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Special Representations, Nathanson's Lambda Sequences and Explicit Bounds

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by

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Abstract

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Let X be a group with identity e, we define A as an infinite set of generators for X, and let (X, d) be the metric space with word length d_A induced by A. Nathanson showed that if P is a nonempty finite set of prime numbers and A is the set of positive integers whose prime factors all belong to P, then the metric space $(\mathbf{Z}, d_A)$ has infinite diameter. Nathanson also studied the $\lambda_A(h)$ sequences, where $\lambda_A(h)$ is defined as the smallest positive integer y with $d_A(e, y) = h$, and he posed the problem to compute $\lambda_A(h)$ and estimate its growth rate. We will give explicit forms for $\lambda_p(h)$ for any fixed odd integer $p > 1$. We will also solve the open problems of computing the term $\lambda_{2,3}(4)$,

provide an explicit lower bound for $\lambda_{2,3}(h)$ and classifying $\lambda_{2,p}(h)$ for $p > 1$ any odd integer and $h \in \{1, 2, 3\}$.

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For Najalia, Devaj and Vairaj

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Introduction

In this thesis we study some important problems in additive number theory that were raised by Nathanson. Nathanson studied Nets in groups and shortest length special g-adic representations, and minimal additive complements in 2010 and 2011. Nathanson created an algorithm that generated the special g-adic representations of shortest length and proved several properties of these representations. For a group X with identity e, and A an infinite set of generators for X, with (X, d) the metric space with word length d_A induced by A, Nathanson showed that if P is a nonempty finite set of prime numbers and A is the set of positive integers whose prime factors all belong to P, then the metric space $(\mathbf{Z}, d_A)$ has infinite diameter. Nathanson also studied the $\lambda_A(h)$ sequences, where $\lambda_A(h)$ is defined as the smallest positive integer y with $d_A(e, y) = h$, and he posed the problem to compute $\lambda_A(h)$ and estimate its growth rate.

The lambda sequence has deep rooted applications in geometric group theory and few terms are known. In the first Chapter we will answer the above question posed by Nathanson in the instance when P consists of any fixed prime number. We give explicit expressions for $\lambda_p(h)$, for any fixed odd integer $p > 1$ with respect to the generating set A_p. We also give upper bounds for $\lambda_q(h)$, for any fixed even integer $q > 2$ with respect to the generating set A_q.

In chapter 2, we solve a conjecture of Conway and Nathanson that claims that the value of $\lambda_{2,3}(4) = 150$. We show this to be the true by showing the insolubility of certain Diophantine equations. In 2011, Hajdu and Tijdeman tackled the important problem of establishing lower and upper bounds of $\lambda_P(h)$. They established non-trivial upper and lower bounds. We establish an explicit lower bound for $\lambda_{2,3}(h)$ in this chapter. We also derive several properties of $\lambda_2(h)$ and $\lambda_3(h)$ and create an arsenal of techniques to find additional lambda terms.

In Chapter 3 we compute the terms of $\lambda_{2,5}(h)$ when $h \in \{1, 2, 3, 4\}$, and classified $\lambda_{2,p}(h)$ for $p > 1$ any odd integer and $h \in \{1, 2, 3\}$. We make the startling discovery that $\lambda_{2,n}(h)$ for odd $n \geq 23$ attain the limit points 11 and 15 infinitely often and take on no other values. Our work rests on results derived from Baker's method on logarithmic forms to show the insolubility of certain Diophantine equations and many other great mathematicians. It is fitting to end this introduction with the fact that a crucial result of Erdős and others on the Carmichael function [4] was used by Hajdu and Tijdeman in deriving the upper bound for $\lambda_P(h)$.

Chapter 1

Explicit lambda expressions.

1.1 Introduction

We will consider the additive Abelian group of integers with respect to the generating set $A_g = \{0\} \cup \{\pm g^j, j \in \mathbf{N}_0\}$ where $g \in P$, and P denotes the set of prime numbers. In the field of Geometric Group theory and in particular the study of h-nets in the additive group A_g, Nathanson created an important algorithm that produces the shortest length representation of integers with respect to A_g. Pivotal to this study is an understanding of word lengths with respect to the additive group A_g. In particular, Nathanson raised the following question stated as Problem N1 below in [9].

