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Generalizing Liouville-type Problems for Differential 1-Forms from L^q Spaces to Non-L^q Spaces

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Abstract

We obtain Liouville-type results for closed and p-pseudo-coclosed differential 1-forms ω with energy of $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B(x_0;r)} |\omega|^q dv < \infty$ (that is, 2-finite growth), which extends finite q-energy ($\int_M |\omega|^q dv < \infty$) in L^q spaces to infinite q-energy ($\int_M |\omega|^q dv = \infty$) in non-$L^q$ spaces. In particular, we recapture mathematicians' vanishing results of Liouville-type theorem for ω with finite q-energy in L^q spaces. Our method in this paper provides a successful way to work on Liouville-type problems for differential forms with a variety of energy conditions in broad spaces.

Mathematics Subject Classification: 53C23, 53C21

Keywords: closed 1-forms, p-pseudo-coclosed 1-forms, L^q space, Ricci curvature, Sobolev inequality, Bochner-Weitzenböck formula

1 Introduction

The study of Liouville-type problems and vanishing results of Liouville-type problems for differential forms or maps on curved Riemannian manifolds has been one of the most valuable and challenging research topics in the mathematical society. Most research findings of Liouville-type problems are restricted with only finite q-energy in L^q spaces on curved manifolds into the following two basic categories:

1. For differential forms in L^q spaces: Regarding vanishing results of Liouville-type problems on complete non-compact manifolds with non-negative Ricci curvature, Greene-Wu [3] proved vanishing theorems for harmonic 1-forms in L^q $(1 < q < \infty)$ spaces in 1981, and Zhang [14] obtained his vanishing results for closed and p-coclosed 1-forms $(p > 1)$ in L^q $(0 < q < \infty)$ spaces in 2001.

2. For maps in L^q spaces: In 1976, Schoen and Yau [7] established Liouville-type results for harmonic maps on complete non-compact manifolds with non-negative Ricci curvature. In 1995, Cheung and Leung [2] showed their Liouville theorems for p-harmonic maps $(p \geq 2)$ in L^q $(q = p - 1)$ spaces when the target manifold was Cartan-Hadamand, i.e., a complete simply connected manifold with non-negative sectional curvatures. In 1999, Kawai [4] derived Liouville-type theorems for p-harmonic maps $(p \geq 2)$ in L^q $(q = p)$ spaces, which were homotopic to constant maps from p-parabolic manifolds to non-positively curved targets. In 2001, Zhang [14] proved Liouville-type theorems for p-harmonic maps $(p > 1)$ in L^q $(0 < q < \infty)$ spaces. In 2008, Pigola, Rigoli, and Setti [5] studied Liouville-type theorems for p-harmonic maps $(p \geq 2)$ from the domains of manifolds satisfying Poincaré-Sobolev Inequality to the target spaces of non-positively curved manifolds.

Regarding to Liouville-type problems with finite q-energy in L^q spaces, various methods have been studied and numerous results have been obtained by mathematicians. One natural question arises: Can we generalize Liouville-type problems under a variety of energy conditions in broad spaces? Extending Liouville-type results from finite q-energy in L^q spaces to infinite q-energy in non-L^q spaces and obtaining vanishing results of Liouville-type problems for maps or differential forms become interesting and challenging research topics.

In our paper, for closed and p-pseudo-coclosed 1-forms ω, we explore an extension of Liouville-type results from finite q-energy in L^q spaces to energy of $\liminf_{r\to\infty} \frac{1}{r^2} \int_{B(x_0;r)} |\omega|^q dv < \infty$ (i.e. 2-finite growth) including both L^q spaces (i.e. $\int_M |\omega|^q dv < \infty$) and non-$L^q$ spaces (i.e. $\int_M |\omega|^q dv = \infty$). Here, ω with 2-finite growth is p-finite for $p = 2$, which is one of 5 cases of p-balanced growth. Definitions and examples of 5 cases in p-balanced growth including different kinds of infinite q-energy in non-L^q spaces can be found in Wei, Li and Wu's work (cf. [9, 10]). In particular, for harmonic forms, Wei and Wu have successfully extended Liouville-type problems from finite q-energy in L^q spaces to infinite q-energy in non-L^q spaces with p-balanced growth when $p = 2$ (cf. [11]). Our method in this paper provides a successful way to work on Liouville-type problems for differential 1-forms in a variety of energy with p-balanced growth for $p = 2$.

