{AB} + a_{43}|\Psi^-\rangle_{CD} \otimes |\Psi^+\rangle_{AB} + a_{44}|\Psi^-\rangle_{CD} \otimes |\Psi^-\rangle_{AB}
\]

where \( \sum_{j=1}^{4} |a_{4j}|^2 = 1 \).

Assuming that the four possible outcomes are equally likely, that is, that on average

\[
|a_{41}|^2 = |a_{42}|^2 = |a_{43}|^2 = |a_{44}|^2 = \frac{1}{4},
\]

the average value of the conditional probability \( P(r_X) \) is

\[
\langle P(r_X) \rangle = \langle |a_{41}|^2 \rangle + \langle |a_{42}|^2 \rangle + \langle |a_{43}|^2 \rangle = \frac{3}{4}.
\] (3-7)

Similarly, using the above reduced expansion of \( |\Omega\rangle \) corresponding to the condition that \( U = Y \), that is,

\[
|\Omega\rangle = |\Omega_Y\rangle = a_{14}|\Phi^+\rangle_{AB} \otimes |\Psi^-\rangle_{CD} + a_{24}|\Phi^-\rangle_{CD} \otimes |\Psi^-\rangle_{AB} + a_{34}|\Psi^+\rangle_{CD} \otimes |\Psi^-\rangle_{AB} + a_{44}|\Psi^-\rangle_{CD} \otimes |\Psi^-\rangle_{AB}
\]

where \( \sum_{j=1}^{4} |a_{ij}|^2 = 1 \)

and assuming that the possible outcomes are equally likely, we have

\[
\langle P(r_Y) \rangle = \langle |a_{14}|^2 \rangle + \langle |a_{24}|^2 \rangle + \langle |a_{34}|^2 \rangle = \frac{3}{4}.
\] (3-8)
3.5: Measurement operators and average success probability

Following the usual notations, we introduce measurement operators $\Pi_X$ and $\Pi_Y$ such that

$$P(r_X) = \langle \Omega_X | \Pi_X | \Omega_X \rangle$$

(3-9)

and

$$P(r_Y) = \langle \Omega_Y | \Pi_Y | \Omega_Y \rangle$$

(3-10)

With these notations, the average success probability has the expression

$$P_s = \eta_1 \langle \Omega_X | \Pi_X | \Omega_X \rangle + \eta_2 \langle \Omega_Y | \Pi_Y | \Omega_Y \rangle$$

(3-11)

and the average failure probability is $P_f = 1 - P_s$.

Note that in this unambiguous discrimination strategy we have the double constraint

$$\langle \Omega_X | \Pi_Y | \Omega_X \rangle = \langle \Omega_Y | \Pi_X | \Omega_Y \rangle = 0$$

(3-12)

The above four-term expansions of $|\Omega_X\rangle$ and $|\Omega_Y\rangle$, together with relations (3-4), (3-5), (3-9), (3-10), and (3-12) lead to the following expressions for the measurement operators:

$$\Pi_X = |\Psi^-\rangle_{AB} \otimes |\Phi^+\rangle_{CD} \langle \Phi^+ | \otimes_{AB} \langle \Psi^- | + |\Psi^-\rangle_{AB} \otimes |\Phi^-\rangle_{CD} \langle \Phi^- | \otimes_{AB} \langle \Psi^- | +$$

$$|\Psi^-\rangle_{AB} \otimes |\Psi^+\rangle_{CD} \langle \Psi^+ | \otimes_{AB} \langle \Psi^- |$$

(3-13)
\[ \Pi_Y = \left| \Phi^+ \right\rangle_{AB} \otimes \left| \Psi^- \right\rangle_{CD} \left\langle \Psi^- \right| \otimes_{AB} \left( \Phi^+ \right)_A \otimes \left| \Psi^- \right\rangle_{CD} \left\langle \Phi^- \right| \otimes_{AB} \left( \Psi^+ \right) \] 

\[ \Pi_I = I - \Pi_X - \Pi_Y \] (3-15)