Problem N1(Nathanson [9]). *Let P be a finite or infinite set of prime numbers and consider the additive group $\mathbf{Z}$ of integers with generating set*

$$A_P = \{0\} \cup \{\pm p^j : p \in P \, and \, j \in \mathbf{N}_0\}.$$

Let l_P and d_P denote, respectively, the corresponding word length function and metric induced on $\mathbf{Z}$*. For every positive integer h, let $\lambda_P(h)$ denote the smallest positive integer of length h, that is, the smallest integer that can be represented as the sum of exactly h elements of A_P, but that cannot be represented as the sum or difference of fewer than h elements of A_P. Compute the function $\lambda_P(h)$.*

In this chapter we will answer the above question raised by Nathanson in the instance when P consists of any prime number. We will give explicit expressions for $\lambda_p(h)$, for any fixed prime number p with respect to the generating set A_p. We will in fact give explicit expressions for $\lambda_p(h)$, where $p > 1$ is any odd integer.

We will present the relevant definitions and notations in section 1.1 of this chapter which will set the stage for our new results. In the remaining sections we will consider the properties of the word lengths and the $\lambda_p(h)$ values. We will pay particular attention to the instances when $p \in \{2, 3\}$. In the final sections we will generalize our results to give explicit expressions for $\lambda_p(h)$, for any fixed prime number p with respect to the generating set A_p. We will also provide upper bounds for $\lambda_j(h)$, where j is any even integer greater than 2.

Convention: Throughout this thesis, $\mathbf{Z}$ will denote the set of integers, $\mathbf{N}_0$ the nonnegative integers and $\mathbf{N}$ the set of positive integers.

1.2 Word-length, metrics and such

In the study of the special representation of the integers as sums and differences of powers of g, where $g \in \ P$ and A_g, and P is the set of primes. We will consider the additive group G of integers $\mathbf{Z}$ with generators A_g for each g, where

$$A_g = \{0\} \cup \{\, \pm g^j, j = 0, 1, 2, 3, ...\}.$$

It is clear that A_g is symmetric since $a \in A_g$ if and only if $-a \in A_g$. The word length function $l_g : G \to \mathbf{N_0}$ is defined as follows: For $x \in G$ and $x \neq 0$, define $l_g(x) = r$, if r is the smallest positive integer such that there exist $a_i's \in A_g$, for $i = 1, 2, ..., r$ and $x = a_1 + a_2 + ... + a_r$. We will define $l_g(0) = 0$ trivially. The integer $l_g(x)$ is called the word length with respect to A_g.

Let $(\mathbf{Z}, d_g)$ be a metric space, where d_g is the metric induced on $\mathbf{Z}$. For $z \in \mathbf{Z}$ and $r \geq 0$, the sphere with center z and radius r is the set

$$S_z(r) = \{x \in \mathbf{Z} : d_g(x, z) = r\}.$$

Let l_P and d_P denote, respectively, the corresponding word length function and metric induced on $\mathbf{Z}$. For every positive integer h, let $\lambda_P(h)$ denote the smallest positive integer of length h, that is, the smallest integer that can be represented as the sum of exactly h elements of A_P, but that cannot be represented as the sum of fewer than h elements of A_P.

1.3 A Special g-adic representation algorithm for $g \in \{2, 3\}$.

For any fixed integer $g \geq 2$, if we consider the additive group $\mathbf{Z}$ and the generating set A_g with an associated word length $l_g(n)$, we can represent every integer as a sum of elements of A_g. We call such a representation a g-adic representation. This is not to be confused with the standard g-adic representation of integers. In studying the metric geometry of the group $\mathbf{Z}$, with generating set A_g, it is a useful tool to have an algorithm to compute the g-adic length $l_g(n)$ of an integer n in the metric $(\mathbf{Z}, d_g)$. Nathanson constructed a special g-adic representation that is unique and has shortest length with respect to the generating set A_g. Nathanson's algorithm is pivotal in our work to generate $\lambda_p(h)$ for a fixed odd integer $p > 1$.

We begin with two of Nathanson's theorems, which illustrate how to generate special g-adic representations of integers when $g \in \{2, 3\}$.

Theorem 1.1 (Nathanson [9]). *Every integer n has a unique representation as a finite sum and difference of distinct powers of 2, in which no two consecutive powers of 2 occur. We call such an enumeration a special 2-adic representation of shortest length. This theorem is also an important tool in our derivation of certain lower and upper bounds in chapter 2.*

Based on the above Theorem, every integer n can be written in the special 2-adic form $n = \sum_{i=0}^{\infty} \varepsilon_i 2^i$, where $\varepsilon_i \in \{-1, 0, 1\}$ of minimal length. We now outline how we can generate the unique special 2-adic form guaranteed by

Nathanson's Theorem. We begin by first writing $|n|$ as a standard binary expansion. It is well known that every positive integer m can be written in the unique binary form $m = \sum_{i=0}^{\infty} \varepsilon_i 2^i$, where $\varepsilon_i \in \{0, 1\}$. We will now proceed to shorten our binary expansion by using the following algorithm:

