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Matthew Alpert
Harvard University

Elie Feder
CUNY Kingsborough Community College

Heiko Harborth
Technische Universitat Braunschweig

Sheldon Klein
Rambam Mestiva High School

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The Maximum Rectilinear Crossing Number of the n Dimensional Cube Graph

Matthew Alpert*, Elie Feder†, Heiko Harborth‡, and Sheldon Klein§

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Abstract

We find and prove the maximum rectilinear crossing number of the three-dimensional *cube graph* (Q_3). We demonstrate a method of drawing the n-cube graph, Q_n, with *many* crossings, and thus find a lower bound for the maximum rectilinear crossing number of Q_n. We conjecture that this bound is sharp. We also prove an upper bound for the maximum rectilinear crossing number of Q_n.

1 Introduction

A *drawing* of the graph G with vertex set $V(G)$ and edge set $E(G)$ is defined as a representation of G in a plane such that the elements of $V(G)$ correspond to points in the plane and the elements of $E(G)$ correspond to continuous arcs. We assume that each arc connects two vertices and that any pair of arcs has at most one point in common, either a vertexpoint or a crossing. A *rectilinear drawing* is a drawing of a graph in which all edges are represented as straight line segments in the plane. A *crossing* is defined to be the intersection of exactly two edges not at a vertex. The *crossing number* of an abstract graph G, denoted $cr(G)$, is defined as the minimum number of edge crossings over all nonisomorphic drawings of G. The *rectilinear crossing number* of a graph G, denoted $\overline{cr}(G)$, is defined as

*Harvard University, Cambridge, MA, USA. *E-mail address:* `mna851@aol.com`

†Department of Mathematics and Computer Science, Kingsborough Community College-CUNY, Brooklyn, NY, USA. *E-mail address:* `efeder@kbcc.cuny.edu`

‡Diskrete Mathematik, Technische Universitaet, Braunschweig, Germany. *E-mail address:* `H.Harborth@tu-bs.de`

§Rambam Mesivta High School, Lawrence, NY, USA. *E-mail address:* `mathrocket@gmail.com`

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the minimum number of edge crossings over all nonisomorphic rectilinear drawings of G. Analogously, the *maximum crossing number*, denoted by $\mathrm{CR}(G)$, is defined as the maximum number of edge crossings over all nonisomorphic drawings of G. The *maximum rectilinear crossing number* of a graph G, $\overline{\mathrm{CR}}(G)$, is defined as the maximum number of crossings over all nonisomorphic rectilinear drawings of G.

The maximum crossing number and maximum rectilinear crossing number have been studied for several classes of graphs (see [7–11, 13, 14]). Most relevant to this paper are studies of the maximum rectilinear crossing number of C_n and of $R_{n,d}$, the class of *d-regular* graphs of order n, i.e., graphs where each of the n vertices has degree d.

It has been shown in [7] that

$$\overline{\mathrm{CR}}(C_n) = \begin{cases} \frac{1}{2}n(n-3) & \text{if } n \text{ is odd,} \\ \frac{1}{2}n(n-4)+1 & \text{if } n \text{ is even.} \end{cases}$$

The parts of this result which are relevant to this paper are $\overline{\mathrm{CR}}(C_4) = 1$ and $\overline{\mathrm{CR}}(C_6) = 7$.

An example of a 2-regular graph of order n is C_n. In [1] we extended the result on cycles to the class of d-regular graphs of order n. Our major result was that

$$\overline{\mathrm{CR}}(R_{n,d}) = \frac{1}{24}nd(3nd - 2d^2 - 6d + 2) \text{ if } n + d \equiv 1 \pmod 2.$$

Other results are proved and conjectured for other cases of the d-regular graph. The n-cube graph Q_n, or the hypercube, is one specific graph in the class of n-regular graphs of order 2^n. Various crossing numbers for the n-cube graph have been explored. In [2–6] the crossing number of the n-cube, is studied. In [12] the maximum crossing number of n-cube graphs is explored and in [10] the maximum crossing number of the 3-cube is proved to be 34. This paper is the first study of the maximum rectilinear crossing number of cube graphs. It is a continuation of our study of the maximum rectilinear crossing number of d-regular graphs, with specific focus on the n-cube graph.