The average success probability is

\[ \left\langle P_s \right\rangle = \eta_1 \left\langle \Omega_X | \Pi_X | \Omega_X \right\rangle + \eta_2 \left\langle \Omega_Y | \Pi_Y | \Omega_Y \right\rangle \] (3-16)

Using relations (3-7), (3-8), (3-9), and (3-10) we find that \( \left\langle P_s \right\rangle = \frac{3}{4} (\eta_1 + \eta_2) = \frac{3}{4} \) (3-17)

3.6: Average values of the conditional probabilities using Haar integrals

Consider the expression \( |\Omega\rangle = X^A U^B Y^C |\Psi^- \rangle_{AB} \otimes |\Psi^- \rangle_{CD} \) of the output state. That is,

\[ |\Omega\rangle = X^A U^B U^C Y^D \left[ \frac{1}{\sqrt{2}} \left( |0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B \right) \right] \otimes \left[ \frac{1}{\sqrt{2}} \left( |0\rangle_C \otimes |1\rangle_D - |1\rangle_C \otimes |0\rangle_D \right) \right] = \]

\[ \frac{1}{2} X^A U^B U^C Y^D \left( |0\rangle_A \otimes |1\rangle_B \otimes |0\rangle_C \otimes |1\rangle_D - |0\rangle_A \otimes |1\rangle_B \otimes |1\rangle_C \otimes |0\rangle_D - |1\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C \otimes |1\rangle_D + \right) \]

This can be expressed as follows:

\[ |\Omega\rangle = \frac{1}{2} \left( X^A |0\rangle_A \otimes U^B |0\rangle_B \otimes U^C |0\rangle_C \otimes Y^D |1\rangle_D - X^A |0\rangle_A \otimes U^B |1\rangle_B \otimes U^C |1\rangle_C \otimes Y^D |0\rangle_D \right) \]

\[ - X^A |1\rangle_A \otimes U^B |0\rangle_B \otimes U^C |0\rangle_C \otimes Y^D |1\rangle_D + X^A |1\rangle_A \otimes U^B |1\rangle_B \otimes U^C |1\rangle_C \otimes Y^D |0\rangle_D \]
Relation (1-1) indicates that any 2X2 unitary matrix \( U \) can be expressed in the form

\[
U = \begin{pmatrix}
\cos \frac{\alpha}{2} - i \sin \frac{\alpha}{2} \cos \theta & -\sin \frac{\alpha}{2} \sin \theta (\sin \varphi + i \cos \varphi) \\
\sin \frac{\alpha}{2} \sin \theta (\sin \varphi - i \cos \varphi) & \cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2} \cos \theta
\end{pmatrix}
\]

In the above expression of \( \Omega \), we will do the following;

1. express in this form the matrices of the transformations \( X, Y, \) and \( U \). Each of the parameters \( \alpha, \theta, \) and \( \phi \) will have an index indicating the corresponding matrix, as was done for the matrices of \( V \) and \( W \) in section... . In the matrix of \( X \), for example, the parameters will appear as \( X_\alpha, X_\theta, \) and \( X_\phi \):

2. use the notations

\[
a_X = \cos \frac{\alpha_X}{2} - i \sin \frac{\alpha_X}{2} \cos \theta_X
\]

and

\[
b_X = \sin \frac{\alpha_X}{2} \sin \theta_X (\sin \phi_X + i \cos \phi_X),
\]

and similar notations for the matrices of the transformations \( Y \) and \( U \);

3. replace the vectors \( |0\rangle \) and \( |1\rangle \) by their matrix forms \( |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

These changes, together with some algebra, yield an expansion of \( \Omega \) in the 16-dimensional computational basis. This expansion, whose coefficients depend on \( a_X, b_X, a_Y, b_Y, a_U, \) and \( b_U \) and their complex conjugates, is of the form
\[|\Omega\rangle = \sum_{j=1}^{16} C_j \left( a_x, b_x, a_y, b_y, a_u, b_u, a_x^*, b_x^*, a_y^*, b_y^*, a_u^*, b_u^* \right) |j\rangle \] (3-20)