2 Preliminary

Let M be an n-dimensional complete non-compact Riemannian manifold and $B(x_0; r)$ (or $B(r)$) denote the geodesic ball of radius r centered at a point $x_0 \in M$.

Let $\mathcal{A}^k(\rho) = C(\wedge^k T^*M \otimes V)$ be the space of smooth k-forms on M with values in the vector bundle $\rho : V \to M$. Let $d : \mathcal{A}^k(\rho) \to \mathcal{A}^{k+1}(\rho)$ be the exterior differential operator and $d^* : \mathcal{A}^k(\rho) \to \mathcal{A}^{k-1}(\rho)$ be the adjoint differential operator of d given by $d^* = -\sum_{j=1}^n i(e_j)\nabla_{e_j}$ where $\{e_j\}$ is a local orthonormal frame at $x \in M$, and $i(X)$ is the interior product by X given by $(i(X)\nu)(Y_1, \cdots, Y_{k-1}) = \nu(X, Y_1, \cdots, Y_{k-1})$ for any $X \in T_x(M), \nu \in \mathcal{A}^k(\rho)$ and $Y_l \in T_x(M), 1 \le l \le k-1$. In particular, if $\nu \in \mathcal{A}^1(\rho)$, d^* is also defined by $d^*\nu = -\text{trace}\nabla\nu = -\text{div}\nu$. The norm of v is denoted by $|\nu| = \langle \nu, \nu \rangle^{\frac{1}{2}}$. The Hodge Laplacian Δ is defined on the V-valued differential forms by

$$\Delta = -(dd^* + d^*d) : \mathcal{A}^k(V) \to \mathcal{A}^k(V).$$

More details can be found in [12].

Definition 2.1. *A differential form ξ is said to be closed if $d\xi = 0$.*

Definition 2.2. *A differential form ξ is said to be p-pseudo-coclosed ($p > 1$) if*

$$d^*(|\xi|^{p-2}\xi) = 0.$$

For example, the differential of a p-harmonic function is a closed and p-pseudo-coclosed 1-form.

Definition 2.3. *A differential form ξ is said to be with finite q-energy in L^q space if*

$$\int_M |\xi|^q dv < \infty$$

for $q > 0$.

We recall the definitions of p-balanced growth (including p-finite growth, p-mild growth, p-obtuse growth, p-moderate growth, and p-small growth) and its counter-part p-imbalanced growth (including respectively p-infinite growth, p-severe growth, p-acute growth, p-immoderate growth, and p-large growth) as follows (cf. [9, 10]):

Definition 2.4. *A function or a differential form f has* p-finite growth (*or,* is p-finite) *if there exists $x_0 \in M$ such that*

$$\liminf_{r\to\infty} \frac{1}{r^p} \int_{B(x_0;r)} |f|^q dv < \infty \tag{1}$$

and has p-infinite growth (*or,* is p-infinite) *otherwise.*

A function or a differential form f has p-mild growth (*or,* is p-mild) *if there exist $x_0 \in M$, and a strictly increasing sequence of $\{r_j\}_0^\infty$ going to infinity, such that for every $l_0 > 0$, we have*

$$\sum_{j=\ell_0}^{\infty} \left(\frac{(r_{j+1} - r_j)^p}{\int_{B(x_0;r_{j+1})\setminus B(x_0;r_j)} |f|^q dv} \right)^{\frac{1}{p-1}} = \infty\,, \tag{2}$$

and has p-severe growth (*or,* is p-severe) *otherwise.*

A function or a differential form f has p-obtuse growth (*or,* is p-obtuse) *if there exists $x_0 \in M$ such that for every $a > 0$, we have*

$$\int_a^\infty \left(\frac{1}{\int_{\partial B(x_0;r)} |f|^q ds} \right)^{\frac{1}{p-1}} dr = \infty\,, \tag{3}$$

