The Orchard Crossing Number of an Abstract Graph

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ABSTRACT. We introduce the Orchard crossing number, which is defined in a similar way to the well-known rectilinear crossing number. We compute the Orchard crossing number for some simple families of graphs. We also prove some properties of this crossing number.

Moreover, we define a variant of this crossing number which is tightly connected to the rectilinear crossing number, and compute it for some simple families of graphs.

1. INTRODUCTION

Let G be an abstract graph. Motivated by the Orchard relation, introduced in [8, 9], one can define the *Orchard crossing number* of G, in a similar way to the well-known *rectilinear crossing number* of an abstract graph G (denoted by $\overline{\mathrm{cr}}(G)$, see [3, 12]).

The Orchard crossing number is interesting for several reasons. First, it is based on the Orchard relation which is an equivalence relation on the vertices of a graph, with at most two equivalence classes (see [8]). Moreover, since the Orchard relation can be defined for higher dimensions too (see [8]), hence the Orchard crossing number may be also generalized to higher dimensions.

Second, a variant of this crossing number is tightly connected to the well-known rectilinear crossing number (see Proposition 2.7).

Third, one can find real problems which the Orchard crossing number can represent. For example, design a network of computers which should be constructed in a manner which allows possible extensions of the network in the future. Since we want to avoid (even future) crossings of the cables which are connecting between the computers, we need to count not only the present crossings, but also the separators (which might come to cross in the future).

In the current paper, we define the Orchard crossing number for an abstract graph G, and we compute it for some simple families of graphs. We also deal with some simple properties of this crossing number. Some more properties can be found at [11].

We also define some variant of this crossing number, which is tightly connected to the rectilinear crossing number, and we compute it for some simple families of graphs.

The paper is organized as follows. In Section 2, we present the Orchard relation, define the Orchard crossing number and its variant, and give some examples. In Section 3, we compute the Orchard crossing number for the complete graph K_n. Section 4 deals with the Orchard crossing number of the star graph $K_{n,1}$. In Section 5, we compute the Orchard crossing number for the wheel graph $W_{n,1}$. Section 6 deals with some partial results about complete bipartite graphs $K_{n,m}$. In Section 7, we discuss the computation of the Orchard crossing number for a particular case of a union of two graphs.

2. The Orchard relation and the Orchard crossing numbers

We start with some notations. A finite set $\mathcal{P} = \{P_1, \cdots, P_n\}$ of n points in the plane $\mathbb{R}^2$ is a *generic configuration* if no three points of $\mathcal{P}$ are collinear.

A line $L \subset \mathbb{R}^2$ *separates* two points $P, Q \in (\mathbb{R}^2 \setminus L)$ if P and Q are in different connected components of $\mathbb{R}^2 \setminus L$. Given a generic configuration $\mathcal{P}$, denote by $n(P, Q)$ the number of lines defined by pairs of points in $\mathcal{P} \setminus \{P, Q\}$, which separate P and Q.

In this situation, one can define:

Definition 2.1 (Orchard relation)**.** *For two distinct points P, Q of a generic configuration $\mathcal{P}$, we set $P \sim Q$ if we have*

$$n(P, Q) \equiv (n - 3) \pmod 2.$$

One of the main results of [8] is that this relation is an equivalence relation, having at most two equivalence classes. Moreover, this relation can be used as an (incomplete) distinguishing invariant between generic configurations of points.

In order to define the *Orchard crossing number*, we need some more notions.

Definition 2.2 (Rectilinear drawing of an abstract graph G)**.** *Let $G = (V, E)$ be an abstract graph. A rectilinear drawing of a graph G, denoted by $R(G)$, is a generic subset of points V' in the affine plane, in bijection with V. An edge $(s, t) \in E$ is represented by the straight segment $[s', t']$ in $\mathbb{R}^2$.*

One then can associate a *crossing number* to such a drawing:

Definition 2.3. *Let $R(G)$ be a rectilinear drawing of the abstract graph $G = (V, E)$. The* crossing number *of $R(G)$, denoted by $n(R(G))$, is:*

$$n(R(G)) = \sum_{(s,t) \in E} n(s, t)$$

Note that the sum is taken only over the edges of the graph, whence $n(s, t)$ counted in *all* the lines generated by pairs of points of the configuration.