Where \[|j\rangle = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C \otimes |0\rangle_D, \quad |0\rangle_A \otimes |0\rangle_B \otimes |1\rangle_C \otimes |1\rangle_D, \ldots, |1\rangle_A \otimes |1\rangle_B \otimes |1\rangle_C \otimes |1\rangle_D\]

and \[\sum_{j=1}^{16} |C_j|^2 = 1\]

From expansion (3-20), we find the expansion of \[|\Omega\rangle\] in the double Bell basis by using the transformation formulas

\[|0\rangle_A \otimes |0\rangle_B = \frac{1}{\sqrt{2}} \left( |\Phi^+\rangle_{AB} + |\Phi^-\rangle_{AB} \right), \quad |1\rangle_A \otimes |1\rangle_B = \frac{1}{\sqrt{2}} \left( |\Phi^+\rangle_{AB} - |\Phi^-\rangle_{AB} \right),\]

\[|0\rangle_A \otimes |1\rangle_B = \frac{1}{\sqrt{2}} \left( |\Psi^+\rangle_{AB} + |\Psi^-\rangle_{AB} \right), \quad |1\rangle_A \otimes |0\rangle_B = \frac{1}{\sqrt{2}} \left( |\Psi^+\rangle_{AB} - |\Psi^-\rangle_{AB} \right),\]

and similar formulas obtained by substituting \(C\) and \(D\) for \(A\) and \(B\). This new expansion has the form

\[|\Omega\rangle = a_{11} |\Phi^+\rangle_{AB} \otimes |\Phi^+\rangle_{CD} + a_{12} |\Phi^+\rangle_{AB} \otimes |\Phi^-\rangle_{CD} + a_{13} |\Phi^+\rangle_{AB} \otimes |\Psi^+\rangle_{CD} + a_{14} |\Phi^+\rangle_{AB} \otimes |\Psi^-\rangle_{CD} + a_{21} |\Phi^-\rangle_{AB} \otimes |\Phi^+\rangle_{CD} + a_{22} |\Phi^-\rangle_{AB} \otimes |\Phi^-\rangle_{CD} + a_{23} |\Phi^-\rangle_{AB} \otimes |\Psi^+\rangle_{CD} + a_{24} |\Phi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD} + a_{31} |\Psi^\rangle_{AB} \otimes |\Phi^+\rangle_{CD} + a_{32} |\Psi^\rangle_{AB} \otimes |\Phi^-\rangle_{CD} + a_{33} |\Psi^\rangle_{AB} \otimes |\Psi^+\rangle_{CD} + a_{34} |\Psi^\rangle_{AB} \otimes |\Psi^-\rangle_{CD} + a_{41} |\Psi^-\rangle_{AB} \otimes |\Phi^+\rangle_{CD} + a_{42} |\Psi^-\rangle_{AB} \otimes |\Phi^-\rangle_{CD} + a_{43} |\Psi^-\rangle_{AB} \otimes |\Psi^+\rangle_{CD} + a_{44} |\Psi^-\rangle_{AB} \otimes |\Psi^-\rangle_{CD}\]

and the coefficients \(a_{ij}\) are functions of \(a_x, b_x, a_y, b_y, a_u, b_u\), and their complex conjugates.
According to relations (3-4) and (3-5), the probabilities associated with the conclusive measurement results are \( P(r_x) = |a_{41}|^2 + |a_{42}|^2 + |a_{43}|^2 \) and \( P(r_y) = |a_{14}|^2 + |a_{24}|^2 + |a_{44}|^2 \). In these relations, the right-hand sides are functions of \( \alpha_a, \beta_a, \alpha_b, \beta_b, \alpha_u, \beta_u \) and their complex conjugates. Using relations (3-18) and (3-19), and the similar relations for \( Y \) and \( U \), we transform the right-hand sides of relations (3-4) and (3-5) so that they become functions of the angular parameters \( \alpha_x, \theta_x, \phi_x, \alpha_y, \theta_y, \phi_y, \alpha_u, \theta_u, \phi_u \). Then we average the terms in those right-hand sides by using Haar integrals. Since the sets of variables \( (\alpha_x, \theta_x, \phi_x), (\alpha_y, \theta_y, \phi_y), \) and \( (\alpha_u, \theta_u, \phi_u) \) are independent, the average value of each term is obtained as a product of three Haar integrals, each integral being of the form

