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# Sum of Mobius Functions over the Shifted Primes

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***Abstract*:** This article provides an asymptotic result for the summatory Mobius function $\sum_{p\leq x}\mu(p+a)=O\left(x\log^{-c}x\right)$ over the shifted primes, where $a\neq 0$ is a fixed parameter, and $c>1$ is a constant.

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# 1 Introduction

The Mobius function $\mu:\mathbb{N}\longrightarrow\{-1,0,1\}$ is defined by

$$\mu(n)=\begin{cases}(-1)^v & n=p_1p_2\cdots p_v\\ 0 & n\neq p_1p_2\cdots p_v,\end{cases} \tag{1}$$

where the $p_i\geq 2$ are primes. The autocorrelation of the Mobius function

$$\sum_{n\leq x}\mu(n)\mu(n+a) \tag{2}$$

is a topic of current research in several area of Mathematics, [13], [11], [8], et alii. The same autocorrelation functions of multiplicative functions over the shifted primes reduce to standard arithmetic averages over the shifted primes, for example, (2) reduce to the followings:

$$\sum_{p\leq x}\mu(p)\mu(p+a)=-\sum_{p\leq x}\mu(p+a). \tag{3}$$

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The best result is $\sum_{p\leq x}\mu(p+a)=(1-\delta)\pi(x)$, where $\delta>0$ is a constant, and it is expected that $\sum_{p\leq x}\mu(p+a)=o(\pi(x))$, see [5, Theorem 1], and [7] for extensive details on recent developments. This note proposes the first nontrivial upper bound.

**Theorem 1.1.** *Let $c>1$ be a constant, and let $x>1$ be a large number. If $a\neq 0$ is a fixed integer, then*

$$\sum_{p\leq x}\mu(p+a)=O\left(\frac{x}{(\log x)^c}\right).$$

The essential foundational topics are covered in Section 2 to Section **??**, and the proof of the main result is assembled in Section 6. Last but not least observe that an autocorrelation function of degree 3,

$$\sum_{n\leq x}\mu(n)\mu(n+a)\mu(n+b), \tag{4}$$

where $a,b\neq 0$ are fixed integers, reduces to an autocorrelation function of degree 2,

$$-\sum_{p\leq x}\mu(p+a)\mu(p+b) \tag{5}$$

over the shifted primes. Accordingly, these two problems are equivalent.

## 2 Standard Results for the Mobius Function

A variety of results for the average orders of the Mobius function are stated in this section. A complete proof for a new result in Theorem 5.1 for the average order over arithmetic progression is included.

There are many sharp bounds of the summatory function of the Mobius function, say, $O(xe^{-\sqrt{\log x}})$, and the conditional estimate $O(x^{1/2+\varepsilon})$ presupposes that the nontrivial zeros of the zeta function $\zeta(\rho)=0$ in the critical strip $\{0<\Re e(s)<1\}$ are of the form $\rho=1/2+it, t\in\mathbb{R}$. However, the simpler notation will be used here.

**Theorem 2.1.** *If $C>0$ is a constant, and $\mu$ is the Mobius function, then, for any large number $x>1$,*

$$\sum_{n\leq x}\mu(n)=O\left(\frac{x}{\log^C x}\right).$$

*Proof.* See [4, p. 6], [10, p. 182]. ■

A few results for the average orders over short intervals are proved in the literature. For the new developments, confer [9, Theorem 1.1].

**Theorem 2.2.** *Let $C>0$ be a constant, and let $\theta>7/12$ be a small real number. If $x>1$ is a large number, and $H\geq x^{\theta}$, then*

$$\sum_{x\leq n\leq x+H}\mu(n)=O\left(\frac{H}{\log^C x}\right).$$

The standard proof for the summatory Mobius function over an arithmetic progression is linked to the Siegel-Walfisz Theorem for primes in arithmetic progressions, the upper bounds are proved or discussed in [6, p. 424], [10, p. 385], et alii.

**Theorem 2.3.** *Let $x \geq 1$ be a large number, and let $q \ll (\log x)^B$, where $B \geq 0$ is an arbitrary constant. If $1 \leq a < q$ are relatively prime integers, then,*

$$\sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \mu(n) = O\left(\frac{x}{\log^C x}\right),$$

*where $C = C(B) > 0$ is a constant.*

*Proof.* A sketch of the proof appears in [10, p. 385]. ■

# 3 Equivalent Twisted Exponential Sums

One of the earliest result for twisted sums is stated below, and the recent version over short intervals appears in [9, Theorem 1.5].