Now, we can define the *Orchard crossing number* of an abstract graph $G = (V, E)$:

Definition 2.4 (Orchard crossing number). *Let $G = (V, E)$ be an abstract graph. The* Orchard crossing number *of G, OCN(G), is*

$$\text{OCN}(G) = \min_{R(G)}(n(R(G)))$$

Note that the usual intersection of two edges contributes two separations: Each edge separates the two points at the ends of the other edge. Hence, one have the following important observation:

Remark 2.5. *Let G be an abstract graph. Then:*

$$\overline{\text{cr}}(G) \leq \frac{1}{2}\text{OCN}(G)$$

where the rectilinear crossing number of an abstract graph $\overline{\text{cr}}(G)$ is defined to be the smallest number of crossings between edges in a rectilinear drawing of the graph G.

Hence, by computing the Orchard crossing number of a graph, one gets also some upper bounds for the rectilinear crossing number (these bounds are rather bad, see the table after Proposition 3.1. The reason for the bad bounds is that the best drawing with respect to the Orchard crossing number is rather worse with respect to the rectilinear crossing number, see Proposition 3.1).

A variant of the Orchard crossing number is the *maximal Orchard crossing number*:

Definition 2.6 (Maximal Orchard crossing number). *Let $G = (V, E)$ be an abstract graph. The* maximal Orchard crossing number *of G, MOCN(G), is*

$$\text{MOCN}(G) = \max_{R(G)}(n(R(G)))$$

This variant is extremely interesting due to the following result:

Proposition 2.7. *The drawing which yields the maximal Orchard crossing number for complete graphs K_n is the same as the drawing which attains the rectilinear crossing number of K_n.*

The importance of this result is that it might be possible that the computation of the maximal Orchard crossing number is easier than the computation of the rectilinear crossing number.

Proof. Let us look on quadruples of points: there are only two possibilities to draw four points in a generic position, see Figure 1.
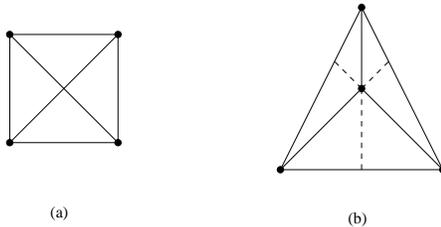

(a)

(b)

FIGURE 1. Two generic configurations of 4 points

Note that $\overline{\mathrm{cr}}(K_4) = 0$, since we can draw K_4 without any crossings of the edges (type (b)). Hence, for getting the minimal rectilinear drawing for K_n with respect to the rectilinear crossing number, we would like to have as many as possible quadruples of points arranged as in type (b).

On the other hand, for getting the maximal number of crossings with respect to the Orchard crossing number, we would like again to have as many as possible quadruples of points arranged as in type (b), since type (a) contributes 2 crossings (each internal edge separates the two ends of the other internal edge) while type (b) contributes 3 crossings (one on each edge of the convex hull). Hence, the rectilinear drawing which yields the maximal number of crossings with respect to the Orchard crossing number, will also give the minimal number of crossings with respect to the rectilinear crossing number.   □

2.1. **Some examples.** In this subsection, we give some examples for computing the Orchard crossing numbers for some small graphs.

The Orchard crossing number of K_4 is 2, since there are only two generic drawings of four points as presented in Figure 1. Type (a) has 2 crossings, while type (b) has 3 crossings. Therefore:

$$\mathrm{OCN}(K_4) = 2.$$

$$\mathrm{MOCN}(K_4) = 3.$$

For the complete bipartite graph $K_{2,2}$, we have more possibilities for rectilinear drawings, since except for the two choices for drawings, we have also to choose the two pairs of points. Figure 2(a) shows a drawing of $K_{2,2}$ without crossings at all (we distinguish between the two sets of two points by their colors). On the other hand, Figure 2(b) shows a rectilinear drawing of $K_{2,2}$ with 2 crossings. Hence, we get that

$$\mathrm{OCN}(K_{2,2}) = 0.$$

$$\text{MOCN}(K_{2,2}) = 2.$$
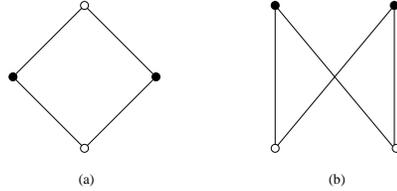

(a)   (b)

FIGURE 2. Generic rectilinear drawings of $K_{2,2}$

Note that $K_{2,2} \cong C_4$ where C_4 is the cycle graph on 4 vertices, and it is shown in [11] that the graphs whose Orchard crossing number equals 0 are the following:

(1) The cycle graphs, C_n, and their subgraphs.
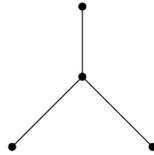(2) The graph on 4 vertices presented in Figure 3.

