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The Maximum Rectilinear Crossing Number of the Petersen Graph

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Abstract

We prove that the maximum rectilinear crossing number of the *Petersen graph* is 49. First, we illustrate a picture of the Petersen graph with 49 crossings to prove the lower bound. We then prove that this bound is sharp by carefully analyzing the ten C_6's which occur in the Petersen graph and their properties.

1 Introduction

A *drawing* of the graph G with vertex set $V(G)$ and edge set $E(G)$ is defined as a representation of G in a plane such that the elements of $V(G)$ correspond to points in the plane and the elements of $E(G)$ correspond to continuous arcs. We assume that each arc connects two vertices and that any pair of arcs has at most one point in common, either a vertexpoint or a crossing. A *rectilinear drawing* is a drawing of a graph in which all edges are represented as straight line segments in the plane. A *crossing* is defined to be the intersection of exactly two edges not at a vertex. The *crossing number* of an abstract graph G, denoted $cr(G)$, is defined as the minimum number of edge crossings over all nonisomorphic drawings of G. The *rectilinear crossing number* of a graph G, denoted $\overline{cr}(G)$, is defined as the minimum number of edge crossings over all nonisomorphic rectilinear drawings of G. Analogously, the *maximum crossing number*, denoted by $\mathrm{CR}(G)$, is defined as the maximum number of edge crossings over all nonisomorphic drawings of G. The *maximum rectilinear crossing number* of a

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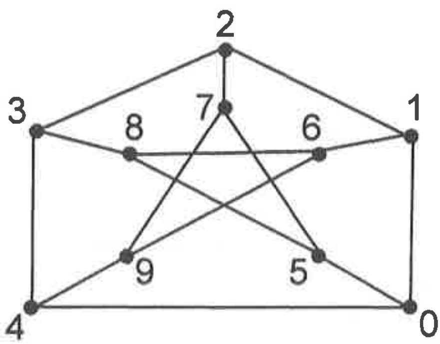

Figure 2: The Petersen graph.

Every I-pair occurs exactly once in one of the C_6's. Every II-pair occurs in exactly two of the C_6's.

Let $\overrightarrow{\mathrm{ij}}$ denote the straight line determined by the vertices i and j and being oriented from i to j.

Two further lemmas are needed in the proof.

Lemma 2.3. *If two cycles C_6 of P have a non-crossing II-pair in common then one of these cycles has at most three crossing I-pairs or the rectilinear drawing of P has at most* 49 *crossings.*

Proof. Observe that the common II-pair of two cycles C_6 has an opposite cyclic orientation in these two cycles.

Consider 2 cycles where the edges of the common II-pair do not intersect one another. Lemma 2.2 guarantees for each cycle two of all six I-pairs to be without crossings. To avoid a third non-crossing I-pair one edge of each additional edge pair has to intersect the non-adjacent edge of the II-pair and both additional edge pairs of each of the two cycles have to intersect one another. However, this is possible only in the following special case of a drawing of P-e (Figure 3).

For simplicity, cycles (0) and (3) may be used where edge 38 lies to the right of $\overrightarrow{01}$. Then the $\overrightarrow{38}$ in this direction has to intersect the edge 01 and vertices 2,5 and vertices 4,6 have to lie to the right and to the left of $\overrightarrow{38}$, respectively. (If 3 and 8 are exchanged then this corresponds to the exchange of cycles (0) and (3).)

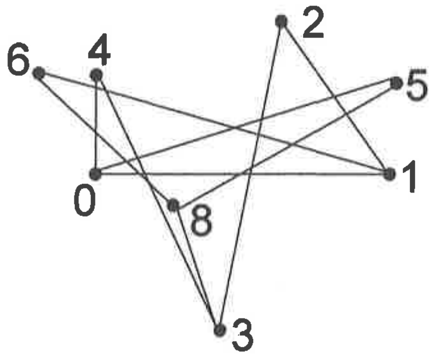

Figure 3: Cycles (0) and (3) share a non-crossing II-pair 01/38.