and has p-acute growth (*or,* is p-acute) *otherwise.*

A function or a differential form f has p-moderate growth (*or,* is p-moderate) *if there exist $x_0 \in M$, and $F(r) \in \mathcal{F}$,such that*

$$\limsup_{r\to\infty} \frac{1}{r^p F^{p-1}(r)} \int_{B(x_0;r)} |f|^q dv < \infty\,, \tag{4}$$

and has p-immoderate growth (*or,* is p-immoderate) *otherwise, where*

$$\mathcal{F} = \{F : [a,\infty) \longrightarrow (0,\infty) | \int_a^\infty \frac{dr}{rF(r)} = \infty \quad for \quad some \quad a \geq 0\}\,. \tag{5}$$

(*Notice that the functions in $\mathcal{F}$ are not necessarily monotone.*)

A function or a differential form f has p-small growth (*or,* is p-small) *if there exists $x_0 \in M$, such that for every $a > 0$,we have*

$$\int_a^\infty \left(\frac{r}{\int_{B(x_0;r)} |f|^q dv} \right)^{\frac{1}{p-1}} dr = \infty\,, \tag{6}$$

and has p-large growth (*or,* is p-large) *otherwise.*

In Definition 2.4, q denotes a real number whose value will be specified in the context in which Definition 2.4 is used.

Here we introduce the following two lemmas, which are used in the proof of our main theorem in section 3.

Lemma 2.5. *(Sobolev Inequality (Theorem 10.4 in [6])) Let M be an n-dimensional complete non-compact Riemannian manifold with non-negative Ricci curvature. If $n > 2$, then there exists $\beta > 0$ depending on n such that for any geodesic ball $B(x;r)$ we have*

$$\beta(vol(B(x;r)))^{\alpha}\frac{1}{r^2}\left(\int_{B(x;r)}|\varphi|^{\frac{2}{1-\alpha}}dv\right)^{1-\alpha} \leq \int_{B(x;r)}|\nabla\varphi|^2 dv \tag{7}$$

for every compactly supported smooth function φ on $B(x;r)$ and $\alpha = \frac{2}{n}$.

If $n \leq 2$, then the above inequality holds for any $0 < \alpha < 1$.

Lemma 2.6. *(Infinite Volume of Manifold (cf. [1, 13, 8])) Every complete non-compact Riemannian manifold with non-negative Ricci curvature must have an infinite volume.*

3 Main Results with Proofs

Here we give proofs of the following two lemmas which play an important role in proof of the main theorem.

Lemma 3.1. *Suppose ξ is a differential form on an n-dimensional manifold M. Then:*

$$d^*(f\xi) = fd^*\xi - i(\nabla f)\xi \tag{8}$$

for any $f \in C^\infty(M)$. In particular, if ξ is a differential 1-form on M, we have:

$$\begin{array}{rcl} d^*(f\xi) & = & fd^*\xi - \langle df, \xi\rangle \\ fd^*\xi & = & d^*(f\xi) + \langle df, \xi\rangle \end{array} \tag{9}$$

Proof. Let $\{e_i\}$ be an orthonormal frame field on M.

$$\begin{array}{rcl} d^*(f\xi) & = & -\sum_{j=1}^n i(e_j)\nabla_{e_j}(f\xi) \\ & = & -\sum_{j=1}^n i(e_j)(f\nabla_{e_j}\xi + (e_jf)\xi) \\ & = & f(-\sum_{j=1}^n i(e_j)\nabla_{e_j}\xi) - i(\sum_{j=1}^n (e_jf)e_j)\xi \\ & = & fd^*\xi - i(\nabla f)\xi \end{array}$$

where we use the definition of d^* given by $d^* = -\sum_{j=1}^n i(e_j)\nabla_{e_j}$. In particular, if ξ is a differential 1-form, we have $i(\nabla f)\xi = \langle df, \xi\rangle$. □