FIGURE 3. An example of a graph with $\text{OCN}(G) = 0$

3. ORCHARD CROSSING NUMBERS FOR COMPLETE GRAPHS K_n

For the complete graphs K_n, we have the following proposition:

Proposition 3.1. *Let $n \in \mathbb{N}$. Then*

$$\text{OCN}(K_n) = 2\binom{n}{4},$$

which is obtained by placing all vertices of K_n in a convex position.

Proof. Since there are only two generic rectilinear drawings of four points (as presented in Figure 1), any quadruple of points can contribute 2 (type (a)) or 3 (type (b)) to the total number of crossings.

Since we want to minimize the total number of crossings, we have to require that every quadruple of points will be in a convex position. This immediately implies that all n points have to be in a convex position as needed.

The computation of the crossing number is straight-forward: from the configuration of n points, we can choose $\binom{n}{4}$ different quadruples of points,

and each quadruple contributes 2 to the total number of crossings, which implies that the total number of crossings is $2\binom{n}{4}$. $\square$

Based on the last proposition, we compare in the following table between the Orchard crossing numbers and the rectilinear crossing numbers for the complete graph K_n, for $n \leq 12$.

n	$\text{OCN}(K_n)$	$\overline{\text{cr}}(K_n)$
4	2	0
5	10	1
6	30	3
7	70	9
8	140	19
9	252	36
10	420	62
11	660	102
12	990	153

Note that up to $n = 27$, $\overline{\text{cr}}(K_n)$ is known, and for $n \geq 28$, only bounds are known (see [1, 3, 4, 6, 7]).

4. Orchard crossing numbers for the star graph $K_{n,1}$

In this section, we deal with the Orchard crossing number of the graph $K_{n,1}$ (which is also called the *star graph*). Assume that we have n black points and one white point, and the white point is connected to all the black points. Then, the following holds:

Proposition 4.1. *The configuration of $n+1$ points which attains the minimal number of Orchard crossings for the graph $K_{n,1}$ consists of n black points which are the vertices of a regular n-gon, and the white point is located at the center of this polygon for odd n, or close to its center (i.e. it is not separated by a line from the center), for even n (see Figure 4 for examples of $n = 7, 8$).*

Hence, we have:

$$\text{OCN}(K_{n,1}) = \begin{cases} \frac{n(n-1)(n-3)}{8} & \text{odd } n \\ \frac{n(n-2)^2}{8} & \text{even } n \end{cases}$$

Proof. As in the proof of Proposition 3.1, we start by looking on quadruples of points. We have three cases of three black points and one white point (we exclude the case of quadruples of four black points, since it contributes nothing in any position):

(1) A white point inside a triangle of black points (see Figure 5(a)). This quadruple contributes nothing to the total number of crossings.
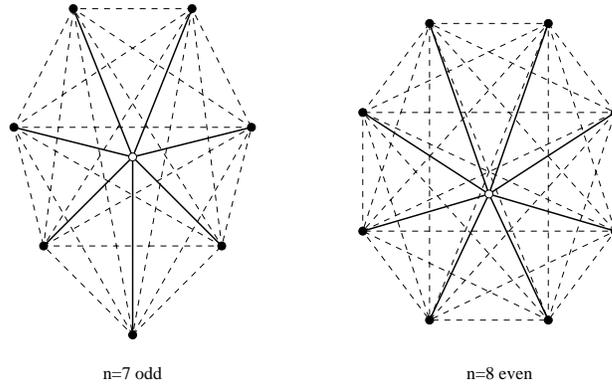
FIGURE 4. Examples of $K_{n,1}$ for odd n and for even n

(2) Three black points and one white point in a convex position (see Figure 5(b)). This quadruple contributes 1 to the total number of crossings.
(3) A black point inside a triangle consists of one white point and two black points (see Figure 5(c)). This quadruple contributes 2 to the total number of crossings.