(a) If there are t consecutive sums of powers of 2, i.e. $\varepsilon_i = \varepsilon_{i+1} = \cdots = \varepsilon_{i+t-1} = 1$ for $t \geq 2$, then we rewrite $\sum_{j=i}^{i+t-1} 2^j = 2^{i+t} - 2^i$. Notice that this shortens the number of terms from t to 2 for $t > 2$. After performing the above step on all consecutive blocks of powers of 2, if we end up with s consecutive negative terms, of powers of 2 then by a similar argument we see that $\varepsilon_i = \varepsilon_{i+1} = \cdots = \varepsilon_{i+s-1} = -1$ and $-\sum_{j=i}^{i+s-1} 2^j = -2^{i+s} + 2^i$.

(b) If we have two consecutive terms with opposite signs, say the k and $k+1$ terms, then we combine them in the following manner: $-2^k + 2^{k+1} = 2^k$ or $2^k - 2^{k+1} = -2^k$ to convert them into a single term.

(c) If identical powers of 2's occurs with opposite signs after implementing the above steps, cancel them. If a term of the form $m2^j$ occurs, where $|m| = 2^r$, then coalesce them into 2^{j+r}. If $m \neq 2^j$ then write the binary expansion of $|m|$ and multiply each power of two in this expansion by 2^j. If $m < 0$, take the negative of your answer.

After repeating these steps a finite number of times, there will be no consecutive terms and the process will terminate to give us a minimum special 2-adic expansion of our integer. We will now give an explicit example. It must also be noted that if we began with a negative integer, to obtain its special expansion, we must take the negative of its minimal expansion. We will illustrate this procedure with the integer -473.

$$
\begin{aligned}
|-473| = 473 &= (2^8 + 2^7 + 2^6) + (2^4 + 2^3) + 2^0 \\
&= (2^9 - 2^6) + (2^5 - 2^3) + 2^0 \\
&= 2^9 + (-2^6 + 2^5) - 2^3 + 2^0 \\
&= 2^9 - 2^5 - 2^3 + 2^0
\end{aligned}
$$

It follows that $-473 = -2^9 + 2^5 + 2^3 - 2^0$ and $l_2(-473) = 4$.

Theorem 1.2 (Nathanson [9])**.** *Every integer n has a unique representation as a finite sum and difference of distinct powers of 3 of minimal length. We call such an enumeration a special 3-adic representation of shortest length. Every integer n can be written in the special 3-adic form $n = \sum_{i=0}^{\infty} \varepsilon_i 3^i$, where $\varepsilon_i \in \{-1, 0, 1\}$.*

We now outline how we can generate the unique special 3-adic form guaranteed by Nathanson's Theorem. We begin by first writing $|n|$ as a standard ternary expansion. Replace all 2's in the ternary expansion by $(3 - 1)$ and simplify to get the unique, minimal 3-adic form. It must also be noted that if we began with a negative integer, to obtain its special expansion, we must take the negative of its minimal expansion. We will illustrate this procedure with the integer 151.

$$\begin{aligned}151 &= 3^4 + (2)3^3 + 3^2 + (2)3^1 + 3^0 = 3^4 + (3-1)3^3 + 3^2 + (3-1)3^1 + 3^0\\ &= 3^4 + 3^4 - 3^3 + 3^2 + 3^2 - 3^1 + 3^0 = (2)3^4 - 3^3 + (2)3^2 - 3^1 + 3^0\\ &= (3-1)3^4 - 3^3 + (3-1)3^2 - 3^1 + 3^0 = 3^5 - 3^4 - 3^2 - 3^1 + 3^0\end{aligned}$$

It follows that $151 = 3^5 - 3^4 - 3^2 - 3^1 + 3^0$ and $l_3(151) = 5$

The following proposition is useful. It shows that if a ternary expansion has $\varepsilon_i' 3^k$ as its highest power of 3, then the special 3-adic expansion can have either 3^k or 3^{k+1} as its highest power.

