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The Maximum Rectilinear Crossing Number of the Wheel Graph

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Abstract

We find and prove the maximum rectilinear crossing number of the *wheel graph*. First, we illustrate a picture of the wheel graph with many crossings to prove a lower bound. We then prove that this bound is sharp. The treatment is divided into two cases for n even and n odd.

1 Introduction

A *drawing* of the graph G with vertex set $V(G)$ and edge set $E(G)$ is defined as a representation of G in a plane such that the elements of $V(G)$ correspond to points in the plane and the elements of $E(G)$ correspond to continuous arcs. We assume that each arc connects two vertices and that any pair of arcs has at most one point in common, either a vertexpoint or a crossing. A *rectilinear drawing* is a drawing of a graph in which all edges are represented as straight line segments in the plane. A *crossing* is defined to be the intersection of exactly two edges not at a vertex. The *crossing number* of an abstract graph G, denoted $cr(G)$, is defined as the minimum number of edge crossings over all nonisomorphic drawings of G. The *rectilinear crossing number* of a graph G, denoted $\overline{cr}(G)$, is defined as the minimum number of edge crossings over all nonisomorphic rectilinear drawings of G. Analogously, the *maximum crossing number*, denoted by $\mathrm{CR}(G)$, is defined as the maximum number of edge crossings over all nonisomorphic drawings of G. The *maximum rectilinear crossing number* of a graph G, $\overline{\mathrm{CR}}(G)$, is defined as the maximum number of crossings over all nonisomorphic rectilinear drawings of G.

The maximum crossing number and maximum rectilinear crossing number have been studied for several classes of graphs (see [1–10]). Most relevant to this paper are studies of the maximum rectilinear crossing number of C_n and of $R_{n,d}$, the class of *d-regular* graphs of order n, i.e., graphs where each of the n vertices has degree d.

It has been shown in [3] that

$$\overline{\mathrm{CR}}(C_n) = \begin{cases} \frac{1}{2}n(n-3) & \text{if } n \text{ is odd,} \\ \frac{1}{2}n(n-4)+1 & \text{if } n \text{ is even.} \end{cases}$$

The maximum crossing number for the wheel graph is discussed in [8]. This paper finds the maximum rectilinear crossing number for the wheel graph. The wheel graph W_n consists of a cycle of order n together with an additional *central* vertex which is connected to every vertex of the cycle by edges or *spokes* (See Figure 1).

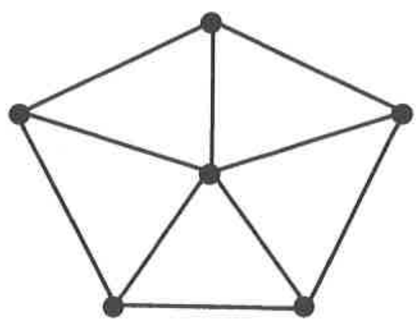

Figure 1: The graph W_5.

Section 2 illustrates drawings of the wheel graph for even and for odd n and thereby finds a lower bound for the maximum rectilinear crossing number of the wheel graph. Section 3 proves that this bound is sharp. Section 4 concludes with a conjecture motivated by the results in this paper.

2 Lower Bound for $\overline{CR}(W_n)$

In this section we construct drawings of W_n for n even and for n odd with many crossings. The number of crossings in these drawings provide us with a lower bound for the maximum rectilinear crossing number of the wheel graph.

Proposition 2.1.

$$\overline{\mathrm{CR}}(W_n) \geq \begin{cases} \frac{2n^2-5n-1}{2} & \text{if } n \text{ is odd,} \\ n^2-3n+1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. We begin with the case of odd n. Draw n vertices on a convex n-gon. Beginning at any vertex, draw diagonals of length $\frac{n-1}{2}$ until returning to the starting vertex. This yields a drawing of C_n with $\frac{n(n-3)}{2}$ crossings. Next, place another vertex on the boundary of the convex n-gon and connect this vertex to the n vertices of the cycle to complete the drawing of W_n (see Figure 2).