(a)   (b)   (c)
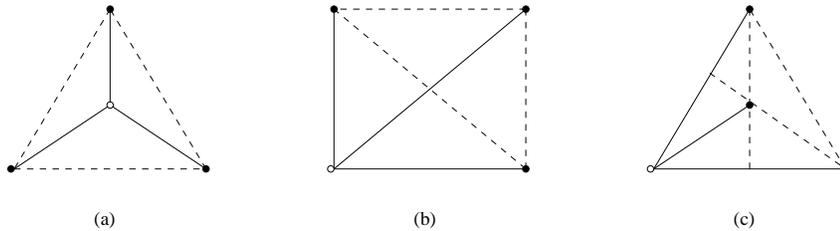
FIGURE 5. Three types of quadruples of points with three black points and one white point

Our aim is to minimize the number of quadruples of types (2) and (3) as a whole. Later, we will minimize the number of quadruples of type (3) as opposed to those of type (2). In order to compute the number of quadruples of types (2) and (3) in a given drawing of $K_{n,1}$, we label the black points $b_1, b_2, \ldots, b_n$ and the white point w.

For each $1 \leq i \leq n$, let L_i be the line which connects b_i to w. This line divides the remaining $n - 1$ black points into two subsets - one contains k_i points, and the other contains $(n - 1) - k_i$ points. For each pair of points, b_j and b_k, situated on the same side of L_i, the quadruple b_i, b_j, b_k, w, is of type (2) or (3) and hence contributes at least one

crossing. We have $\binom{k_i}{2} + \binom{(n-1)-k_i}{2}$ such pairs of points. Thus in total we have $\sum_{i=1}^{n} \left(\binom{k_i}{2} + \binom{(n-1)-k_i}{2} \right)$ quadruples of types (2) or (3). (Note that if b_j and b_k are separated by L_i, then the quadruple b_i, b_j, b_k, w, is of type (1) or (2) and hence contributes at most one crossing.)

Note that each quadruple is counted twice due to the following reason. Consider the quadruple b_i, b_j, b_k, w. Assume, without loss of generality, that b_i, b_j and b_k are oriented in clockwise fashion around w. Then, both lines L_i and L_k give this quadruple. However, the line L_j separates b_i from b_k and therefore will not give this quadruple. So we divide the above number of quadruples by 2.

Since each quadruple of type (2) or type (3) contributes at least one crossing, we have at least

$$\frac{\sum_{i=1}^{n} \left(\binom{k_i}{2} + \binom{(n-1)-k_i}{2} \right)}{2}$$

crossings.

This sum is minimized if $k_i = \frac{n-1}{2}$ for all i, i.e. every line connecting the white point to a black point divides the remaining black points into two subsets with the same number of points (or different by 1 for even n). This condition will be satisfied only if the n points are evenly distributed around the white point.

In order to determine the distance of the n black points to the white point, we consider types (2) and (3). In order to avoid any quadruple of type (3), we have to ensure that each quadruple b_i, b_j, b_k, w is in a convex position. This will be accomplished only if the n black points are all in a convex position around the white point. In this case, every quadruple which contains a white point, will be of type (2) and therefore will contribute only one crossing. This gives us the requested minimal rectilinear drawing.

Therefore, we can compute the number of crossings as follows. For odd n, we have $k_i = \frac{n-1}{2}$ for all $1 \le i \le n$. Thus, we have:

$$\begin{aligned} \mathrm{OCN}(K_{n,1}) &= \frac{\sum_{i=1}^{n} \left(\binom{\frac{n-1}{2}}{2} + \binom{(n-1)-\frac{n-1}{2}}{2} \right)}{2} = \\ &= n\binom{\frac{n-1}{2}}{2} = \frac{n\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)}{2} = \\ &= \frac{n(n-1)(n-3)}{8}. \end{aligned}$$

For even n, we have $k_i = \frac{n}{2}$ for all $1 \leq i \leq n$. Thus, we have:

$$
\begin{aligned}
\mathrm{OCN}(K_{n,1}) &= \frac{\sum_{i=1}^{n} \left(\binom{\frac{n}{2}}{2} + \binom{(n-1)-\frac{n}{2}}{2} \right)}{2} = \\
&= \frac{\sum_{i=1}^{n} \left(\binom{\frac{n}{2}}{2} + \binom{\frac{n-2}{2}}{2} \right)}{2} = \\
&= n \cdot \frac{\frac{\left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right)}{2} + \frac{\left(\frac{n-2}{2}\right)\left(\frac{n-4}{2}\right)}{2}}{2} = \\
&= \frac{n(n-2)(n+n-4)}{16} = \\
&= \frac{n(n-2)(2n-4)}{16} = \frac{n(n-2)^2}{8}.
\end{aligned}
$$

$\square$

In the following lemma, we present a different way to count the number of Orchard crossings in the minimal rectilinear drawing, as in Proposition 4.1.