In this special case of $P - e$, one already counts 17 crossings. To complete $P - e$ to P, vertices 7 and 9 have to be added. Together with the 10 edges of $P - e$ there are c_7 crossings on 72 and 75, c_9 crossings on 94 and 96, and c_{79} crossings on 79. With at most 4 crossings among 72, 75, 94, and 96 it follows for the total number C of crossings that $C \leq 17 + 4 + c_7 + c_9 + c_{79}$.

The three cases are distinguished that

1. 7 and 9 lie to the left of $\overrightarrow{01}$,
2. 7 and 9 lie to the right of $\overrightarrow{01}$,
3. 7 lies to the right and 9 lies to the left of $\overrightarrow{01}$.

For Case 1 it follows $c_7 \leq 10$, $c_9 \leq 10$, $c_{79} \leq 8$, and thus $C \leq 21 + 10 + 10 + 8 = 49$.

For Case 2 it follows $c_7 \leq 12$, $c_9 \leq 12$, $c_{79} \leq 4$, and thus $C \leq 21 + 12 + 12 + 4 = 49$.

For Case 3 there are $c_7 \leq 12$, $c_9 \leq 10$, and $c_{79} \leq 8$. If $c_7 \leq 10$ then $C \leq 49$.

Only if 7 lies to the left of $\overrightarrow{50}$ and to the right of $\overrightarrow{58}$ then $c_7 = 11$ is possible, however, this implies $c_{79} \leq 7$ so that $C \leq 21 + 11 + 10 + 7 = 49$. Only if 7 lies to the left of $\overrightarrow{58}$ and to the right of $\overrightarrow{23}$ then $c_7 = 12$ is possible. Then $c_9 = 10$ implies $c_{79} \leq 6$ and $C \leq 21 + 12 + 10 + 6 = 49$. Otherwise it holds $c_9 \leq 8$ and thus $C \leq 21 + 12 + 8 + 8 = 49$.

Altogether, the special drawing of $P - e$ has at most 49 crossings.

□

Lemma 2.4. *For the part of a C_6 in Figure 4 this C_6 can have (i) at most 6 crossings if the position of z is to the right of $\overrightarrow{xy}$ or to the left of $\overrightarrow{rs}$ and (ii) at most 5 crossings if z lies to the left of $\overrightarrow{xy}$ and to the right of $\overrightarrow{rs}$.*

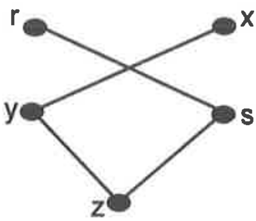

Figure 4: Part of a C_6.

Proof. The I-pairs xy/sz and rs/yz do not have crossings. For (i) a sixth vertex t determines with tr/xy or tx/rs a third non-crossing I-pair and there remain at most 6 crossings from all 9 non-adjacent edge pairs. For (ii) each of the edges tx and tr can intersect only one of the two edges yz and sz so that there remain at most 5 crossings. □

Now consider a new graph PC having the ten C_6's as vertices and edges if the two C_6's have a II-pair of edges of P in common (see Figure 5). The graph PC again is a Petersen graph.

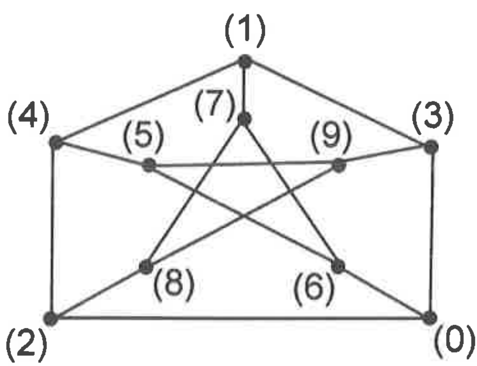

Figure 5: The graph PC.

Every C_6 has at most 4 crossing I-pairs and at most 3 crossing II-pairs. This gives $\overline{CR}(P) \leq 10 \cdot 4 + 15 = 55$ since the $10 \cdot 3$ crossing II-pairs occur twice.

If at least 6 of the ten C_6's have less than 7 crossings then $\overline{CR}(P) \leq 55 - 6 = 49$ since either they all have at most 3 crossing I-pairs or in any pair of them having a non-crossing II-pair by Lemma 2.3 at most 3 crossing I-pairs occur in one C_6 of the pair.