Proposition 1.3. *If a ternary expansion has $\varepsilon_k' 3^k$ as its highest power of* 3*, where $\varepsilon_k' \in \{1, 2\}$then the special* 3*-adic expansion can have either 3^k or 3^{k+1} as its highest power.*

Proof. Without loss of generality, we may assume that m is a positive integer. Let its ternary expansion be $m = \sum_{i=0}^{k} \varepsilon_i' 3^i$, where $\varepsilon_i' \in \{0, 1, 2\}$. Starting from left to right we replace the first 2 by $(3-1)$ and simplify. If this occurs at the 3^j term, we will rewrite it in the following way: $2 \cdot 3^j = -1 \cdot 3^j + 3^{j+1}$. The term 3^{j+1} will have the coefficient $1 + \varepsilon_{j+1}'$, where $1 + \varepsilon_{j+1}' \in \{1, 2, 3\}$. If the coefficient of 3 occurs, then the coefficient of 3^{j+1} becomes 0 and we add 1 to the coefficient of the 3^{j+2} term. As the process is continued, we will have that the last term 3^k in the ternary expansion, will have a coefficient of either $1, 2$ or 3. In the event that its coefficient is 1, the ternary and special 3-adic representations have the same last terms. In the event that its coefficient is 2, we have the special 3-adic expansion with last two terms $-3^k + 3^{k+1}$ and if the coefficient is 3, then the last two terms are $0 \cdot 3^k + 3^{k+1}$. It follows that when the last term in the ternary expansion of a number has coefficients 2 or 3, the special 3-adic expansion has as its last term $1 \cdot 3^{k+1}$.

□

1.4 Additional Proofs of the uniqueness of the special 3-adic representation of positive integers

We will provide two new proofs of the unique partitions depicted by Theorem 1.2 above that are different in approach from the proof given by Nathanson in [9]. The first is by induction and the other, by making use of generating functions. Before we begin we would also like to point out that there is a third proof of this in the first part of Theorem 141 in [7, p.116].

Proof. (1). In the first proof we will use induction to show that every integer n can be written in the special 3-adic form $n = \sum_{i=0}^{\infty} \varepsilon_i 3^i$, where $\varepsilon_i \in \{-1, 0, 1\}$. If we assume that $n = \sum_{i=0}^{\infty} \varepsilon_i 3^i$, is a special 3-adic expansion of n then $-n = \sum_{i=0}^{\infty}(-\varepsilon_i)3^i$ is a special 3-adic representation of $-n$, so it suffices to show that every nonnegative integer has a unique 3-adic expansion. It is trivially clear that $0 = \sum_{i=0}^{\infty} 0 \cdot 3^i$ is a special minimal 3-adic length of 0.

For any positive integer n, we will show that $n = 3^t + \sum_{i=0}^{t-1} \varepsilon_i 3^i$ is a unique special 3-adic expansion of n. We will use induction on t to prove that any positive integer can be written in the special 3-adic form above.

We will show that for any positive integer k, all positive integers $n < 3^k/2$ can be written in the form $n = 3^t + \sum_{i=0}^{t-1} \varepsilon_i 3^i$, where $t < k$. For $k = 1$, we need to show the above form for integers $< 3/2 = 1.5$. In this case we have only the integer 1 which can be written as $1 = 3^0$. Now we will assume that all integers from the interval $[1, 3^k/2)$ can be written in the special 3-adic form. We need to show that any integer L in the interval $\left(3^k/2, 3^{k+1}/2\right)$ can be written in the special 3-adic form, that is $3^k/2 < L < 3^{k+1}/2$. Now let $(L' = L - 3^k)$ from which it follows that $|L'| < 3^k/2$ and from the induction step we have that $|L'| = 3^s + \sum_{i=0}^{s-1} \varepsilon_i 3^i$ for $s < k$. It follows that $L = 3^k + L' = 3^k \pm \sum_{i=0}^{s} \varepsilon_i 3^i$. We have shown that n can be written in the special 3-adic form.

We will now prove the uniqueness of these special forms. Let $n = \sum_{i=0}^{p} \varepsilon_i 3^i = \sum_{i=0}^{q} \varepsilon'_i 3^i$, which are two different expansions of n with say $q < p$. That is, we have that $3^p + \varepsilon_{p-1} 3^{p-1} + \cdots + \varepsilon_1 3 + \varepsilon_0 = 3^q + \varepsilon'_{q-1} 3^{q-1} + \cdots + \varepsilon'_1 3 + \varepsilon'_0$. We can rewrite this expression as

$$3^p + \varepsilon_{p-1} 3^{p-1} + \cdots + \varepsilon_1 3 + \varepsilon_0 - \varepsilon'_0 = 3^q + \varepsilon'_{q-1} 3^{q-1} + \cdots + \varepsilon'_1 3$$

from which it follows that $3 | (\varepsilon_0 - \varepsilon'_0)$, which means that $\varepsilon_0 = \varepsilon'_0 \in \{-1, 0, 1\}$.