Lemma 4.2.

$$
\mathrm{OCN}(K_{n,1}) = \begin{cases} \frac{n(n-1)(n-3)}{8} & \text{odd } n \\ \frac{n(n-2)^2}{8} & \text{even } n \end{cases}
$$

Proof. Since there are no quadruples of type (3) (since the only internal point is white), all the crossings come from quadruples of type (2), which contribute one crossing for each quadruple. So, we have to count the number of quadruples of type (2): quadruples with three black points and one white point in a convex position.

In the odd case, since the white point is in the center of the n-gon, for having a quadruple of type (2), all the black points should be within $180°$ of each other. For counting the number of such triples, we fix one black point. The other two points should be within $180°$ from this point. This gives us $\frac{n-1}{2}$ options to choose two points from. We can do this in $\binom{\frac{n-1}{2}}{2}$ ways. Now since we could have started with any of the n points, we have to multiply it by n. So, we get:

$$
n\binom{\frac{n-1}{2}}{2} = \frac{n \cdot \frac{n-1}{2} \frac{n-3}{2}}{2} = \frac{n(n-1)(n-3)}{8},
$$

as needed.

Now we turn to the even case. This case is more complicated, since if we choose two antipodal black points, we have to cut the number of quadruples by 2, due to the central white point which is not located exactly at the center

(see Figure 4, $n = 8$). So, we split the count of triples into two cases: triples which do not include antipodal points, and triples which include antipodal points.

We start with the case which does not include antipodal points: Fix a black point. We have to choose two black points out of $\frac{n-2}{2}$ options. Hence, we have $\binom{\frac{n-2}{2}}{2}$ possibilities , and we have in total:

$$n\binom{\frac{n-2}{2}}{2} = \frac{n(n-2)(n-4)}{8}$$

crossings.

Now we count the number of triples which include antipodal points: We have $\frac{n}{2}$ pairs of points in antipodal position. The third point can be any of the other $n - 2$ points. Since we only count half of these triples (as explained above), we have $\frac{n(n-2)}{4}$ crossings.

Summing up the two cases, we get:

$$\frac{n(n-2)(n-4)}{8} + \frac{n(n-2)}{4} = \frac{n(n-2)(n-2)}{8},$$

again as needed.   □

5. Orchard crossing number of a wheel

In this section, we deal with the Orchard crossing number of a *wheel*, which is an abstract graph on $n + 1$ points, which is composed of a cycle of n points and one other point connected to each point of the cycle (see Figure 6 for an example). We denote such a graph by $W_{n,1}$
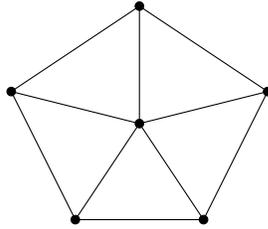

FIGURE 6. The wheel graph $W_{5,1}$

Before stating the general case, a simple observation yields the following facts:

Remark 5.1.

$$\mathrm{OCN}(W_{3,1}) = 2$$
$$\mathrm{OCN}(W_{4,1}) = 6$$

Proof. Note that $W_{3,1} = K_4$, and hence the result follows.

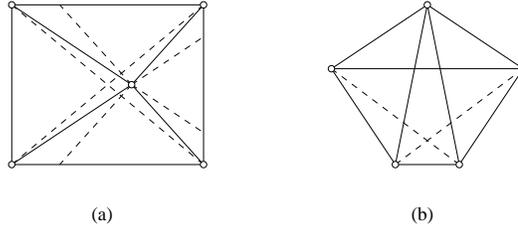For $W_{4,1}$, we have two optimal drawings, each of them has 6 crossings, see Figure 7.

(a)   (b)

FIGURE 7. Two optimal drawings for $W_{4,1}$

$\square$

Here we present the general case:

Proposition 5.2. *Let $n \geq 5$. Let $G = W_{n,1}$ be a graph consisting of $n+1$ vertices configured as a wheel.*

The configuration of $n + 1$ points which attains the minimal number of Orchard crossings consists of n points which are the vertices of a regular n-gon (these points form the cycle), and the other point is located at the center of this polygon for odd n, or close to its center (i.e. it is not separated by a line from the center) for even n.