Since the Petersen graph P contains at most 4 independent vertices it now can be assumed that two adjacent vertices in PC both have 7 crossings.

There are two possibilities to fix a II-pair of C_6 with 7 crossings (see Figure 6).

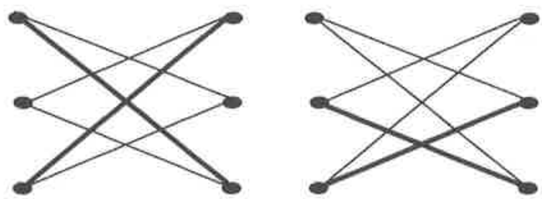

Figure 6: Different II-pairs in C_6 with 7 crossings.

Due to symmetry there are only three types of drawings for two in PC adjacent C_6's, both having 7 crossings. Using the labellings of Figures 2 and 5, for simpilicity one may choose cycles (0) and (3) to obtain the three cases (a), (b), and (c) in Figure 7.

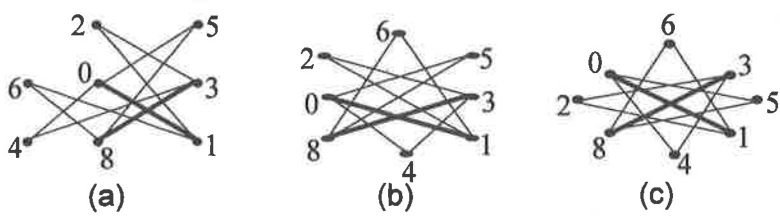

Figure 7: Two C_6's with 7 crossings and with a common II-pair.

In Case (b) the vertex 6 can be moved to any position to the right of $\overrightarrow{10}$ and to the left of $\overrightarrow{83}$. In Case (c) vertices 4, 6, 2, and 5 can be moved to any position to the right of $\overrightarrow{83}$ and $\overrightarrow{01}$, of $\overrightarrow{38}$ and $\overrightarrow{10}$, of $\overrightarrow{01}$ and $\overrightarrow{38}$, and of $\overrightarrow{10}$ and $\overrightarrow{83}$, respectively.

Case (a): The parts of cycles (1),(9), (6), (2), (4), and (5) in Figure 7(a) are as in Figure 8.

If (1) or (6) has 7 crossings then (3) and (1) or (0) and (6), respectively, are as in Case (b). Thus it can be assumed that (1) and (6) each has at

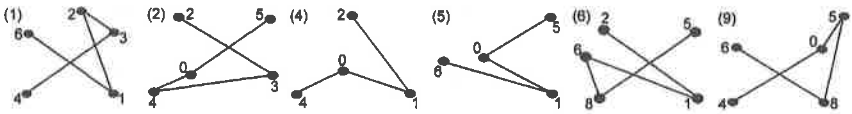

Figure 8: Parts of C_6's in Case (a).

most 6 crossings. Assume that (9) and (2) each has at most 5 crossings. Then there are at most $55 - 6 = 49$ crossings, since (1), (6), (9), and (2) are independent in PC.

By Lemma 2.4 cycle (9) has at most 6 crossings. Assume (9) has 6 crossings. Vertex 5 cannot lie to the right of $\overrightarrow{68}$ since 1 lies to the left of $\overrightarrow{68}$ and 85 intersects 61. Thus by Lemma 2.4 vertices 5 and 9 have to lie to the left of $\overrightarrow{40}$. Since 12 intersects 05, subsequently vertex 2 lies to the left of $\overrightarrow{40}$. Then (4) has at most 6 crossings, too, since either 49/12 is a non-crossing II-pair or due to Lemma 2.4 otherwise. These arguments hold correspondingly for (2) and (5) instead of (9) and (4). Then with (1), (6), (9), (4), (2), and (5) there are 6 cycles, each with at most 6 crossings.

Cases (b) and (c): The parts of cycles (1), (9), (2), and (6) in Figure 7(b) and 7(c) are as in Figure 9 and Figure 10, respectively.