We can now rewrite the above expression as

$$3^p + \varepsilon_{p-1}3^{p-1} + \cdots + \varepsilon_1 3 = 3^q + \varepsilon'_{q-1}3^{q-1} + \cdots + \varepsilon'_1 3,$$

and dividing by 3 gives us

$$3^{p-1} + \varepsilon_{p-1}3^{p-2} + \cdots + \varepsilon_1 = 3^{q-1} + \varepsilon'_{q-1}3^{q-2} + \cdots + \varepsilon'_1.$$

We can now repeat this argument to see that $\varepsilon_i = \varepsilon'_i \in \{-1, 0, 1\}$ for all $i \in \{0, 1, 2, ...q\}$ from which the uniqueness follows.

□

Proof. (2). We will use a generating function approach to show that every integer n can be written in the 3-adic form $n = \sum_{i=0}^{\infty} \varepsilon_i 3^i$, where $\varepsilon_i \in \{-1, 0, 1\}$ uniquely. Let

$$n_{j,3}(x) = \prod_{k=0}^{j} \left(1 + x^{3^k} + x^{-3^k}\right)$$

Clearly $n_{j,3}(x)$ has 3^{j+1} powers of x. Each term of $n_{j,3}(x)$ has an exponent that consists of sums and differences of powers of three. The coefficients of each term is also 1. We claim that the exponents of the 3^{j+1} terms of the $x's$ represents all integers in the closed interval $\left[-\left(\frac{3^{j+1}-1}{2}\right), \frac{3^{j+1}-1}{2}\right]$ uniquely. Observe that

$$\left(1 + x^{3^k} + x^{-3^k}\right) = \frac{1 + x^{3^k} + x^{2(3^k)}}{x^{3^k}} = \frac{x^{3^{k+1}} - 1}{(x^{3^k})(x^{3^k} - 1)}$$

from which it follows that

$$\begin{aligned}
n_{j,3}(x) &= \prod_{k=0}^{j}\left(1+x^{3^k}+x^{-3^k}\right)\\
&= \prod_{k=0}^{j}\frac{x^{3^{k+1}}-1}{(x^{3^k})(x^{3^k}-1)}\\
&= \frac{x^{3^{j+1}}-1}{(x-1)x^{(1+3+3^2+\ldots+3^j)}}\\
&= \frac{x^{3^{j+1}}-1}{(x-1)x^{\frac{3^{(j+1)}-1}{2}}}\\
&= x^{-\left(\frac{3^{(j+1)}-1}{2}\right)}\sum_{k=1}^{3^{j+1}}x^{(3^{j+1}-k)}\\
&= \sum_{k=1}^{3^{j+1}}x^{\left(\frac{3^{j+1}+1}{2}-k\right)}\\
&= \sum_{k=\frac{-3^{j+1}+1}{2}}^{\frac{3^{j+1}-1}{2}}x^k
\end{aligned}$$

From this we see that the exponents of the $x's$ in $n_{j,3}(x)$ include all the integers from $\left[-\left(\frac{3^{j+1}-1}{2}\right), \frac{3^{j+1}-1}{2}\right]$. The uniqueness of the representation is guaranteed by the fact that the coefficients of the $x's$ are all ones.

□

The generating function technique above gives us a nice way to represent the integers uniquely as sums and differences of powers of threes. We will illustrate this with some examples.

Before we begin; we will define the $(t+1)$ tuple $[\delta_0, \delta_1, \delta_2, \cdots, \delta_t]$ as equivalent to the sum $\sum_{i=0}^{t} \delta_i 3^i$ for $\delta_i \in \{-1, 0, 1\}$.

Example 1.4. *For* $j = 1$; $n_{1,3}(x) = \prod_{k=0}^{1} \left(1 + x^{3^k} + x^{-3^k}\right)$, *generates unique representations for all integers in the closed interval* $[-4, 4]$ *by the second proof above.*

We illustrate this,

$$\begin{aligned} n_{1,3}(x) &= \prod_{k=0}^{1} \left(1 + x^{3^k} + x^{-3^k}\right) \\ &= (1 + x + x^{-1})(1 + x^3 + x^{-3}) \\ &= x^0 + x^{-1} + x^1 + x^{1-3} + x^{-1+3} + x^{-3} + x^3 + x^{-1-3} + x^{1+3} \\ &= \sum_{\substack{\delta_i \in \{-1,0,1\} \\ i=0,1}} x^{[\delta_0, \delta_1]} \end{aligned}$$

and the nine exponents of the $x's$, namely $[\delta_0, \delta_1]$ represents the unique partitions of the integers in the interval $[-4, 4]$.

Example 1.5. *For* $j = 9$, $n_{9,3}(x) = \prod_{k=0}^{9} \left(1 + x^{3^k} + x^{-3^k}\right)$, *generates unique partitions for all integers in the closed interval* $[-29524, 29524]$.