In Section 2, we find the value of $\overline{\mathrm{CR}}(Q_3)$. In Section 3.1, we derive a lower bound for $\overline{\mathrm{CR}}(Q_n)$ by an inductive construction of a drawing for Q_n, for all n. In Section 3.2, we prove an upper bound for $\overline{\mathrm{CR}}(Q_n)$ and in Section 3.3, we conjecture that our lower bound is sharp. We cite some computational results that support this conjecture.

2 The three dimensional cube graph

Clearly, $\overline{\mathrm{CR}}(Q_2) = 1$. We thus begin by studying $\overline{\mathrm{CR}}(Q_3)$.

2.1 Construction and lower bound of $\overline{\mathrm{CR}}(Q_3)$

To find the lower bound of $\overline{\mathrm{CR}}(Q_3)$, we construct the drawing in Figure 1 to obtain the following proposition.

Proposition 2.1.

$$\overline{\mathrm{CR}}(Q_3) \geq 28.$$

Proof. A careful count of the crossings in Figure 1 gives 28 crossings.

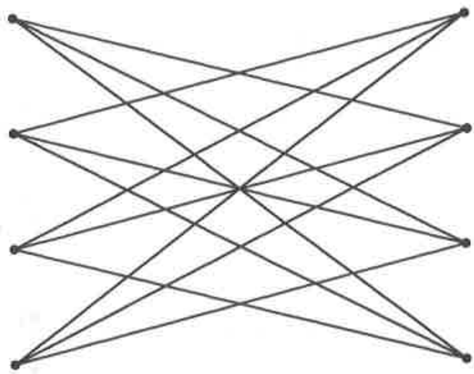

Figure 1: The 28-crossing realization of Q_3.

□

2.2 Upper bound of $\overline{\mathrm{CR}}(Q_3)$

We now show that this bound is sharp.

Proposition 2.2.

$$\overline{\mathrm{CR}}(Q_3) = 28.$$

Proof. Crossings can only result from the intersection of nonadjacent edges. A nonadjacent edge pair can contribute at most one crossing in a rectilinear graph. Therefore, we will count the number of nonadjacent edge pairs, and then determine how many of these pairs can intersect.

Every vertex of Q_3 has degree 3. If we consider a given edge, there are two edges adjacent to it on each side, totaling four adjacent edges. As each edge cannot intersect itself nor its four adjacent edges, we can subtract these five from the total of twelve edges in the cube, resulting in seven nonadjacent edge pairs per edge. Since each pair is counted twice, once for each edge of the pair, we have $(12 \cdot 7)/2 = 42$ pairs of nonadjacent edges in Q_3. This is our first upper bound.

Of the 42 nonadjacent edge pairs, 12 may be accounted for through C_4s. Each of the 6 faces of the cube in Figure 2 is a C_4. Each C_4 contains two pairs of nonadjacent edges. Thus, the six distinct C_4s contain 12 pairs of nonadjacent edges. The other 30 pairs may be accounted for by considering

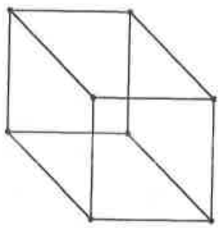

Figure 2: The conventional drawing of a cube showing 6 faces.

the nonadjacent edge pairs of four specific C_6s. Consider the C_6s that are formed by removing opposite vertices of the cube and their incident edges as shown in Figure 3. Each C_6 has 9 nonadjacent edge pairs. The nonadjacent edge pairs of these C_6s are distinct from those of the C_4s (only adjacent pairs of edges are shared with the C_4s), but each C_6 shares one pair of nonadjacent edges with each other C_6. Specifically, each pair of edges opposite from each other in a C_6 is shared by two of the C_6s as seen in Figure 3. There are 6 such pairs of opposite edges in Q_3. Therefore, if we multiply the nine nonadjacent edge pairs in each C_6 by the four C_6s, and subtract the 6 pairs that have been counted twice, we obtain 30 nonadjacent edge pairs contributed by the chosen C_6s. In this manner, we include every pair of nonadjacent edges in the Q_3.