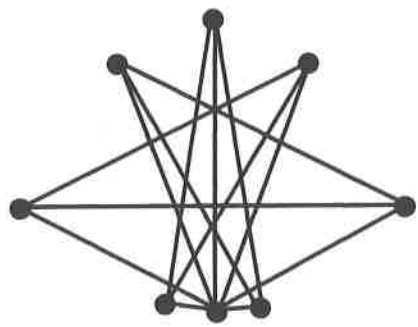

Figure 2: Drawing of W_7 with 31 crossings. The central vertex is the bottom, center vertex.

We now count how many new crossings are added by the n additional edges. For $i = 0, \ldots, \frac{n-3}{2}$ the 2 edges connecting the central vertex to the vertices whose distance around the n-gon are $i+1$ contribute $2i$ crossings each. Finally, the edge connecting the central vertex to its opposite vertex contributes $n-2$ crossings. Thus, the total number of crossings in this drawing is

$$\begin{aligned} & 2\left(\sum_{i=0}^{(n-3)/2} 2i\right) + (n-2) + \frac{n(n-3)}{2} = \\ = \quad & \frac{(n-3)(n-1)}{2} + (n-2) + \frac{n(n-3)}{2} = \\ = \quad & \frac{2n^2-5n-1}{2} \end{aligned}$$

We now construct a drawing for even n. Again, start with n vertices of a convex n-gon. Starting with any vertex (always moving clockwise around the n-gon) draw diagonals of the following lengths : $\frac{n-2}{2}$ diagonals of length $\frac{n-2}{2}$, one diagonal of length $\frac{n}{2}$, $\frac{n-2}{2}$ diagonals of length $\frac{n+2}{2}$, and one diagonal of length $\frac{n}{2}$. We thus have a drawing of C_n with $\frac{n(n-4)}{2}+1$ crossings. Then add a central point on the n-gon in between two adjacent endvertices of the diagonals of length $\frac{n}{2}$. Connect this new vertex to the n vertices of the cycle to get a drawing of W_n (see Figure 3).

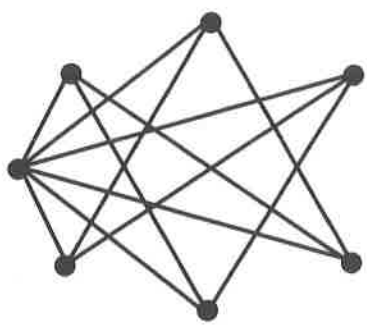

Figure 3: Drawing of W_6 with 19 crossings. The central vertex is the leftmost vertex.

We now consider how many crossings are added by the n new edges. For $i = 0, \ldots, \frac{n-2}{2}$ the 2 edges connecting the central vertex to the vertices whose distance around the n-gon are $i+1$ contribute $2i$ crossings each. We thus have the total number of crossings in this drawing is

$$
\begin{aligned}
& 2\left(\sum_{i=0}^{\frac{n-2}{2}} 2i\right) + \frac{n(n-4)}{2} + 1 \\
= \quad & \frac{n(n-2)}{2} + \frac{n(n-4)}{2} + 1 \\
= \quad & n^2 - 3n + 1
\end{aligned}
$$

□

3 Upper Bound for $\overline{CR}(W_n)$

In this section we prove that the lower bound obtained in Proposition 2.1 is sharp. The following exact value of $\overline{\text{CR}}(W_n)$ will thus be proved.

Proposition 3.1.

$$\overline{\text{CR}}(W_n) = \begin{cases} \frac{2n^2-5n-1}{2} & \textit{if } n \textit{ is odd,} \\ n^2 - 3n + 1 & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. The lower bound follows from Proposition 2.1, so we proceed by proving that this expression is an upper bound. Note that W_n has $2n$ edges, n of which are on the cycle and n of which are spokes. Each edge on the cycle cannot intersect itself nor the 4 edges adjacent to it. Each spoke cannot intersect itself nor the $n-1$ other spokes nor the 2 edges of the cycle which it is adjacent to. Thus, a first upper bound is

$$\overline{\text{CR}}(W_n) \leq \frac{n(2n-5) + n(n-2)}{2} = \frac{3n^2 - 7n}{2}.$$

For W_n where n is even there are additional missed crossings due to the even cycle. In [3], [6] it is proved that

$$\overline{\text{CR}}(C_n) = \frac{n(n-4)}{2} + 1$$

for n even. Since C_n has $\frac{n(n-3)}{2}$ nonadjacent edge pairs which can potentially cross, there are

$$\frac{n(n-3)}{2} - (\frac{n(n-4)}{2} + 1) = \frac{n-2}{2}$$

additional missed crossings. We thus have an upper bound of $\frac{1}{2}(3n^2-8n-2)$ for n even.