\[
\langle C(\alpha, \theta, \phi) \rangle = \frac{1}{4\pi^2} \int_0^{2\pi} d\alpha \int_0^\pi d\theta \int_0^{2\pi} d\phi C(\alpha, \theta, \phi) \sin^2 \frac{\alpha}{2} \sin \theta
\]

It is found that \( \langle |a_{41}|^2 \rangle = \langle |a_{42}|^2 \rangle = \langle |a_{43}|^2 \rangle = \langle |a_{14}|^2 \rangle = \langle |a_{24}|^2 \rangle = \langle |a_{44}|^2 \rangle = \frac{1}{4} \). These results confirm relations (3-7) and (3-8).
4.1: Generalization

The argument used above, to find the conditional probabilities, can be extended to the case of an arbitrary number of data copies, provided that this number is even. Below we are using a “crude” approach, which results in the same conditional probabilities as in the case of two copies of data. Clearly, with such an approach the use of many copies of data does not pay. A different, rewarding approach is used in [47].

4.2: Problem formulation and solution

Let \( N = 2n \) be the number of copies of \( U \). As our input state we will choose the tensor product of \( n + 1 \) singlet states. We will transform this input state with the tensor product of \( 2n + 2 \) one-qubit operators, with \( X \) and \( Y \) being respectively the first and last operators, and the \( 2n \) copies of \( U \) in between. Thus the output state will be as follows:

\[
|\Omega\rangle = X^{A_1} U^{A_5} U^{A_9} \ldots U^{A_{2n-1}} U^{A_{2n}} U^{A_{2n+1}} Y^{A_{2n+2}} |\Psi^+\rangle_{A_1 A_3} \otimes |\Psi^-\rangle_{A_3 A_4} \otimes \ldots \otimes |\Psi^-\rangle_{A_{2n-1} A_{2n}} \otimes |\Psi^-\rangle_{A_{2n+1} A_{2n+2}}
\]

The states of the \( n - 1 \) two-qubit subsystems \((A_3 A_4), \ldots, (A_{2n-1} A_{2n})\) will always remain the same because, for each of these states, the operators acting on the two parts are identical.
If it is given that \( U = X \) then the state of subsystem \((A_1, A_2)\) will remain the same (in addition to the states of the above mentioned \( n - 1 \) subsystems), so that there will be only four possible outcomes for the output state. These four outcomes correspond to the four possibilities (the four Bell states) for the state of subsystem \((A_{2n+1}, A_{2n+2})\). Out of these four outcomes, three are conclusive. The inconclusive outcome is the one where the state of subsystem \((A_{2n+1}, A_{2n+2})\) is also the singlet. It follows that the (conditional) probability of success is \( \frac{3}{4} \).

If it is given that \( U = Y \), the argument is similar, leading to the same result.

Thus the above approach does not yield anything (it does not increase the success probability). As indicated above, a different approach is used in [47].
Conclusions

We have considered the discrimination between two unknown unitary transformations, the minimum-error strategy based comparison of such transformations, and their unambiguous discrimination with multiple copies of data. In these situations, where no knowledge about the individual transformations is available, we solve problems by using symmetry and averaging.

In this work we abundantly used Bell states as our input states. The reason for this preference is that these states are maximally entangled. In fact, studies have shown that the use of entangled states as input states, when discriminating among unitary transformations, leads to better results [43].

We began with the discrimination problem. We considered, on one hand, two unknown unitary transformations, V and W, which were guaranteed to be different; and on the other hand, a third unitary transformation U. The transformation U was also unknown, but was guaranteed to be the same as either V or W, with different prior probabilities. Our task was to find out which of V and W was the same as U. This operator discrimination problem, somewhat surprisingly, turned out to have the same results as the analogous problem with quantum states. The latter has already been treated.