Now that we have accounted for all 42 nonadjacent edge pairs in Q_3, let us consider the number of possible crossings amongst these pairs. Each of the six C_4s can only have one intersection. By Lemma 2.3, each of the four C_6s has at most 4 non-opposite edge pairs which intersect. Additionally, the four C_6s have 6 opposite edge pairs which can intersect. In total, we have at most $6 + 4 \cdot 4 + 6 = 28$ possible crossings.

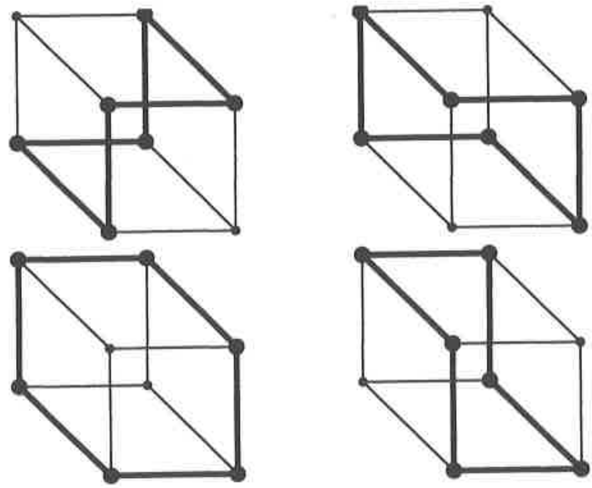

Figure 3: In bold are the four C_6s which are formed by the removal of pairs of opposite vertices and their incident edges.

□

Lemma 2.3. *Any C_6 has at most 4 pairs of non-opposite edge pairs which intersect.*

Proof. Suppose there are at least five non-opposite edge pairs in C_6 which intersect. Each of these edge pairs is determined by that edge e_i of C_6 being adjacent to both edges of the pair. Start with edge e_3 of five consecutive edges e_1, e_2, e_3, e_4, e_5 of C_6 which determine a crossing pair. Then e_2 has to intersect e_4 (see Figure 4). Next e_1 and e_5 both have to intersect e_3. Since then e_6 is in one halfplane determined by e_3, and e_2 and e_4 are in the other one, neither e_6 and e_2 nor e_6 and e_4 can intersect, a contradiction. □

3 The n-dimensional cube graph: lower bound of $\overline{\mathrm{CR}}(Q_n)$

In this section we discuss the maximum rectilinear crossing number of the n-cube.

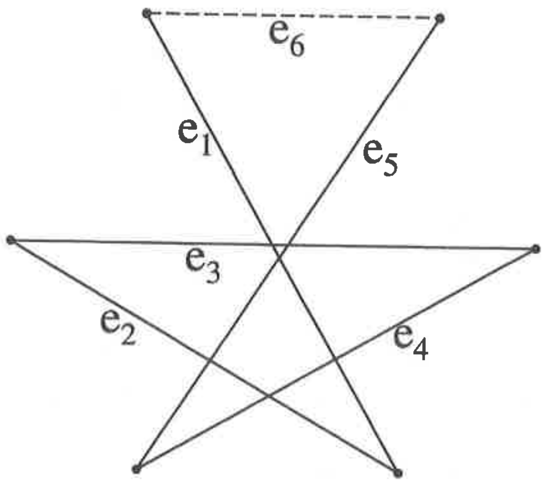

Figure 4: Intersecting non-opposite edge pairs of C_6.

3.1 Construction and lower bound of $\overline{\mathrm{CR}}(Q_n)$

We illustrate an iterative construction of Q_n to yield a lower bound for $\overline{\mathrm{CR}}(Q_n)$.