Lemma 3.2. *Suppose a differential* 1*-form* ω *is closed and* p*-pseudo-coclosed on* M*. Let* $\eta \geq 0$ *and* $\eta \in C_0^\infty(M)$ *(that is, a compactly supported non-negative smooth function on* M*), and* $\phi = \eta|\omega|^m$ *where* $m \geq p \geq 2$*. Then*

$$\begin{aligned} & \int_M \phi^2 \langle \triangle\omega, \omega\rangle dv \\ = \ & \frac{(p-2)(2m+2-p)}{4} \int_M \eta^2 |\omega|^{2m-4} \langle d|\omega|^2, \omega\rangle^2 dv \\ & +(p-2)\int_M \eta|\omega|^{2m-2}\langle d|\omega|^2,\omega\rangle\langle d\eta,\omega\rangle dv \\ \geq \ & -(p-2)\int_M \eta|\nabla\eta||\omega|^{2m}|\nabla|\omega|^2| dv. \end{aligned} \tag{10}$$

Proof. Since ω is closed, i.e. $d\omega = 0$, then

$$\begin{aligned} \Delta\omega &= -(dd^*\omega + d^*d\omega) \\ &= -dd^*\omega. \end{aligned}$$

Let $\phi = \eta|\omega|^m$ for $m \geq p \geq 2$, then

$$\begin{aligned} & \int_M \phi^2\langle\triangle\omega,\omega\rangle dv \\ = \ & \int_M \eta^2|\omega|^{2m}\langle -dd^*\omega,\omega\rangle dv \\ = \ & -\int_M \langle \eta^2|\omega|^m dd^*\omega, |\omega|^m\omega\rangle dv \\ = \ & -\int_M \langle d(\eta^2|\omega|^m d^*\omega), |\omega|^m\omega\rangle dv + \int_M \langle d^*\omega\cdot d(\eta^2|\omega|^m), |\omega|^m\omega\rangle dv \\ = \ & -\int_M \langle \eta^2|\omega|^m d^*\omega, d^*(|\omega|^m\omega)\rangle dv + \int_M \langle d^*\omega\cdot d(\eta^2|\omega|^m), |\omega|^m\omega\rangle dv. \end{aligned} \tag{11}$$

According to (9) and the assumption that ω is p-pseudo-coclosed, that is, $d^*(|\omega|^{p-2}\omega) = 0$, we obtain, due to $m+2-p \geq 2$

$$\eta^2|\omega|^m d^*\omega = d^*(\eta^2|\omega|^m\omega) + \langle d(\eta^2|\omega|^m),\omega\rangle, \tag{12}$$

$$\begin{aligned} d^*(|\omega|^m\omega) &= d^*(|\omega|^{m+2-p}|\omega|^{p-2}\omega) \\ &= |\omega|^{m+2-p}d^*(|\omega|^{p-2}\omega) - \langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle \\ &= -\langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle, \end{aligned} \tag{13}$$

$$\begin{aligned} d^*(\eta^2|\omega|^m\omega) &= d^*(\eta^2|\omega|^{m+2-p}|\omega|^{p-2}\omega) \\ &= \eta^2|\omega|^{m+2-p}d^*(|\omega|^{p-2}\omega) - \langle d(\eta^2|\omega|^{m+2-p}), |\omega|^{p-2}\omega\rangle \\ &= -\langle d(\eta^2|\omega|^{m+2-p}), |\omega|^{p-2}\omega\rangle. \end{aligned} \tag{14}$$

Via (12), (13) and (14), we have

$$\begin{aligned}
&-\int_M \langle \eta^2|\omega|^m d^*\omega, d^*(|\omega|^m\omega)\rangle dv \\
= &\int_M (d^*(\eta^2|\omega|^m\omega) + \langle d(\eta^2|\omega|^m), \omega\rangle)\langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle dv \\
= &\int_M (-\langle d(\eta^2|\omega|^{m+2-p}), |\omega|^{p-2}\omega\rangle + \langle d(\eta^2|\omega|^m), \omega\rangle)\langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle dv \\
= &\int_M (\eta^2\langle d|\omega|^m, \omega\rangle - \eta^2\langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle)\langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle dv. \quad (15)
\end{aligned}$$

Meanwhile,

$$\int_M \langle d^*\omega \cdot d(\eta^2|\omega|^m), |\omega|^m\omega\rangle dv = \int_M \langle |\omega|^m d^*\omega \cdot d(\eta^2|\omega|^m), \omega\rangle dv. \quad (16)$$