**Theorem 3.1.** ([3]) *If $\alpha$ is a real number, and $D > 0$ is an arbitrary constant, then*

$$\sup_{\alpha \in \mathbb{R}} \sum_{n \leq x} \mu(n) e^{i2\pi\alpha n} < \frac{c_1 x}{(\log x)^D},$$

*where $c_1 = c_1(D) > 0$ is a constant depending on $D$, as the number $x \to \infty$.*

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a function, and let $q \in \mathbb{N}$ be a large integer. The finite Fourier transform

$$\hat{f}(s) = \frac{1}{p} \sum_{0 \leq t \leq p-1} f(t) e^{i\pi st/p} \tag{6}$$

and its inverse are used here to derive a summation kernel function.

**Definition 3.1.** Let $p$ be a prime, and let $\omega = e^{i2\pi/p}$ be a root of unity. The *finite summation kernel* is defined by the finite Fourier transform identity

$$\mathcal{K}(f(n)) = \frac{1}{p} \sum_{0 \leq t \leq p-1,} \sum_{0 \leq s \leq p-1} \omega^{t(n-s)} f(s) = f(n). \tag{7}$$

This simple identity is used to derive some equivalent exponential sums as illustrated below.

**Lemma 3.1.** *Let $q > 1$ and $1 \leq u < q$ be integers. If $a \neq 0$, and $D > 0$ is an arbitrary constant, then*

$$\sum_{n \leq x} \mu(n+a) e^{i2\pi un/q} = \sum_{n \leq x} \mu(n+a) e^{i2\pi n/q} + O\left(\frac{x}{(\log x)^D}\right),$$

*where the implied constant depends only on $D$, as the number $x \to \infty$.*

*Proof.* Let $1 < q \leq x$ be an integer, let $p \geq x$ be a large prime, and let $u \neq 0$.

Applying the finite summation kernel to $f(n) = \mu(n+a)e^{i2\pi un/q}$, see Definition 3.1, the twisted exponential sum has the form

$$\sum_{n \leq x} \mu(n+a)e^{i2\pi un/q} = \frac{1}{p} \sum_{n \leq x,} \sum_{0 \leq t \leq p-1,} \sum_{1 \leq s \leq p-1} \mu(s+a)\omega^{t(n-s)} e^{i2\pi us/q}. \tag{8}$$

The term $t = 0$ contributes

$$E_0(x) = \frac{1}{p} \sum_{n \leq x,} \sum_{1 \leq s \leq p-1} \mu(s+a)e^{i2\pi us/q} \ll \frac{x}{p} \frac{x}{(\log x)^{D_0}} \ll \frac{x}{(\log x)^{D_0}}, \tag{9}$$

where $D_0 > 0$, and the implied constant depends only on $D_0$, this follows from $x \leq p$, and Theorem 3.1. Rearranging it yields

$$\begin{aligned} \sum_{n \leq x} \mu(n+a)e^{i2\pi un/q} &= \frac{1}{p} \sum_{n \leq x,} \sum_{1 \leq t \leq p-1,} \sum_{1 \leq s \leq p-1} \mu(s+a)\omega^{t(n-s)} e^{i2\pi us/q} + E_0(x) \\ &= \frac{1}{p} \sum_{1 \leq t \leq p-1} \left( \sum_{1 \leq s \leq p-1} \mu(s+a)\omega^{-ts} e^{i2\pi us/q} \right) \\ &\qquad \times \sum_{n \leq x} \omega^{tn} + E_0(x). \end{aligned} \tag{10}$$

Similarly, for $u = 1$, the term $t = 0$ contributes

$$E_1(x) = \frac{1}{p} \sum_{n \leq x,} \sum_{1 \leq s \leq p-1} \mu(s+a)e^{i2\pi s/q} \ll \frac{x}{p} \frac{x}{(\log x)^{D_1}} \ll \frac{x}{(\log x)^{D_1}}, \tag{11}$$

where $D_1 > 0$, and the implied constant depends only on $D_1$, this follows from $x \leq p$, and Theorem 3.1. Accordingly, the twisted exponential sum has the form

$$\begin{aligned} \sum_{n \leq x} \mu(n+a)e^{i2\pi n/q} &= \frac{1}{p} \sum_{n \leq x,} \sum_{1 \leq t \leq p-1,} \sum_{1 \leq s \leq p-1} \mu(s+a)\omega^{t(n-s)} e^{i2\pi s/q} + E_1(x) \\ &= \frac{1}{p} \sum_{1 \leq t \leq p-1} \left( \sum_{1 \leq s \leq p-1} \mu(s+a)\omega^{-ts} e^{i2\pi s/q} \right) \\ &\qquad \times \sum_{n \leq x} \omega^{tn} + E_1(x). \end{aligned} \tag{12}$$