Hence, we have:

$$\mathrm{OCN}(W_{n,1}) = \begin{cases} \frac{n(n^2 - 4n + 11)}{8} & \text{odd } n \\ \frac{n(n^2 - 4n + 12)}{8} & \text{even } n \end{cases}$$

First we will prove that this drawing minimizes the crossings for a wheel. Notice that a wheel $W_{n,1}$ is very similar to $K_{n,1}$. The only difference is that in $W_{n,1}$, we have an additional cycle on the n external points.

For $K_{n,1}$, we know that the above configuration minimizes the number of crossings. Thus, all we have to show is that the addition of the cycle maintains the minimality. Hence, we have to consider how many crossings are added by the addition of the cycle in the optimal configuration, and how many are added in other configurations.

There are two types of edges in our configuration:

(1) the edges on the cycle, and
(2) the edges which connect the center and an external point.

Let us consider them one at a time.

(1) The edges of the cycle do not separate between any pair of points in the configuration.

(2) Each edge connecting the center and an external point will inter-
    sect the cycle once (where the continuation of the edge crosses the
    convex hull). This intersection contributes 1 to the total number
    of crossings (because this edge separates the vertices on the cycle).

Since we have n external points, we conclude that the addition of the cy-
cle contributes n crossings in total. Now let us consider how many crossings
the addition of the cycle contributes in other configurations.

Lemma 5.3. *Let $R(W_{n,1})$ be a rectilinear drawing for the wheel $W_{n,1}$.
Then, we have for $n \geq 5$:*

$$n(R(W_{n,1})) \geq \mathrm{OCN}(K_{n,1}) + n,$$

Proof. We split the proof into two cases, depending on the position of the
central point of the wheel in the drawing.

Case 1: If the central point of the wheel is inside the convex hull generated
by the points of the cycle, then for any edge connecting the center and
an external point, the extension of this edge separates two points on the
convex hull of the cycle. This occurs where the extension of the edge exits
the interior of the cycle. Since we have n points on the cycle, we add at
least n crossings. Hence, we have:

$$n(R(W_{n,1})) \geq n(R(K_{n,1})) + n \geq \mathrm{OCN}(K_{n,1}) + n,$$

(where $R(K_{n,1})$ is the corresponding drawing for $K_{n,1}$).

Case 2: If the central point of the wheel is outside the convex hull generated
by the points of the cycle, we just show that

$$n(R(W_{n,1})) \geq \mathrm{OCN}(K_{n,1}) + n.$$

by a direct computation: in case that the central point of the wheel is
outside the convex hull generated by the points of the cycle, each quadruple
involved the central point and three points of the cycles contributes at least
one crossing (see types (b) and (c) in Figure 5). Since we have $\binom{n}{3}$ such
quadruples, we have $n(R(W_{n,1})) \geq \binom{n}{3}$. On the other hand, for $n \geq 5$,

$$\binom{n}{3} \geq \mathrm{OCN}(K_{n,1}) + n$$

since for odd n (the even case is similar), we have:

$$\binom{n}{3} - (\mathrm{OCN}(K_{n,1}) + n) =$$

$$= \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)(n-3)}{8} - n =$$

$$= \frac{4n(n-1)(n-2) - 3n(n-1)(n-3) - 24n}{24} =$$

$$= \frac{n^3 - 25n}{24} \geq 0$$

$\square$

Hence, we conclude that the rectilinear drawing of the wheel with the minimal number of Orchard crossings is as stated in Proposition 5.2.

To verify that the values of $\mathrm{OCN}(W_{n,1})$ which stated in Proposition 5.2 are correct, we simply have to take the values for $\mathrm{OCN}(K_{n,1})$ and add n for the n crossings contributed by the edges of the cycle (as above).

6. Complete bipartite graphs of type $K_{n,m}$

The case of complete bipartite graphs of type $K_{n,n}$ is complicated, and we leave its long proof of the following proposition to a different paper [10]:

Proposition 6.1.

$$\mathrm{OCN}(K_{n,n}) = 4n\binom{n}{3},$$

which is achieved by the regular $2n$-gon, alternating in colors.