Applying (9) and $d^*(|\omega|^{p-2}\omega) = 0$ again, we have

$$\begin{aligned}
|\omega|^m d^*\omega &= d^*(|\omega|^m\omega) + \langle d|\omega|^m, \omega\rangle \\
&= d^*(|\omega|^{m+2-p}|\omega|^{p-2}\omega) + \langle d|\omega|^m, \omega\rangle \\
&= |\omega|^{m+2-p} d^*(|\omega|^{p-2}\omega) - \langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle + \langle d|\omega|^m, \omega\rangle \\
&= \langle d|\omega|^m, \omega\rangle - \langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle. \quad (17)
\end{aligned}$$

Plugging (17) into (16), we have

$$\begin{aligned}
&\int_M \langle d^*\omega \cdot d(\eta^2|\omega|^m), |\omega|^m\omega\rangle dv \\
= &\int_M \langle |\omega|^m d^*\omega \cdot d(\eta^2|\omega|^m), \omega\rangle dv \\
= &\int_M (\langle d|\omega|^m, \omega\rangle - \langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle)\langle d(\eta^2|\omega|^m), \omega\rangle dv \\
= &\int_M \langle d|\omega|^m, \omega\rangle \cdot \{\eta^2\langle d|\omega|^m, \omega\rangle + |\omega|^m\langle d\eta^2, \omega\rangle\} dv \\
&\quad - \int_M \langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle \cdot \{\eta^2\langle d|\omega|^m, \omega\rangle + |\omega|^m\langle d\eta^2, \omega\rangle\} dv. \quad (18)
\end{aligned}$$

Therefore, based on (11), (15) and (18), we have, if $m \geq p \geq 2$:

$$\begin{aligned}
&\int_M \phi^2 \langle \triangle\omega, \omega\rangle dv \\
= &\int_M (\eta^2\langle d|\omega|^m, \omega\rangle - \eta^2\langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle)\langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle dv \\
&\quad + \int_M \langle d|\omega|^m, \omega\rangle \cdot \{\eta^2\langle d|\omega|^m, \omega\rangle + |\omega|^m\langle d\eta^2, \omega\rangle\} dv \\
&\quad - \int_M \langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle \cdot \{\eta^2\langle d|\omega|^m, \omega\rangle + |\omega|^m\langle d\eta^2, \omega\rangle\} dv \\
= &\int_M \eta^2(\langle d|\omega|^m, \omega\rangle^2 - \langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle^2) dv \\
&\quad + \int_M (\langle d|\omega|^m, \omega\rangle - \langle d|\omega|^{m+2-p}, |\omega|^{p-2}\omega\rangle)|\omega|^m\langle d\eta^2, \omega\rangle dv \\
= &\frac{(p-2)(2m+2-p)}{4}\int_M \eta^2|\omega|^{2m-4}\langle d|\omega|^2, \omega\rangle^2 dv \\
&\quad +(p-2)\int_M \eta|\omega|^{2m-2}\langle d|\omega|^2, \omega\rangle\langle d\eta, \omega\rangle dv \\
\geq &-(p-2)\int_M \eta|\nabla\eta||\omega|^{2m}|\nabla|\omega|^2| dv.
\end{aligned}$$

□

Here we give the detailed proof of our main theorem.

Theorem 3.3. *Let ω be a closed and p-pseudo-coclosed differential 1-form on a complete non-compact manifold M with non-negative Ricci curvature. Then $|\omega| =$ constant if there exists $x_0 \in M$ such that for $q \in [2p+2, \infty)$ with $p \geq 2$*

$$\liminf_{r\to\infty} \frac{1}{r^2}\int_{B(x_0;r)} |\omega|^q dv < \infty \quad \textit{(i.e. 2-finite growth).}$$

In particular, $\omega = 0$ if

$$\int_M |\omega|^q dv < \infty \quad \textit{(i.e. } \omega \in L^q(M)\textit{).}$$