Taking the difference of (10) and (12) returns

$$\begin{aligned} R(x) &= \sum_{n \leq x} \mu(n+a)e^{i2\pi un/q} - \sum_{n \leq x} \mu(n+a)e^{i2\pi n/q} \qquad (13) \\ &= \frac{1}{p} \sum_{1 \leq t \leq p-1} \left( \sum_{1 \leq s \leq p-1} \mu(s+a)\omega^{-ts} e^{i2\pi us/q} - \sum_{1 \leq s \leq p-1} \mu(s+a)\omega^{-ts} e^{i2\pi s/q} \right) \\ &\qquad \times \sum_{n \leq x} \omega^{tn} + E_0(x) + E_1(x). \end{aligned}$$

By Lemma 3.2, the inner sum satisfies the upper bound

$$\left|\sum_{n\le x}\omega^{tn}\right|\le\frac{2p}{\pi t}. \tag{14}$$

And by Lemma 3.3, the two middle sums have the upper bound

$$\begin{aligned}
&\left|\sum_{1\le s\le p-1}\mu(s+a)\omega^{-ts}e^{i2\pi us/q}-\sum_{1\le s\le p-1}\mu(s+a)\omega^{-ts}e^{i2\pi s/q}\right| \qquad (15)\\
\le&\left|\sum_{1\le s\le p-1}\mu(s+a)\omega^{-ts}e^{i2\pi us/q}\right|+\left|\sum_{1\le s\le p-1}\mu(s+a)\omega^{-ts}e^{i2\pi s/q}\right|\\
\ll&\frac{x}{(\log x)^{D_2}},
\end{aligned}$$

where $D_2>0$, and the implied constant depends only on $D_2$, Together, these estimates lead to

$$\begin{aligned}
|R(x)|\le&\frac{1}{p}\sum_{1\le t\le p-1}\left|\sum_{1\le s\le p-1}\mu(s+a)\omega^{-ts}e^{i2\pi us/q}-\sum_{1\le s\le p-1}\mu(s+a)\omega^{-ts}e^{i2\pi s/q}\right|\\
&\times\left|\sum_{n\le x}\omega^{tn}\right|+E_0(x)+E_1(x)\\
\ll&\frac{1}{p}\sum_{1\le t\le p-1}\left(\frac{x}{(\log x)^{D_2}}\right)\cdot\left(\frac{2p}{\pi t}\right)+\frac{x}{(\log x)^{D_0}}+\frac{x}{(\log x)^{D_1}}\\
\ll&\left(\frac{x}{(\log x)^{D_2-1}}\right)+\frac{x}{(\log x)^{D_0}}+\frac{x}{(\log x)^{D_1}}\\
\ll&\frac{x}{(\log x)^{D}}, \qquad (16)
\end{aligned}$$

where $D=\min\{D_0,D_1,D_2-1\}$, and the implied constant depends only on $D$. ∎

**Lemma 3.2.** *Let $p$ be a large prime, and let $1<q<p$ be an integer. If $\omega^{nt}=e^{i2\pi nt/p}$, then*

$$\left|\sum_{1\le n\le x}\omega^{nt}\right|\le\frac{2p}{\pi t},$$

*as the number $p\to\infty$.*

*Proof.* Summing over the variable $n\le x$ returns

$$\sum_{1\le n\le x}\omega^{nt}=\sum_{1\le n\le x}e^{i2\pi nt/p}=\frac{1-e^{\frac{i2\pi t(x+1)}{p}}}{1-e^{\frac{i2\pi t}{p}}}. \tag{17}$$

Routine calculations lead to

$$\begin{aligned}
\left|\sum_{1\le n\le x}e^{i2\pi nt/p}\right|&\le\left|\frac{1-e^{\frac{i2\pi t(x+1)}{p}}}{1-e^{\frac{i2\pi t}{p}}}\right| \qquad (18)\\
&\le\frac{2p}{\pi t}.
\end{aligned}$$

■

**Lemma 3.3.** *Let $p$ be a large prime, and let $1 < q < p$ be an integer. If $\omega^{-ts} = e^{-i2\pi ts/p}$, and $u/q \neq 0$ is a real number, then*

$$\sum_{1 \leq s \leq p-1} \mu(s+a)\omega^{-ts} e^{i2\pi us/q} \ll \frac{x}{(\log x)^{D_2}},$$