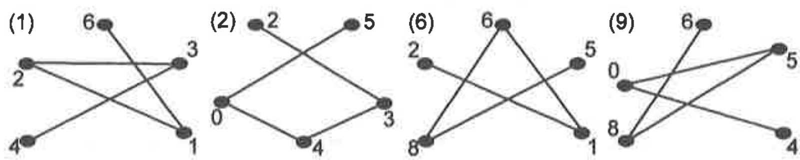

Figure 9: Parts of C_6's in Case (b).

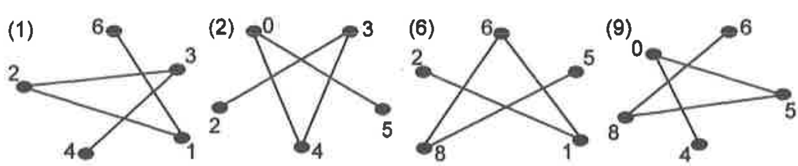

Figure 10: Parts of C_6's in Case (c).

For cycle (2) in Figure 9 vertex 4 lies to the left of $\overrightarrow{50}$ since vertex 8 lies to the left of $\overrightarrow{50}$ and 04 intersects 58. Correspondingly, vertex 4 lies to the right of $\overrightarrow{23}$ since vertex 1 lies to the right of $\overrightarrow{23}$ and 34 intersects 01. Thus by Lemma 2.4 there are at most 5 crossings in (2) for Case (b).

For (1) and (9) in both cases consider vertex 9 which lies in one halfplane determined by the straight line through 4 and 6. It follows for all possible positions of vertex 6 in Case (b) and of vertices 4, 6, 2, and 5 in Case (c) that one of the cycles (1) and (9) has at most 5 crossings. Correspondingly, (2) or (6) in Case (c) has at most 5 crossings if vertex 7 and the straight line through 2 and 5 are considered.

Together for Cases (b)and (c) each pair of neighboring cycles of (0) and (3) has one cycle having at most 5 crossings.

Now for (0) in Case (b) let (6) have 7 crossings. Then (0) and (6) are as in Case (b). For (3) in Case (b) let (1) or (9) have 7 crossings. Then (3) and (1) or (3) and (9) are as in Case (b). For (0) in Case (c) let (2) or (6) have 7 crossings. Then (0) and (2) or (0) and (6) are as in Case (b). Also (3) in Case (c) implies (3) and (9) or (3) and (1) in Case (b).

Consider the two pairs of neighboring cycles (2) and (6) of (0) and (9) and (1) of (3). For those two of these 4 cycles which are not known to have at most 5 crossings the three cases are distinguished that

1. both have at most 6,
2. only one has at most 6,
3. both have seven crossings.

For Case 1 there are 4 independent vertices of PC, two with at most 5 crossings and two with at most 6 crossings, that is, together there are 6 crossings less than 55.

For Case 2 there are at least 5 crossings less than 55 from the independent cycles (2), (6), (9), and (1). One of them has 7 crossings and due to Cases (b) and (c) one of its two further neighbors has at most 5 crossings. Each of these further neighbors has one edge in PC to only one of the independent vertices (2), (6), (9), and (1). This guarantees a sixth crossing less than 55 even if a II-pair is counted twice.

Case 3 has only 4 crossings less than 55 from (2), (6), (9), and (1). Two of these four cycles have 7 crossings. Due to Case (b) or (c) for each of these cycles one of the two further neighboring cycles has at most 5 crossings. Two of these further cycles coincide in a cycle being independent of (2), (6), (9), and (1). If this cycle has at most 5 crossings then altogether there are 6 crossings less than 55. Otherwise, there is a path of 4 vertices in PC having at most 5 crossings for each vertex. If less than 3 of them are II-pairs then there remain at least 6 crossings less than 55. If at least 3 of them are II-pairs then consider the three pairs of neighboring cycles of the path. Due to Lemma 2.3 each of them has at least 3 non-crossing edge pairs and if the central pair has a common non-crossing II-pair then due to Lemma 2.3 one of the other pairs has at least 4 non-crossing edge pairs. □

Finally, it may be asked whether there exits a 3-regular graph of order 10 having a smaller maximum rectilinear crossing number than 49.

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