Proposition 3.1.

$$\overline{\mathrm{CR}}(Q_n) \geq 2^{n-2}[(2^{n-1}(n^2 - 2n + 3) - n^2 - 1].$$

Proof. We construct a drawing G_n of Q_n inductively. For Q_2 we have a bipartite realization G_2 as in Figure 5.

Given G_{n-1}, we construct G_n as follows (see Figures 6 and 7 for G_3 and G_4, respectively). We take two copies of G_{n-1} (aligned in a "bipartite manner"), call them $G_{n-1,1}$ and $G_{n-1,2}$, and arrange them so that they form an X (see Figure 5). In other words, let the left side of $G_{n-1,1}$ be above the left side of $G_{n-1,2}$, while the right side of $G_{n-1,2}$ be above the right side of $G_{n-1,1}$. So far each vertex has degree $n-1$. To complete the graph, we connect the top left vertices of $G_{n-1,1}$ to the top right vertices of $G_{n-1,2}$ in the following manner. Connect the top left vertex of $G_{n-1,2}$ to the bottom right vertex of $G_{n-1,2}$. Similarly the i-th vertex from the top on the left side of $G_{n-1,2}$ connects to the i-th vertex from the bottom on the right side of $G_{n-1,2}$. We follow the same procedure to connect the

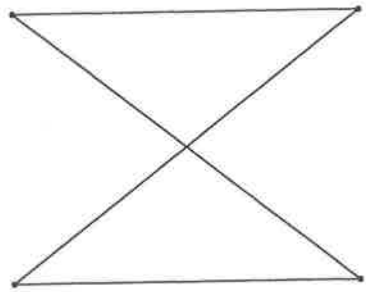

Figure 5: G_2 drawing of Q_2 with 1 intersection.

bottom left vertices of $G_{n-1,2}$ to the bottom right vertices of $G_{n-1,2}$. We thus have G_n.

In order to count the number of intersections in G_n, notice that G_n consists of 2 types of edges: (a) edges which connect two vertices from the same copy of G_{n-1}, (b) edges which connect two vertices from different copies of G_{n-1}. In order to count the intersections, we divide the intersections in G_n into 4 types:
(1) intersections of two edges in the same copy of G_{n-1},
(2) intersections of one edge from one copy of G_{n-1} with one edge from the other copy of G_{n-1},
(3) intersections of two type (b) edges,
(4) intersections of a type (a) edge with a type (b) edge.

Let x_i denote the number of intersections in G_i. Let $x_1 = 0$.
In order to compute x_n, we count the number of intersections of each type in G_n.
(1) Since there are two copies of G_{n-1} we have $2x_{n-1}$ such intersections.
(2) Notice that every edge in one copy of G_{n-1} intersects every edge from the other copy of G_{n-1} (because of the X configuration). Since each G_{n-1} has $(n-1)2^{n-2}$ edges, we have $((n-1)2^{n-2})^2$ such intersections.
(3) Any two type (b) edges on top (bottom) intersect each other. Since we have 2^{n-2} edges on the top (bottom) we have $2\binom{2^{n-2}}{2}$ such intersections.
(4) Without loss of generality, let us consider the intersections between the type (b) edges on the bottom and the type (a) edges. There are 2^{n-2} type (b) edges on the bottom. Each such edge has i vertices below the left endvertex and $2^{n-2}-1-i$ vertices below the right endvertex. In total there are $2^{n-2}-1$ vertices below each type (b) edge on the bottom. Each vertex has $n-1$ type (a) edges emerging from it. Notice that each type (b)