*where $D_2 > 0$, and the implied constant depends only on $D_2$, as the number $p \to \infty$.*

*Proof.* Merge the primitive roots

$$\omega^{-ts} e^{\frac{i2\pi us}{q}} = e^{\frac{-i2\pi ts}{p}} \cdot e^{\frac{i2\pi us}{q}} = e^{\frac{i2\pi s(up-tq)}{pq}} = e^{i2\pi \alpha s}, \tag{19}$$

where $\alpha = (up - tq)/pq \neq 0$ since $q < p$ and $p$ is prime. Summing over the variable $s \geq 1$ returns

$$\begin{aligned} \sum_{1 \leq s \leq p-1} \mu(s+a)\omega^{-ts} e^{i2\pi us/q} &= \sum_{1 \leq s \leq p-1} \mu(s+a) e^{i2\pi \alpha s} \\ &= e^{-i2\pi \alpha a} \sum_{a \leq r \leq p-1+a} \mu(r) e^{i2\pi \alpha r} \\ &\ll \frac{x}{(\log x)^{D_2}}, \end{aligned} \tag{20}$$

where $D_2 > 0$, this follows from Theorem 3.1. ■

# 4 Mobius Sums Over Equivalent Classes

Observe that for any pair of integers $q \geq 4$, and $a \geq 0$, the sequence of consecutive values

$$\mu(1+a) = \mu(2+a) = \mu(3+a) = \mu(4+a) = \cdots = \mu(q+a) = 1, \tag{21}$$

is impossible since $\mu$ is not periodic. Likewise, the sequence of consecutive values in arithmetic progression

$$\mu(q+a) = \mu(2q+a) = \mu(3q+a) = \mu(4q+a) = \cdots = \mu([x/q]q+a) = 1 \tag{22}$$

is impossible. Therefore, the cardinalities of the two subsets of integers

$$\{qm + a \leq x : m \leq x/q\} \quad \text{and} \quad \{m \leq x/q\}, \tag{23}$$

with even number of prime factors, satisfy the relation

$$\sum_{m \leq x/q} \mu^+(qm+a) \leq \sum_{m \leq x/q} \mu^+(m) = \frac{3}{\pi^2}\frac{x}{q} + O\left(\frac{x}{q(\log x/q)^D}\right). \tag{24}$$

A closely related conjecture claims that

$$\sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \mu^2(n) = \frac{6}{\pi^2}\prod_{p \mid q}\left(1 - \frac{1}{p^2}\right)\frac{x}{q} + O\left((x/q)^{1/4+\varepsilon}\right), \tag{25}$$

for $1 \leq q < x^{1-\varepsilon}$, see [2, p. 2].

**Lemma 4.1.** *Let $x$ be a large number, and let $1 < q < x$ be an integer. If $a \geq 0$, then,*

$$\sum_{\substack{n \leq x \\ q | n}} \mu(n+a) \ll \frac{x}{q(\log x/q)^D},$$

*where $D > 0$, and the implied constant depends only on $D$, as the number $x \to \infty$.*

*Proof.* The asymptotic number of squarefree integers with even number of prime factors in a subset of integers (23) of cardinality $x/q$ has the formula

$$\begin{aligned} \sum_{m \leq x/q} \mu^+(qm+a) &= \sum_{m \leq x/q} \frac{\mu^2(qm+a) + \mu(qm+a)}{2} \\ &= c(a,q)\frac{x}{q} + O\left(\frac{x}{q(\log x/q)^D}\right), \end{aligned} \tag{26}$$

where $0 < c(a,q) < 1$ is a constant, see (25). Likewise, the asymptotic number of squarefree integers with odd number of prime factors in a subset of integers (23) of cardinality $x/q$, has the formula

$$\begin{aligned} \sum_{m \leq x/q} \mu^-(qm+a) &= \sum_{m \leq x/q} \frac{\mu^2(qm+a) - \mu(qm+a)}{2} \\ &= c(a,q)\frac{x}{q} + O\left(\frac{x}{q(\log x/q)^D}\right). \end{aligned} \tag{27}$$

The difference yields,

$$\begin{aligned} \sum_{m \leq x/q} \mu(qm+a) &= \sum_{m \leq x/q} \mu^+(qm+a) - \sum_{m \leq x/q} \mu^-(qm+a) \\ &= O\left(\frac{x}{q(\log x/q)^D}\right). \end{aligned} \tag{28}$$

■

The upper bound in Lemma 4.1 is not sharp since

$$\begin{aligned} \sum_{n \leq x} \mu(n) &= \sum_{0 \leq a < q,} \sum_{m \leq x/q} \mu(qm+a) \\ &= O\left(q \cdot \frac{x}{q(\log x/q)^D}\right) \\ &= O\left(\frac{x}{(\log x/q)^D}\right), \end{aligned} \tag{29}$$

which is weaker than $\sum_{n \leq x} \mu(n) = O\left(x(\log x)^{-D}\right)$ for $q > 1$, see Theorem 2.1.