As in Example 1.4 above, we have that

$$n_{9,3}(x) = \prod_{k=0}^{9} \left(1 + x^{3^k} + x^{-3^k}\right)$$
$$= \sum_{\substack{\delta_i \in \{-1,0,1\} \\ i=0,1,\ldots,9}} x^{[\delta_0,\delta_1,\cdots,\delta_9]}$$

From the exponents of the $x's$ we get the unique representation of all 59049 integers in the interval $[-29524, 29524]$. For example the exponent $[-1,-1,0,0,1,1,-1,0,1,0]$ partitions the integer 6152 uniquely as

$$6152 = -3^0 - 3^1 + 3^4 + 3^5 - 3^6 + 3^8.$$

Remarks. It is important to point out that our method above does not show that the representations have shortest length. The representations do coincide with Nathanson's partitions of shortest length. As such Nathanson's Theorem 1.2 is stronger.

It is also useful to note that the generating function technique above will represent the integers in the special 2-adic way, but not uniquely. This is easily seen from the discussion below.

Since

$$n_{j,2}(x) = \prod_{k=0}^{j} \left(1 + x^{2^k} + x^{-2^k}\right) = (1+x^{-1}+x^1)(1+x^{-2}+x^2)...(1+x^{-2^j}+x^{2^j}),$$

chose the highest exponent of x in each of the $j+1$ bracketed terms above and form its sum. i.e. $1 + 2 + 2^2 + ... + 2^j = 2^{j+1} - 1$. Now choose the lowest exponents similarly to obtain the sum $-(2^{j+1} - 1)$. We see from this that the total number of terms in the closed interval $[-(2^{j+1}-1), 2^{j+1}-1]$ is $2(2^{j+1}-1)+1$. It is also clear that we will generate 3^{j+1} terms by expanding

$n_{j,2}(x)$. We claim that $3^{j+1} > 2(2^{j+1} - 1) + 1$ when $j > 1$. We now justify our claim.

$$\begin{aligned} 3^{j+1} = (2+1)^{j+1} =& 2^{j+1} + (j+1)2^j + ... + 1 \\ &> 2^{j+1} + j2^j + 2^j + 1 = (3+j)2^j + 1 \\ &> 4(2^j) - 1. \end{aligned}$$

It follows by an application of the pigeonhole principle that some exponents will be duplicated since we have more terms in the expansion of $n_{j,2}(x)$ than the number of integers in the closed interval $[-(2^{j+1} - 1), 2^{j+1} - 1]$. Consequently the coefficients of the $x's$ are not all ones and the partitions are not unique.

1.5 An Extension of the Generating Function Technique

We will show that every integer from the closed interval $\left[-\frac{(g^k-1)}{2}, \frac{(g^k-1)}{2}\right]$, where $m \in \mathbf{Z}^+$ and $g = 2m + 1$, has a unique representation as a sum and difference of powers of g with coefficients in $\{1, 2, 3, ..., m\}$.

We will consider the Laurent polynomial

$$F_{k,g}(t) = \prod_{i=0}^{k-1} \sum_{j=-m}^{m} t^{j \cdot g^i}.$$

We begin by manipulating $F_{k,g}(t)$ in the manner outlined below.

$$\begin{aligned}
F_{k,g}(t) &= \prod_{i=0}^{k-1} \sum_{j=-m}^{m} t^{j\cdot g^i} \\
&= \prod_{i=0}^{k-1} t^{-m\cdot g^i} \sum_{j=0}^{g-1} t^{j\cdot g^i} \\
&= \prod_{i=0}^{k-1} t^{-m\cdot g^i} \prod_{i=0}^{k-1} \frac{t^{g^{i+1}}-1}{t^{g^i}-1} \\
&= t^{-m\sum_{j=0}^{g-1} g^j} \left(\frac{t^{g^k}-1}{t-1}\right) \\
&= t^{-m\cdot\frac{g^k-1}{g-1}} \left(\frac{t^{g^k}-1}{t-1}\right) \\
&= t^{-m\cdot\frac{g^k-1}{g-1}} \sum_{j=0}^{g^k-1} t^j \\
&= \sum_{m\in J_k} t^m
\end{aligned}$$

Where J_k is the set of integers in the interval $\left[-\frac{(g^k-1)}{2}, \frac{(g^k-1)}{2}\right]$ and $n \in J_k$ has a unique representation in the form $n = \sum_{i=0}^{k-1} c_i g^i$, with coefficients $c_i \in \{-m, -m+1, ..., m-1, m\}$. The uniqueness follows since the coefficients of $F_{k,g}(t)$ when expanded as powers of t are all ones.