In Table 1, we summarize our computational results (based on the database of Aichholzer, Aurenhammer and Krasser [2]) for the Orchard crossing number for some complete bipartite graphs $K_{n,m}$, where $n \neq m$, $n > 1$ and $m > 1$. In this table, the rows will be the different values of n, and the columns will be the different values of m.

n: / m:	2	3	4	5	6
2	0	4	12	26	48
3	4	12	32	63	
4	12	32	64		
5	26	63			
6	48				

TABLE 1. $\mathrm{OCN}(K_{n,m})$ for different values of n and m

6.1. **The maximal Orchard crossing number for complete bipartite graphs** $K_{n,m}$. In this section, we deal with the maximal Orchard crossing number of bipartite graphs $K_{n,m}$. Assume that we have n black points and m white points, and all the white points are connected to all the black points. Then, the following holds:

Proposition 6.2. *The configuration of $n + m$ points which attains the maximal number of Orchard crossings for the graph $K_{n,m}$ consists of n black points and m white points which are organized on two arcs facing each other, where the black points are located on one arc and the white points are located on the other arc (see Figure 8 for an example for $n = m = 4$).*
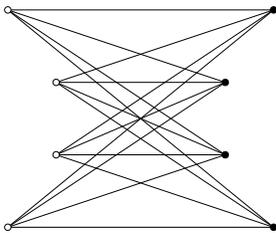

FIGURE 8. The optimal drawing for MOCN($K_{4,4}$)

Hence, we have:

$$\text{MOCN}(K_{n,m}) = 2\binom{n}{2}\binom{m}{2} + 2m\binom{n}{3} + 2n\binom{m}{3}$$

Proof. As in the previous proofs, we start by looking on quadruples of points. We have three types of quadruples with two black points and two white points:

(1) Two black points and two white points in a convex position and the points alternate in colors (see Figure 9(a)). This quadruple contributes nothing to the total number of crossings.
(2) Two black points and two white points in a convex position and the points do not alternate in colors (see Figure 9(b)). This quadruple contributes 2 to the total number of crossings.
(3) Two black points and two white points which are not in a convex position (see Figure 9(c)). This quadruple contributes 2 to the total number of crossings.

On the other hand, as in the proof of Proposition 4.1, we have three types of quadruples with three black points and one white point:

(1) The three black points are in the convex hull and the white point is inside this triangle (see Figure 5(a)). This quadruple contributes nothing to the total number of crossings.
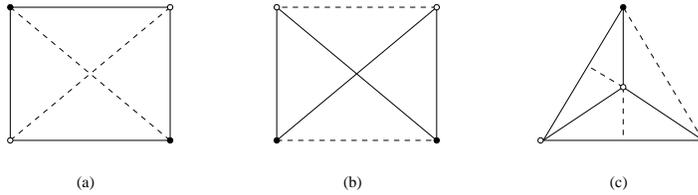
FIGURE 9. Three cases of quadruples of points with two black points and two white points

(2) All the four points are in a convex position (see Figure 5(b)). This quadruple contributes 1 to the total number of crossings.

(3) The convex hull consists of two black points and a white point (see Figure 5(c)). This quadruple contributes 2 to the total number of crossings.

We shall show that the drawing appearing in the formulation of the proposition indeed attains the maximal possible number of Orchard crossings for the graph $K_{n,m}$. Let us start with quadruples with two black points and two white points. The maximal number of crossings which such a quadruple can contribute is 2. In the above drawing, all the quadruples consisting on two black points and two white points indeed contribute 2 (all the quadruples are organized as in type (b)).

Now we move to quadruples with three points of one color and one point of the other color. For such quadruples, the maximal number of crossings is again 2. Note that in the above drawing, all the quadruples of this type indeed contribute 2 (all the quadruples are organized as in type (c)).

Hence, we get that the drawing appearing in the formulation of the proposition indeed attains the maximal possible number of Orchard crossings for $K_{n,m}$.

For computing MOCN($K_{n,m}$), we simply count the number of quadruples of the different types:

(1) We have $\binom{n}{2}\binom{m}{2}$ quadruples with two black points and two white points, since for constructing such a quadruple, we have to choose two black points (out of n) and two white points (out of m). Since each quadruple contributes 2, the total contribution of these quadruples is $2\binom{n}{2}\binom{m}{2}$.

(2) We have $n\binom{m}{3}$ quadruples with one black point and three white points, since for constructing such a quadruple, we have to choose one black point (out of n) and three white points (out of m). Similarly, we have $m\binom{n}{3}$ quadruples with one white point and three black points. Since each quadruple contributes 2, the total contribution of these quadruples is $2m\binom{n}{3} + 2n\binom{m}{3}$.

Summing up the contributions, we get the desired result.   □

7. The Orchard crossing number of a union of two graphs

In this section, we deal with the question: what happens to the Orchard crossing number when we unite two graphs.