The central vertex of W_n is an endvertex for n edges. Let such an endvertex be of type i if the edge incident to it divides the drawing of the graph into two halfplanes, one containing i edges emanating from this vertex, and the other containing $n-i-1$ edges emanating from the same vertex (see Figure 4). By symmetry we only consider $0 \leq i \leq \lfloor\frac{1}{2}(n-1)\rfloor = N$.

Let s be the smallest type of endvertex for this central point in a given drawing. Let y_i be the number of central endvertices of type i. Thus, we have $\sum_{i=s}^{N} y_i = n$.

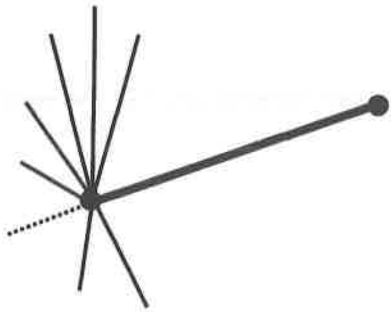

Figure 4: The left endvertex of the bold edge is of type 2 because the *smaller* halfplane determined by this edge contains 2 edges emanating from this vertex.

Similarly each vertex of the cycle in W_n is an endvertex of a spoke. Let such an endvertex be of type j if the edge incident to it divides the drawing of the graph into two halfplanes, one containing j edges emanating from this vertex, and the other containing $2-j$ edges emanating from the same vertex.

We call an edge with i edges from the central endvertex in one halfplane and j edges from the endvertex of the cycle in the same halfplane a type i,j edge. Let $x_{i,j}$ count the number of type i,j edges (see Figure 5).

Notice that y_i is related to $x_{i,j}$ by the following equation:

$$y_i = x_{i,0} + x_{i,1} \tag{1}$$

For a type i,j edge, the i edges in the halfplane of one endvertex cannot intersect the $2-j$ edges in the opposite halfplane emanating from the other endvertex (see Figure 5). The same holds true for the j edges in the halfplane of one endvertex and the $n-i-1$ edges in the opposite halfplane emanating from the other endvertex. Therefore, a given type i,j edge determines $i(2-j)+j(n-i-1)$ pairs of nonintersecting edges. A drawing which maximizes the number of edge crossings should minimize the number M of pairs of nonintersecting edges.

Note that it is true that for a given type i,j edge it may be that the i edges from one endvertex and the j edges from the other endvertex will be in different halfplanes. This will yield $ij+(2-j)(n-i-1)$ nonintersecting

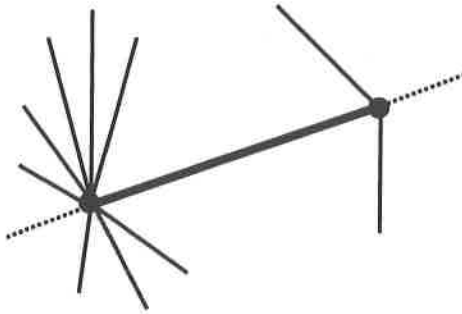

Figure 5: The bold edge is a type 3, 1 edge because the left endvertex has 3 edges emanating from it in the smaller halfplane, and the right endvertex has 1 edge emanating from it in the same halfplane.

edge pairs. However, $i(2-j)+j(n-i-1) \leq ij+(2-j)(n-i-1)$ when $0 \leq i \leq j \leq N$. Therefore, to find a lower bound for M, we can assume that the i and j edges are arranged so that they lie in the same halfplane.