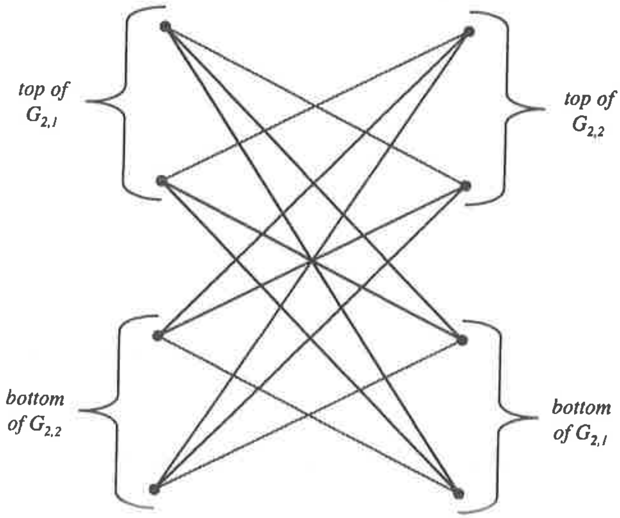

Figure 6: The G_3 drawing of Q_3 with 28 intersections. The drawing is constructed from $G_{2,1}$ and $G_{2,2}$, two copies of G_2.

edge is intersected by all $(n-1)(2^{n-2}-1)$ type (a) edges emerging from vertices below either of its endvertices. Accounting for the type (b) edges on the top as well, we have $(n-1)(2^{n-1})(2^{n-2}-1)$ intersections of this type.

Summing over all types of intersections yields

$$\begin{aligned} x_n &= 2x_{n-1} + (n-1)^2 2^{2n-4} + 2^{n-2}(2^{n-2}-1) + (n-1)2^{n-1}(2^{n-2}-1) \\ &= 2x_{n-1} + 2^{n-3}[(n-1)^2 2^{n-1} + 2(2^{n-1}-1) + 4(n-1)(2^{n-2}-1)] \\ &= 2x_{n-1} + 2^{n-3}(n^2 2^{n-1} - 4n + 2). \end{aligned} \tag{1}$$

We continue to prove by induction that

$$x_n = 2^{n-2}[(2^{n-1}(n^2-2n+3) - n^2 - 1].$$

Assume true for $n-1$. Thus,

$$x_{n-1} = 2^{n-3}[(2^{n-2}((n-1)^2 - 2(n-1) + 3) - (n-1)^2 - 1]$$

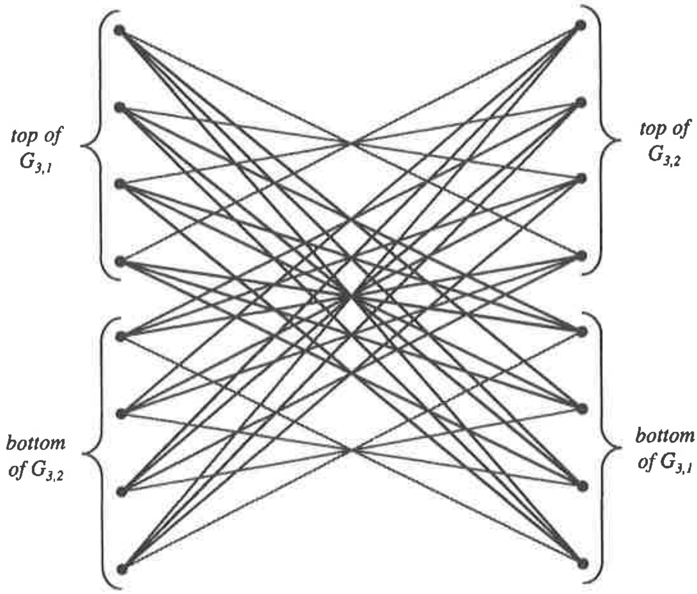

Figure 7: The G_4 drawing of Q_4 with 284 intersections. The drawing is constructed from $G_{3,1}$ and $G_{3,2}$, two copies of G_3.

$$= 2^{n-3}[(2^{n-2}(n^2 - 4n + 6) - n^2 + 2n - 2].$$

By equation (1) we have

$$\begin{aligned} x_n &= 2 \cdot 2^{n-3}[(2^{n-2}(n^2 - 4n + 6) - n^2 + 2n - 2] + 2^{n-3}(n^2 2^{n-1} - 4n + 2) \\ &= 2^{n-2}[(2^{n-1}(n^2 - 2n + 3) - n^2 - 1] \end{aligned}$$

as claimed.