**Theorem 4.1.** *If $x$ is a large real number, and $a < q < x$ is a pair of integers, then, for any $u \neq 0$ and any arbitrary constant $D > 0$,*

$$\sum_{n \leq x} \mu(n+a) e^{i2\pi un/q} = O\left(\frac{x}{(\log x/q)^D}\right),$$

*where the implied constant depends only on $D$, as the number $x \to \infty$.*

*Proof.* Substituting the change of variable $n = qm + v$ returns

$$\begin{aligned} \sum_{n\leq x} \mu(n+a)e^{i2\pi un/q} &= \sum_{0\leq v<q,} \sum_{0\leq m\leq (x-q+)/q} \mu(qm+v+a)e^{i2\pi u(qm+v)/q} \qquad (30) \\ &= e^{i2\pi uv/q} \sum_{0\leq v<q,} \sum_{0\leq m\leq (x-q+a)/q} \mu(qm+v+a). \end{aligned}$$

Apply Lemma 4.1 to the inner sum, and take a sum over all the equivalent classes $a \geq 0$, to obtain the upper bound.

$$\begin{aligned} e^{i2\pi uv/q} \sum_{0\leq v<q,} \sum_{0\leq m\leq (x-q+)/q} &= \sum_{0\leq a<q,} \sum_{m\leq x/q} \mu(qm+a) \qquad (31) \\ &= O\left( q \cdot \frac{x}{q(\log x/q)^D} \right) \\ &= O\left( \frac{x}{(\log x/q)^D} \right), \end{aligned}$$

as claimed. ■

# 5 Mertens Function Over Arithmetic Progressions

A different result for the average order of the Mobius function over arithmetic progressions, sharper than Theorem 2.3 is given below. Unlike the proof based on the Siegel-Walfisz Theorem methodology, this technique is based on completely different ideas, and has no limitation on the parameter $q \geq 1$.

**Theorem 5.1.** *For any fixed integer $a$, and an arbitrary constant $D > 0$. If $q < x$ is a parameter, then*

$$\sum_{\substack{n\leq x \\ q|n}} \mu(n+a) \ll \frac{1}{q} \frac{x}{(\log x/q)^D},$$

*where the implied constant depends only on $D$, as the number $x \to \infty$.*

*Proof.* Fix an integer $a \neq 0$, and let $q < x$. Now, use an indicator function

$$\frac{1}{q} \sum_{0\leq u<q} e^{i2\pi u(n-a)/q} = \begin{cases} 1 & \text{if } n = mq + a, \\ 0 & \text{if } n \neq mq + a, \end{cases} \qquad (32)$$

to remove the congruence $q \mid n$ in the Mertens sum:

$$\begin{aligned} \sum_{\substack{n\leq x \\ q|n}} \mu(n+a) &= \sum_{n\leq x} \mu(n+a) \times \frac{1}{q} \sum_{0\leq u<q} e^{i2\pi u(n-a)/q} \qquad (33) \\ &= \frac{1}{q} \sum_{n\leq x} \mu(n+a) + \frac{1}{q} \sum_{n\leq x} \mu(n+a) \sum_{1\leq u<q} e^{i2\pi u(n-a)/q} \\ &= V_0(x) \ + \ V_1(x). \end{aligned}$$

The first term is bounded by

$$V_0(x)=\frac{1}{q}\sum_{n\le x}\mu(n+a)\ll\frac{1}{q}\frac{x}{(\log x)^{D_0}}, \tag{34}$$

where $D_0>0$, and the implied constant depends only on $D_0$, see Theorem 2.1. Now, use Lemma 3.1 to rewrite the second term in the following way, (it removes the dependence on the variable $u\neq 0$).