Example 1.6. *For $g = 5$, $m = 2$ and $k = 3$ we will use $F_{3,5}(t)$ to generate the unique representations of the integers J_3 from the interval $[-62, 62]$. The* *list of all the representations are given in Appendix A.I.*

1.6 Explicit forms for $\lambda_2(h)$ and $\lambda_3(h)$

Recall that $\lambda_g(h)$ is the smallest positive integer of length h that can be written as the sum or difference of precisely h non-zero terms of A_g, but that cannot be represented as the sum or difference of fewer than h terms. We will now give explicit forms for $\lambda_2(h)$ and $\lambda_3(h)$ with the help of the shortening procedures outlined in Theorems 1.1 and 1.2 above.

Theorem 1.7. *For $h \in \mathbf{N}$ we have that $\lambda_2(h) = \frac{2^{2h-1}+1}{3}$.*

Proof. From Theorem 1.1, we cannot have consecutive powers of two in our special 2-adic expansion. We will consider the $(2h-1)$ tuple
$[\varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{2h-2}]$, where it represents the sum $n = \sum_{i=0}^{2h-2} \varepsilon_i 2^i$ for $\varepsilon_i \in \{-1, 0, 1\}$. To produce the minimum n for which the word length h is obtained, we will modify the $(2h-1)$ tuples. From Theorem 1.1, we must avoid consecutive non-zero $\varepsilon_i' s$ and coupled with the fact that $\sum_{t=0}^{2h-3} \varepsilon_t 2^t \leq \sum_{t=0}^{2h-3} 2^t = 2^{2h-2} - 1 < 2^{2h-2}$; It is clear that if we set $\varepsilon_{2h-2} = 1$, $\varepsilon_{2i} = -1$, for $i = 0, 1, \cdots, h-2$ and $\varepsilon_{2i+1} = 0$, for $i = 0, 1, \cdots, h-2$ we will get the smallest n of word length h.
It follows that the smallest value of n is $2^{2h-2} - \sum_{i=0}^{h-2} 2^{2i} = \frac{2^{2h-1}+1}{3}$. Therefore $\lambda_2(h) = \frac{2^{2h-1}+1}{3}$. □

Remark: The sequence $\lambda_2(h)$ is identified as A007583 in [14].

We will first generate a table of values for $l_3(h)$, where $h = 1$ to 48, before we give an explicit expression for $\lambda_3(h)$. We will use the following bracket notation:

$$[\varepsilon_0, \varepsilon_1, ..., \varepsilon_\infty]_g = \sum_{i=0}^{\infty} \varepsilon_i g^i.$$

Table 1.1 illustrates this. We will suppress the subscript $g = 3$ for brevity in the bracketed expression.

Table 1.1: Special 3-adic lengths

n	$l_3(n)$	$[\varepsilon_0, \varepsilon_1, ..., \varepsilon_\infty]$
$\boxed{1}$	1	$[1]$
$\boxed{2}$	2	$[-1, 1]$
3	1	$[0, 1]$
4	2	$[1, 1]$
$\boxed{5}$	3	$[-1, -1, 1]$
6	2	$[0, -1, 1]$
7	3	$[1, -1, 1]$
8	2	$[-1, 0, 1]$
9	1	$[0, 0, 1]$
10	2	$[1, 0, 1]$
11	3	$[-1, 1, 1]$
12	2	$[0, 1, 1]$
13	3	$[1, 1, 1]$
$\boxed{14}$	4	$[-1, -1, -1, 1]$
15	3	$[0, -1, -1, 1]$
16	4	$[1, -1, -1, 1]$
17	3	$[-1, 0, -1, 1]$
18	2	$[0, 0, -1, 1]$
19	3	$[1, 0, -1, 1]$
20	4	$[-1, 1, -1, 1]$
21	3	$[0, 1, -1, 1]$
22	4	$[1, 1, -1, 1]$
23	3	$[-1, -1, 0, 1]$
24	2	$[0, -1, 0, 1]$