Summing the missed edge crossings over all edges of a drawing we obtain

$$\begin{aligned} M &= \sum_{i=s}^{N}\sum_{j=0}^{1}[i(2-j)+j(n-i-1)]x_{i,j} = \\ &= \sum_{i=s}^{N} 2i \cdot x_{i,0} + \sum_{i=s}^{N} (n-1)x_{i,1}. \end{aligned}$$

In order to minimize M, we begin by multiplying equation (1) by $2i$ and subtracting it from M for all values of i, yielding

$$M - \sum_{i=s}^{N} 2i \cdot y_i = \sum_{i=s}^{N}(n-1-2i)x_{i,1}.$$

This implies

$$
\begin{aligned}
M &= \sum_{i=s}^{N} 2i \cdot y_i + \sum_{i=s}^{N} (n-1-2i)x_{i,1} = \\
&= \sum_{i=s}^{N} (2i \cdot y_i - 4i) + \sum_{i=s}^{N} 4i + \sum_{i=s}^{N} (n-1-2i)x_{i,1} = \\
&= \sum_{i=s}^{N} 2i(y_i - 2) + \sum_{i=s}^{N} 4i + \sum_{i=s}^{N} (n-1-2i)x_{i,1}
\end{aligned} \tag{2}
$$

We consider 2 cases:

Case 1: $s = 0$

By Equation (2) we have

$$
M = \sum_{i=0}^{N} 2i(y_i - 2) + \sum_{i=0}^{N} 4i + \sum_{i=0}^{N} (n-1-2i)x_{i,1}. \tag{3}
$$

Lemma 3.2. *$y_i \geq 2$ for all $i < N$, and $y_N \geq 1$ for n odd.*

Proof. For a given vertex, we begin by proving there is at least one endvertex of type $\frac{n-1}{2}$ for n odd and there are at least two endvertices of type $\frac{n-2}{2}$ for n even. This statement can be proved by induction from n to $n+1$. This statement is obvious for $n = 2$ and $n = 3$, so we begin with the inductive step. Note that in traversing the n edges incident to the central vertex in a clockwise or counterclockwise manner, in moving from edge to edge, edge to extension, extension to edge, and extension to extension, the number of edges in the clockwise following halfplane may change by at most one. This fact will be used numerous times throughout the proof.

Case (a): From odd n to $n + 1$.
We consider the edge whose endvertex is of type $\frac{n-1}{2}$ in the drawing of W_n. When the $(n + 1)$st edge is added, this original endvertex will be the first endvertex of type $\frac{1}{2}[(n+1)-2]$. If the $(n+1)$-st edge is added in this edge's clockwise following halfplane then an immediately following edge or edge extension's endvertex will have type $\frac{1}{2}[(n + 1) - 2]$. Thus, either this edge or the edge corresponding to this extension's endvertex will be the second endvertex of type $\frac{n-1}{2}$.

Case (b): From even n to $n+1$.
Consider an edge whose endvertex is of type $\frac{n-2}{2}$ which has $\frac{n}{2}$ edges in one of its halfplanes and $\frac{n-2}{2}$ in the other. If the $(n+1)$st edge is added in the halfplane with $\frac{n-2}{2}$ edges then the considered endvertex is of type $\frac{1}{2}[(n+1)-1]$. If the $(n+1)$st edge is added in the halfplane with $\frac{n}{2}$ edges then there are $\frac{1}{2}[(n+1)+1]$ edges in this halfplane and $\frac{1}{2}[(n+1)-3]$ edges in the clockwise following halfplane of this edge's extension. Since the number of edges in the clockwise following halfplane can change by at most one when moving from edge line to edge line (edge ray and edge extension), we find that traversing the graph from the edge with $\frac{1}{2}[(n+1)+1]$ edges in the clockwise following halfplane to the extension with $\frac{1}{2}[(n+1)-3]$ there must occur an edge or extension with $\frac{1}{2}[(n+1)-1]$ edges in the clockwise following halfplane. Thus, this edge or the edge corresponding to the extension's endvertex is of type $\frac{1}{2}[(n+1)-1]$.