We then discussed the comparison of two unknown unitary transformations. With no knowledge about the individual transformations, we were interested in determining whether the transformations were the same or different. We used the minimum-error strategy, as the unambiguous strategy for this problem has already been treated. Our results were obtained in the general case of different prior probabilities.

The strategy used for unambiguous discrimination with multiple copies of data led to a very high success probability, for the case of two copies of data. We are currently studying the case of N copies of data.
We have treated the discrimination and comparison of 2 by 2 unitary transformations. Extensions to the present work include the treatment of two higher-order transformations.
Appendix

Below we show the following fact: when the operators acting on the two parts of the singlet state \( |\Psi^- \rangle \) are identical, the state is transformed into itself.

Consider two unitary transformations \( V = e^{i\phi} \begin{pmatrix} a_v & -b_v \\ b_v^* & a_v^* \end{pmatrix} \) and \( W = e^{i\phi_w} \begin{pmatrix} a_w & -b_w \\ b_w^* & a_w^* \end{pmatrix} \). Let us express the state \( VW|\Psi^- \rangle = VW\left[ \frac{1}{\sqrt{2}} (|0\rangle|1\rangle - |1\rangle|0\rangle) \right] \) in the Bell basis.

In the above notations, by \( VW \) we mean the tensor product of these operators (not the ordinary operator product, since \( V \) and \( W \) are 2-dimensional operators while \( |\Psi^- \rangle \) is a 4-dimensional vector).

Using the above matrices, and replacing the vectors \( |0\rangle \) and \( |1\rangle \) by their matrix forms \( |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), we have (omitting the factor \( \frac{1}{\sqrt{2}} e^{i(\phi + \phi_w)} \)):

\[
VW|\Psi^- \rangle = \left( a_v \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes a_w \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) - \left( b_v^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes a_w \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \left( a_v \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes b_v \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) - \left( b_v^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes a_w \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = (a_v |0\rangle + b_v^* |1\rangle) \otimes (b_v |0\rangle + a_v^* |1\rangle) - (b_v^* |0\rangle + a_v |1\rangle) \otimes (a_v |0\rangle + b_v |1\rangle) -
\]

\[
= \left( -b_v |0\rangle + a_v |1\rangle \right) \otimes (a_v |0\rangle + b_v^* |1\rangle) = \left( b_v a_w - a_v b_w \right) |0\rangle \otimes |0\rangle + \left( a_v a_w^* + b_v b_w^* \right) |0\rangle \otimes |1\rangle -
\]

\[
\left( b_v^* b_w + a_v^* a_w \right) |1\rangle \otimes |0\rangle + \left( b_v^* a_w^* - a_v^* b_w \right) |1\rangle \otimes |1\rangle . \]

That is,

\[
VW|\Psi^- \rangle = \left( b_v a_w - a_v b_w \right) |0\rangle \otimes |0\rangle + \left( a_v a_w^* + b_v b_w^* \right) |0\rangle \otimes |1\rangle -
\]

\[
\left( b_v^* b_w + a_v^* a_w \right) |1\rangle \otimes |0\rangle + \left( b_v^* a_w^* - a_v^* b_w \right) |1\rangle \otimes |1\rangle
\]

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Now, suppose that the two operators are identical. Setting $V = W$ in the last equation above yields

$$VV|\Psi^-\rangle = \left( |a_r|^2 + |b_r|^2 \right) \left( |0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle \right)$$

Noting that $|a_r|^2 + |b_r|^2 = 1$ (since $V$ is unitary), and taking into account the factor $\frac{1}{\sqrt{2}} e^{i(\phi_r + \phi_w)}$ that was omitted above, we have $VV|\Psi^-\rangle = e^{i(\phi_r + \phi_w)} |\Psi^-\rangle$. The right-hand side of this equation is the same physical state as $|\Psi^-\rangle$, because a phase factor has no physical significance.
References