$$\begin{aligned}V_1(x) &= \frac{1}{q}\sum_{n\le x}\mu(n+a)\sum_{1\le u<q}e^{i2\pi u(n-a)/q} \\ &= \frac{1}{q}\sum_{1\le u<q}e^{-i2\pi au/q}\sum_{n\le x}\mu(n+a)e^{i2\pi un/q} \\ &\le \frac{1}{q}\sum_{1\le u<q}e^{-i2\pi au/q}\left(\sum_{n\le x}\mu(n+a)e^{i2\pi n/q}+\frac{c_1x}{(\log x)^{D_1}}\right),\end{aligned} \tag{35}$$

where $D_1>0$, and $c_1>0$ is a constant depending on $D_1$. By Theorem 4.1, the inner sum

$$\sum_{n\le x}\mu(n+a)e^{i2\pi n/q}\le\frac{c_2x}{(\log x/q)^{D_2}}, \tag{36}$$

where $D_2>0$, and $c_2$ is a constant depending on $D_2$. Hence, the second term has the following upper bound.

$$\begin{aligned}|V_1(x)| &\ll \frac{1}{q}\left|\sum_{1\le u<q}e^{-i2\pi au/q}\right|\left|\sum_{n\le x}\mu(n+a)e^{i2\pi n/q}+\frac{c_1x}{(\log x)^{D_1}}\right| \\ &\ll \frac{1}{q}|-1|\left(\frac{c_2x}{(\log x/q)^{D_2}}+\frac{c_1x}{(\log x)^{D_1}}\right) \\ &\ll \frac{1}{q}\frac{x}{(\log x/q)^{D_2}},\end{aligned} \tag{37}$$

where the factor $\sum_{1\le u<q}e^{-i2\pi au/q}=-1$. Summing (34) and (37), and setting $D=\min\{D_1,D_2\}$ completes the proof. ■

The evidence generated by random numerical experiments are within the expected conditional estimate

$$\sum_{\substack{n\le x\\ q|n}}\mu(n+a)\ll\frac{1}{q}x^{1/2+\varepsilon}. \tag{38}$$

# 6 Proof of the Main Result

The analysis of the plain average order over the shifted primes

$$\sum_{p\le x}\mu(p+a) \tag{39}$$

seems to be unmanageable. But, the introduction of the weighted prime indicator function, (vonMangoldt function),

$$\Lambda(n) = \begin{cases} \log n & \text{if } n = p^k, \\ 0 & \text{if } n \neq p^k, \end{cases} \tag{40}$$

where $p^k$ is a prime power, the identity

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d, \tag{41}$$

see [1, Theorem 2.11], and the result in Theorem 5.1 change everything.

*Proof.* (**Theorem 1.1**) To remove the reference to primes, insert the vonMangoldt function to obtain the equivalent form

$$\sum_{n \leq x} \Lambda(n)\mu(n+a). \tag{42}$$

Applying the identity (41) and reversing the order of summation produce the followings.

$$\begin{aligned} \sum_{n \leq x} \Lambda(n)\mu(n+a) &= -\sum_{n \leq x} \mu(n+a) \sum_{d|n} \mu(d) \log d \\ &= -\sum_{d \leq x} \mu(d) \log(d) \sum_{\substack{n \leq x \\ d|n}} \mu(n+a). \end{aligned} \tag{43}$$

Letting $q = d$, and applying Theorem 5.1 yield:

$$\begin{aligned} \left| \sum_{d \leq x} \mu(d) \log(d) \sum_{\substack{n \leq x \\ d|n}} \mu(n+a) \right| &\leq \sum_{d \leq x} |\mu(d) \log(d)| \left| \sum_{\substack{n \leq x \\ d|n}} \mu(n+a) \right| \\ &\ll \sum_{d \leq x} \log(d) \left( \frac{1}{d} \frac{x}{(\log x/d)^D} \right) \\ &\ll \frac{x}{(\log x)^D} \sum_{d \leq x} \frac{\log d}{d}, \end{aligned} \tag{44}$$

where $D > 0$ is an arbitrary constant. The estimate of the finite sum

$$\sum_{n \leq x} \frac{\log n}{n} \ll (\log x)^2, \tag{45}$$

is a routine calculation. Thus, replacing this estimate yields

$$\left|\sum_{d\leq x}\mu(d)\log(d)\sum_{\substack{n\leq x\\ d|n}}\mu(n+a)\right| \ll \frac{x}{(\log x)^D}\sum_{d\leq x}\frac{\log d}{d} \tag{46}$$
$$\ll \frac{x}{(\log x)^D}(\log x)^2$$
$$\ll \frac{x}{(\log x)^c},$$

where $D-2\geq c>0$ is a constant. ■

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