□

3.2 Upper bound of $\overline{\mathrm{CR}}(Q_n)$

Here we prove an upper bound for the maximum rectilinear crossing number of Q_n, based on the number of nonadjacent edge pairs and the numbers of Q_2s and Q_3s.

Proposition 3.2.

$$\overline{\mathrm{CR}}(Q_n) \leq n2^{n-3}[n2^n - \frac{1}{3}(4n^2 + 3n - 1)].$$

Proof. Crossings can only result from the meeting of nonadjacent edges, and a nonadjacent edge pair can contribute at most one crossing in a rectilinear graph. Let's call the number of nonadjacent edge pairs in an n-cube graph p_n. First, we establish the value of p_n for all n-cubes

$$p_n = n2^{n-3}[n2^n - 2(2n-1)].$$

Every vertex has degree n, so the end-vertices of an edge are each incident to $n-1$ other edges, and there are $2(n-1)$ edges adjacent to each edge. The number of nonadjacent edge pairs that contain a given edge is the number of all the edges in the graph, $n2^{n-1}$, minus those adjacent to it and the edge itself, that is, $2n-1$. Therefore, the number of nonadjacent edge pairs that contain a given edge is $n2^{n-1} - (2n-1)$. We multiply this by the number of edges. Since this counts each edge pair twice, once for each of its edges, we divide by 2 to get p_n.

Now we will eliminate as many pairs of edges as we can that cannot cross. Every Q_2 in a Q_n has at most one crossing, but two nonadjacent edge pairs that are counted in p_n. Therefore, we can subtract the number of Q_2s in the Q_n from p_n. By Lemma 3.3, the number of Q_2s per Q_n is $\binom{n}{2}2^{n-2}$. Additionally, we can use Proposition 2.1 to eliminate more crossings due to the Q_3s in Q_n. By Lemma 3.3, the number of Q_3s in Q_n is $\binom{n}{3}2^{n-3}$. Each Q_3 has 42 pairs of nonadjacent edges, for which a maximum of 28 cross. This leaves 14 nonadjacent edges in each Q_3 which cannot cross. Since each Q_3 contains six C_4s, six of these 14 are already counted above in the C_4s. Thus each Q_3 contributes $42 - 28 - 6 = 8$ nonintersecting nonadjacent edge pairs. Any pair of Q_3s has a vertex, an edge and a Q_2 in common, or is disjoint. Therefore, by subtracting 8 per Q_3, we are not counting any missed crossings twice. We can therefore subtract $8\binom{n}{3}2^{n-3}$ crossings from p_n. We thus have a maximum of

$$n2^{n-3}[n2^n - 2(2n-1)] - \binom{n}{2}2^{n-2} - 8\binom{n}{3}2^{n-3}$$

$$= n2^{n-3}[n2^n - \frac{1}{3}(4n^2 + 3n - 1)]$$

possible crossings as claimed. □

Lemma 3.3. *The number of subgraphs Q_i in a Q_n is*

$$\binom{n}{i}2^{n-i}.$$

Proof. Every set of i edges at a vertex of a Q_n determines a Q_i. For example, in a geometric cube, Q_3, every set of two edges at a vertex defines a square, Q_2, and every one edge is a Q_1. There are $\binom{n}{i}$ such sets per vertex, and 2^n vertices per Q_n. The product of these counts each Q_i repeatedly, once for each of its 2^i vertices. Therefore, we divide by 2^i to obtain $\binom{n}{i}\frac{2^n}{2^i} = \binom{n}{i}2^{n-i}$ subgraphs Q_i per Q_n. □

3.3 Conjecture on the upper bound of $\overline{\mathrm{CR}}(Q_n)$

In the general case of the cube graph we present the following conjecture.

Conjecture 3.4. *The bound in Proposition 3.1 is sharp.*

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