We deal with a property of a union of two special graphs:

Proposition 7.1. *Let G and H be two graphs on the* same *set of vertices. Assume that G and H get the minimal number of Orchard crossings in the same rectilinear drawing C. Assume also that H is contained in the complement of G (so they have no edge in common).*

Then, the configuration C is the minimal configuration with respect to the Orchard crossing number for $G \cup H$, and:

$$\mathrm{OCN}(G \cup H) = \mathrm{OCN}(G) + \mathrm{OCN}(H).$$

Proof. Assume on the contrary that there is a better rectilinear drawing for $G \cup H$. Since in the better rectilinear drawing, $\mathrm{OCN}(G \cup H) < \mathrm{OCN}(G) + \mathrm{OCN}(H)$, at least one of the following will occur:

(1) If we delete the edges of H, we get a better rectilinear drawing for G, or

(2) If we delete the edges of G, we get a better rectilinear drawing for H,

which is a contradiction.   □

Remark 7.2. *Although one can decompose $W_{n,1}$ into a disjoint union of $K_{n,1}$ and a cycle of length n, C_n, we can not apply the last proposition, since the cycle of length n has another isolated vertex, and hence its minimal drawing is no more n points in convex position and one point inside (as in Figure 10(a), which has n crossings), but $n+1$ points in a convex position (as in Figure 10(b), which has $n-2$ crossings).*
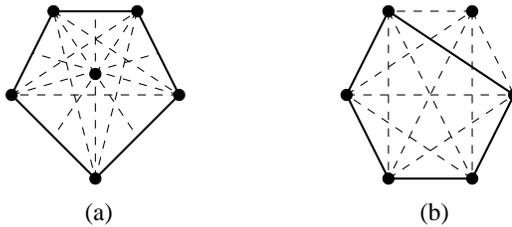

(a)   (b)

FIGURE 10. Two drawings of a cycle of length $n = 5$ with additional isolated vertex

Acknowledgments

References

[1] Abrego, B.M., Fernandez-Merchant, S., Leanos, J. and Salazar, G., *The maximum number of halving lines and the rectilinear crossing number of K_n for $n \leq 27$*, Electronic Notes in Discrete Math. **30** (2008), 261–266.

[2] Aichholzer, O., Aurenhammer, F. and Krasser, H., *Enumerating order types for small point sets with applications*, in: Proc. 17th Ann. ACM Symp. Computational Geometry, 11–18, Medford, Massachusetts, USA, 2001.

[3] Aichholzer, O., Aurenhammer, F. and Krasser, H., *On the crossing number of complete graphs*, in: Proc. 18th Ann. ACM Symp. Computational Geometry, 19–24, Barcelona, Spain, 2002.

[4] Aichholzer, O., Garcia, J., Orden, D. and Ramos, P., *New lower bounds for the number of ($\leq k$)-edges and the rectilinear crossing number of K_n*, Discrete Comput. Geom. **38**(1) (2007), 1–14.

[5] Aichholzer, O. and Krasser, H., *The point set order type data base: A collection of applications and results*, in: Proc. 13th Annual Canadian Conference on Computational Geometry CCCG 2001, 17–20, Waterloo, Ontario, Canada, 2001.

[6] Aichholzer, O. and Krasser, H., *Abstract order type extension and new results on the rectilinear crossing number*, Computational Geometry: Theory and Applications, Special Issue on the 21st European Workshop on Computational Geometry, **36**(1) (2006), 2–15.

[7] Aichholzer, O., Orden, D. and Ramos, P.A., *On the structure of sets attaining the rectilinear crossing number*, In: Proc. 22nd European Workshop on Computational Geometry EuroCG '06, pp. 43–46, Delphi, Greece, 2006.

[8] Bacher, R., *Le cocycle du verger*, C. R. Acad. Sci. Paris, Ser. I, **338**(3) (2004), 187–190.

[9] Bacher, R. and Garber, D., *The Orchard relation of planar configurations of points I*, Geombinatorics **15**(2) (2005), 54–68.

[10] Feder, E. and Garber, D., *On the Orchard crossing number of complete bipartite graphs*, in preparation.

[11] Garber, D., Pinchasi, R. and Sharir, M., *On some properties of the Orchard crossing number of an abstract graph*, in preparation.

[12] Pach, J. and Tóth, G., *Thirteen problems on crossing numbers*, Geombinatorics **9** (2000), 194–207.

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