Proof. Applying Bochner-Weitzenböck formula for ω on M, we have:

$$\begin{aligned}
\tfrac{1}{2}\triangle|\omega|^2 &= \langle \triangle\omega, \omega\rangle + |\nabla\omega|^2 + \mathcal{R}(\omega, \omega) \\
&\geq \langle \triangle\omega, \omega\rangle + |\nabla\omega|^2
\end{aligned} \tag{19}$$

where $\mathcal{R}(\omega, \omega)$ denotes the Ricci curvature of M in the direction of ω, which is non-negative. It follows from (19) and the observation $|\nabla\omega| \geq |\nabla|\omega||$ that

$$\langle \triangle\omega, \omega\rangle + |\nabla|\omega||^2 - \frac{1}{2}\triangle|\omega|^2 \leq 0. \tag{20}$$

Multiplying ϕ^2 on both sides of (20) and applying integration by parts yield:

$$\int_M \phi^2\langle\triangle\omega,\omega\rangle dv + \int_M \phi^2|\nabla|\omega||^2 dv + \frac{1}{2}\int_M \langle\nabla\phi^2,\nabla|\omega|^2\rangle dv \le 0. \tag{21}$$

We choose the test function $\phi = \eta|\omega|^m$ where $m = (q-2)/2 \ge p$ (that is, $q = 2m+2 \ge 2p+2$) and η is a rotationally symmetric Lipschitz continuous function $\eta = \eta(x;s,t), 0 < s < t$ satisfying the following properties:

1. $\eta = 1$ on $B(x;s)$, $\eta = 0$ off $B(x;t)$, and $0 \le \eta \le 1$ on $B(x;t)\backslash B(x;s)$;
2. $|\nabla\eta| \le \frac{C_1}{t-s}$ a.e. on M for some positive constant C_1.

It is clear that

$$\begin{aligned}
& \frac{1}{2}\int_M \langle\nabla\phi^2,\nabla|\omega|^2\rangle dv \\
= & \int_M \phi\langle\nabla\phi,\nabla|\omega|^2\rangle dv \\
= & \int_M \eta|\omega|^m\langle\nabla(\eta|\omega|^m),\nabla|\omega|^2\rangle dv \\
= & \int_M \eta|\omega|^{2m}\langle\nabla\eta,\nabla|\omega|^2\rangle dv + \frac{m}{2}\int_M \eta^2|\omega|^{2m-2}|\nabla|\omega|^2|^2 dv \\
\ge & -\int_M \eta|\nabla\eta||\omega|^{2m}|\nabla|\omega|^2|dv + \frac{m}{2}\int_M \eta^2|\omega|^{2m-2}|\nabla|\omega|^2|^2 dv,
\end{aligned} \tag{22}$$

and

$$\int_M \phi^2|\nabla|\omega||^2 dv = \int_M \eta^2|\omega|^{2m}|\nabla|\omega||^2 dv = \frac{1}{4}\int_M \eta^2|\omega|^{2m-2}|\nabla|\omega|^2|^2 dv. \tag{23}$$

For $m \ge p \ge 2$, via Lemma 3.2, substituting (22), (23), (10) into (21), we have:

$$-(p-1)\int_M \eta|\nabla\eta||\omega|^{2m}|\nabla|\omega|^2|dv + \frac{2m+1}{4}\int_M \eta^2|\omega|^{2m-2}|\nabla|\omega|^2|^2 dv \le 0 \tag{24}$$

that is:

$$\begin{aligned}
& \int_{B(t)} \eta^2|\nabla|\omega|^2|^2|\omega|^{2m-2}dv \\
\le & \frac{4(p-1)}{2m+1}\int_{B(t)\backslash B(s)} \eta|\nabla\eta||\nabla|\omega|^2||\omega|^{2m}dv \\
\le & C(\int_{B(t)\backslash B(s)} \eta^2|\nabla|\omega|^2|^2|\omega|^{2m-2}dv)^{\frac{1}{2}}(\int_{B(t)\backslash B(s)} |\nabla\eta|^2|\omega|^{2m+2}dv)^{\frac{1}{2}} \\
\le & C\cdot\frac{C_1}{t-s}(\int_{B(t)\backslash B(s)} \eta^2|\nabla|\omega|^2|^2|\omega|^{2m-2}dv)^{\frac{1}{2}}(\int_{B(t)\backslash B(s)} |\omega|^{2m+2}dv)^{\frac{1}{2}}
\end{aligned} \tag{25}$$

where the constant $C = \frac{4(p-1)}{2m+1} > 0$.