Using this result and the fact that in moving from edge line to adjacent edge line, the number of edges in the clockwise following halfplane may change by at most one, we can prove that there are two endvertices of each type from the minimal type 0 to the maximal type N. For n odd, we have one endvertex of maximal type $N = \frac{n-1}{2}$. Traversing the n edges starting and ending with the edge of type N from edge line to edge line we must go down to an edge or an extension with 0 edges in the clockwise following halfplane, and then back up to one with N. Thus, we find there are at least two of edges or extensions whose endvertices are of each type from 0 to N. For n even, we have two edges of maximal type $N = \frac{n-2}{2}$. Traversing the n edges from one of the type N edges to the other must go down to an edge or extension with 0 edges in the clockwise following halfplane and back up to one with N. Thus, there are at least two edges or extensions whose endvertices are of each type from 0 to N. It follows that $y_i \geq 2$ for all $i < N$. $\square$

To minimize M, we start with the case of n odd. Since $y_i - 2 \geq 0$ for all $i < N$, $y_N \geq 1$, and $n - 1 - 2i \geq 0$ for all i, by Equation (3) we have

$$\begin{aligned} M &\geq -2N + \sum_{i=0}^{N} 4i = \\ &= -(n-1) + \frac{(n-1)(n+1)}{2} = \frac{(n-1)^2}{2}. \end{aligned}$$

Combining this with our first upper bound we have

$$\begin{aligned}\overline{\mathrm{CR}}(W_n) &\leq \frac{3n^2-7n-(n-1)^2}{2} = \\ &= \frac{2n^2-5n-1}{2}.\end{aligned}$$

For n even, since $y_i - 2 \geq 0$ for all i and $n-1-2i \geq 0$ for all i, by Equation (2) we have

$$M \geq \sum_{i=0}^{N} 4i = \frac{1}{2}n(n-2).$$

Combining this with our first upper bound we have

$$\overline{\mathrm{CR}}(W_n) \leq \frac{3n^2-8n-2-n(n-2)}{2} = n^2-3n+1.$$

Case 2: $s > 0$

By Equation (2) we have

$$\begin{aligned}M &= \sum_{i=s}^{N} 2i(y_i-2) + \sum_{i=s}^{N} 4i + \sum_{i=s}^{N}(n-1-2i)x_{i,1} \\ &= \sum_{i=s}^{N} 2i(y_i-2) + \sum_{i=0}^{N} 4i - \sum_{i=0}^{s-1} 4i + \sum_{i=s}^{N}(n-1-2i)x_{i,1} \\ &\geq \sum_{i=s}^{N} 2s(y_i-2) + \sum_{i=0}^{N} 4i - \sum_{i=0}^{s-1} 4i + \sum_{i=s}^{N}(n-1-2i)x_{i,1} \\ &= C(s,n) + \sum_{i=0}^{N} 4i + \sum_{i=s}^{N}(n-1-2i)x_{i,1}\end{aligned}$$

where

$$C(s,n) = \sum_{i=s}^{N} 2s(y_i-2) - \sum_{i=0}^{s-1} 4i.$$

We now show that $C(s,n) \geq 0$ for all s and n. Since $\sum_{i=s}^{N} y_i = n$, we have

$$\begin{aligned}C(s,n) &= 2s(n-\sum_{i=s}^{N} 2) - \sum_{i=0}^{s-1} 4i \\ &= 2s(n-2(N-s+1)) - 2s(s-1) = 2s(n-2N+s-1).\end{aligned}$$

Since $2s \geq 0$ and $n - 2N \geq 0$ and $s - 1 \geq 0$, we can conclude that $C(s,n) \geq 0$ as well. We now have that

$$M \geq C(s,n) + \sum_{i=0}^{N} 4i + \sum_{i=s}^{N} (n-1-2i)x_{i,1} \geq \sum_{i=0}^{N} 4i.$$

The rest of the proof follows as in Case 1.

□

4 Concluding Remarks

The method of proof presented here for the maximum rectilinear crossing number of the wheel graph is an application of a method of proving maximum rectilinear crossing numbers of graphs which was developed in [1]. The successful application of this method to the wheel graph seems promising for the generalization of this method towards proving the following conjecture which was initially formulated in [1].

Conjecture 4.1. *The maximum rectilinear crossing number of any graph can be realized in a drawing where all the vertices are vertexpoints of a convex polygon.*

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