We define:

$$\begin{aligned} A_j &:\triangleq \frac{1}{r_j^2}\int_{B(r_j)} |\omega|^{2m+2} dv = \frac{1}{r_j^2}\int_{B(r_j)} |\omega|^q dv \\ \eta_j &= \eta(x, r_j, r_{j+1}) \\ Q_{j+1} &:\triangleq \int_{B(r_{j+1})} \eta_j^2 |\nabla|\omega|^2|^2 |\omega|^{2m-2} dv \end{aligned}$$

where $\{r_j\}$ is a strictly increasing sequence going to infinity as $j \to \infty$. By setting $\eta = \eta_j$, $t = r_{j+1}$, $s = r_j$ in (25) and observing the fact $\eta_{j-1} \le \eta_j$ for any j, we obtain:

$$Q_{j+1}^2 \le (CC_1)^2 \frac{(r_{j+1}^2 A_{j+1} - r_j^2 A_j)}{(r_{j+1} - r_j)^2}(Q_{j+1} - Q_j). \tag{26}$$

Since $\liminf\limits_{r\to\infty} \frac{1}{r^2}\int_{B(x_0;r)} |\omega|^q dv < \infty$, we choose $\{r_j\}$ such that $r_{j+1} \ge 2r_j$ and $\lim\limits_{j\to\infty} A_j = \liminf\limits_{r\to\infty} \frac{1}{r^2}\int_{B(x_0;r)} |\omega|^q dv < \infty$. It is clear that $r_{j+1}/2 \ge r_j$, then $r_{j+1} - r_j \ge r_{j+1}/2 > 0$ and

$$\frac{r_{j+1}^2 A_{j+1} - r_j^2 A_j}{(r_{j+1} - r_j)^2} \le 4A_{j+1}, \tag{27}$$

$$Q_{j+1} \le 4(CC_1)^2 A_{j+1}. \tag{28}$$

Since $\{A_j\}$ is convergent, there exists a positive number K such that $A_j \le K$ for any j. Note that $\{Q_{j+1}\}$ is a non-negative, non-decreasing sequence, then in view of (28)

$$\lim_{j\to\infty} Q_{j+1} \le \lim_{j\to\infty} 4(CC_1)^2 A_{j+1} \le 4(CC_1)^2 K$$

Hence, $\{Q_{j+1}\}$ is also bounded. Summing over j in (26) and via (27) and (28), we obtain that

$$\begin{aligned} \sum_{j=1}^N Q_{j+1}^2 &\le \sum_{j=1}^N \left(4(CC_1)^2 A_{j+1}(Q_{j+1} - Q_j)\right) \\ &\le 4(CC_1)^2 K(Q_{N+1} - Q_1) \\ &\le 16(CC_1)^4 K^2. \end{aligned}$$

This shows that $\sum_{j=1}^N Q_{j+1}^2$ is bounded for $\forall N > 1$. Therefore, $Q_j \to 0$ as $j \to \infty$ which implies $|\nabla|\omega|^2|^2|\omega|^{2m-2} \equiv 0$ on M. Due to the fact that $|\nabla|\omega|^2|^2|\omega|^{2m-2} = (\frac{2}{m+1})^2|\nabla|\omega|^{m+1}|^2$, then $\nabla|\omega|^{m+1} \equiv 0$. Note that ω is continuous. Thus, $|\omega| = \mathit{constant}$.

Due to the fact that ω satisfies $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B(x_0;r)}|\omega|^q dv < \infty$ (i.e. 2-finite growth) when $\omega \in L^q(M)$ for $q \geq 2p+2$ with $p \geq 2$, we now claim that $|\omega| = 0$ if $\omega \in L^q(M)$ for $q \geq 2p+2$ with $p \geq 2$. Suppose on the contrary that $|\omega| = k \neq 0$, setting $\varphi = \eta_j|\omega|^{m+1}$ on $B(r_{j+1})$ in Sobolev inequality (7), via Lemma 2.5, observing that $\{r_j\}$ is a non-decreasing sequence going to infinity, we have, for $\beta > 0$:

$$\begin{aligned}
&\beta(vol(B(r_{j+1})))^{\alpha}\frac{1}{r_{j+1}^2}\left(\int_{B(r_{j+1})}\eta_j^{\frac{2}{1-\alpha}}|\omega|^{\frac{2m+2}{1-\alpha}}dv\right)^{1-\alpha}\\
\leq\ & \int_{B(r_{j+1})}|\nabla(\eta_j|\omega|^{m+1})|^2 dv\\
=\ & \int_{B(r_{j+1})\setminus B(r_j)}|\omega|^{2m+2}|\nabla\eta_j|^2 dv\\
\leq\ & \frac{C_1^2}{(r_{j+1}-r_j)^2}(r_{j+1}^2A_{j+1} - r_j^2A_j). \qquad (29)
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
&\beta(vol(B(r_{j+1})))^{\alpha}\frac{1}{r_{j+1}^2}\left(\int_{B(r_{j+1})}\eta_j^{\frac{2}{1-\alpha}}|\omega|^{\frac{2m+2}{1-\alpha}}dv\right)^{1-\alpha}\\
\geq\ & \beta\frac{1}{r_{j+1}^2}(vol(B(r_{j+1})))^{\alpha}|\omega|^{2m+2}\left(\int_{B(r_j)}dv\right)^{1-\alpha}\\
\geq\ & \beta|\omega|^{2m+2}\frac{1}{r_{j+1}^2}vol(B(r_j)). \qquad (30)
\end{aligned}$$

Combining (29) and (30), and choosing $\{r_j\}$ such that $r_{j+1} \geq 2r_j$, via (27), we have:

$$\begin{aligned}
\beta|\omega|^{2m+2}\frac{1}{r_{j+1}^2}vol(B(r_j)) &\leq C_1^2\frac{r_{j+1}^2A_{j+1} - r_j^2A_j}{(r_{j+1}-r_j)^2}\\
&\leq 4C_1^2\frac{1}{r_{j+1}^2}\int_{B(r_{j+1})}|\omega|^{2m+2}dv. \qquad (31)
\end{aligned}$$

That is,

$$\begin{aligned}
\beta|\omega|^{2m+2}vol(B(r_j)) &\leq 4C_1^2\int_{B(r_{j+1})}|\omega|^{2m+2}dv\\
&< \tilde{K}, \quad \text{for some constant } \tilde{K} > 0, \qquad (32)
\end{aligned}$$

since $\omega \in L^q(M)$, where $q = 2m+2 \geq 2p+2$. Then according to Lemma 2.6 that $\lim_{j\to\infty} vol(B(r_j)) = \infty$, as $j \to \infty$, (32) would lead to $\infty \leq \tilde{K}$, a contradiction when $|\omega| = k \neq 0$. Therefore, we obtain the vanishing result for Liouville-type problem when ω is in L^q space. □

4 Discussion

In this paper, we study Liouville-type problems for differential 1-forms by broadening L^q spaces to non-L^q spaces. For a closed and p-pseudo-coclosed differential 1-form ω, we generalize Liouville results from finite q-energy in L^q spaces to energy of $\liminf_{r\to\infty}\frac{1}{r^2}\int_{B(x_0;r)}|\omega|^q dv < \infty$ (that is, p-finite growth where $p = 2$) including both L^q spaces and non-L^q spaces. In particular, we obtain a vanishing theorem for ω with finite q-energy in L^q spaces. We recapture vanishing properties for differential forms in L^q spaces in a different approach. Our follow-up research work will be the study of Liouville-type problems for differential forms in the other 4 cases of p-balanced growth such as p-mild growth, p-obtuse growth, p-moderate growth, and p-small growth. Our method in this paper provides a successful way to work on Liouville-type problems in a variety of energy growth conditions. The investigation of extending L^q spaces to non-L^q spaces in this paper could lead to further study of generalizing L^q spaces to Morrey-Campanato spaces and possibly to Banach spaces. Calculation techniques of estimating inequalities in this paper could be applied to related estimation problems in many mathematical fields such as differential geometry, PDE, real analysis and complex analysis.

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