n	$l_3(n)$	$[\varepsilon_0, \varepsilon_1, ..., \varepsilon_\infty]$
25	3	$[1, -1, 0, 1]$
26	2	$[-1, 0, 0, 1]$
27	1	$[0, 0, 0, 1]$
28	2	$[1, 0, 0, 1]$
29	3	$[-1, 1, 0, 1]$
30	2	$[0, 1, 0, 1]$
31	3	$[1, 1, 0, 1]$
32	4	$[-1, -1, 1, 1]$
33	3	$[0, -1, 1, 1]$
34	4	$[1, -1, 1, 1]$
35	3	$[-1, 0, 1, 1]$
36	2	$[0, 0, 1, 1]$
37	3	$[1, 0, 1, 1]$
38	4	$[-1, 1, 1, 1]$
39	3	$[0, 1, 1, 1]$
40	4	$[1, 1, 1, 1]$
$\boxed{41}$	5	$[-1, -1, -1, -1, 1]$
42	4	$[0, -1, -1, -1, 1]$
43	5	$[1, -1, -1, -1, 1]$
44	4	$[-1, 0, -1, -1, 1]$
45	3	$[0, 0, -1, -1, 1]$
46	4	$[1, 0, -1, -1, 1]$
47	5	$[-1, 1, -1, -1, 1]$
48	4	$[0, 1, -1, -1, 1]$

Theorem 1.8. *For $h \in \mathbf{N}$ we have that $\lambda_3(h) = \frac{3^{h-1}+1}{2}$.*

Proof. From Theorem 1.2, every positive integer n can be written uniquely in the special 3-adic form with minimal length h and $n = 3^{h-1} + \sum_{i=0}^{h-2} \varepsilon_i 3^i$, for $\varepsilon_i \in \{-1, 0, 1\}$. Since $\sum_{i=0}^{h-2} \varepsilon_i 3^i \leq \frac{3^{h-1}-1}{2} < \frac{2.3^{h-1}}{2} = 3^{h-1}$. We infer that to obtain the smallest n with length h, we will set $\varepsilon = -1$ for $i = 1, 2, 3, \cdots, h-2$. It follows that the smallest n with length h is given by $n = 3^{h-1} - \sum_{i=0}^{h-2} 3^i = \frac{3^{h-1}+1}{2}$. Therefore $\lambda_3(h) = \frac{3^{h-1}+1}{2}$. □

Remark: The sequence $\lambda_3(h)$ is identified as A007051 in [15].

Table 1.2 below illustrates the first ten terms of the $\lambda_2(h)$ and $\lambda_3(h)$ sequences.

Table 1.2: $\lambda_2(h)$ and $\lambda_3(h)$

h	1	2	3	4	5	6	7	8	9	10
$\lambda_2(h)$	1	3	11	43	171	683	2731	10923	43691	174763
$\lambda_3(h)$	1	2	5	14	41	122	365	1094	3281	9842

Lemma 1.9.

(a) $\lambda_2(1) = \lambda_3(1)$. *For $h \geq 2$ we have that $\lambda_2(h) > \lambda_3(h)$.*

(b) $\lambda_2(h)$ *is odd for $h \geq 1$.*

(c) $\lambda_g(h)$ *is strictly increasing for* $g \in \{2, 3\}$ *and* $h \in \mathbf{N}$.

(d) For $t \in \mathbf{N}$, $\lambda_3(2t)$ *is even and* $\lambda_3(2t-1)$ *is odd.*

Proof. First we will show (a). It is clear that $\lambda_2(1) = 1 = \lambda_3(1)$. For $h \geq 2$, we have that $2(2^{2h-1}+1) = 4^h + 2 = (3+1)^h + 2 > 3^h + 3 = 3(3^{h-1}+1)$. We have shown that $2(2^{2h-1}+1) > 3(3^{h-1}+1)$ and it follows that $\lambda_2(h) > \lambda_3(h)$.

To prove (b) we will use induction. For the base case: $\lambda_2(1) = 1$. Assume it is true for k, i.e. $\lambda_2(k) = 2j+1$, for some positive integer j; then for $h = k+1$ we have

$$\lambda_2(k+1) = \frac{2^{2k+1}+1}{3} = \frac{4(2^{2k-1})+1}{3},$$

and from the induction hypothesis, we get

$$\lambda_2(k+1) = \frac{4[3(2j+1)-1]+1}{3} = 2r+1; r = 4j+1,$$

which is clearly odd.

To prove (c), we will define two functions, $f_1(x) = \frac{2^{2x-1}+1}{3}$ and $f_2(x) = \frac{3^{x-1}+1}{2}$ for $x \in (0, \infty)$. The first derivatives of these functions are $f_1'(x) = 4^x \ln(\sqrt[3]{2})$ and $f_2'(x) = 3^x \ln(\sqrt[6]{3})$ and they are clearly positive on $x \in (0, \infty)$. This in turn implies that both functions are strictly increasing and part (c) of the lemma follows.

We will prove (d) first for the even case and then the odd case. Observe that $\lambda_3(2t) = \frac{9^t+3}{6}$, and from this it is enough to show that $9^t + 3 \equiv 0 \mod 12$.

Now $9^t+3 \equiv 0 \mod 3$ and $9^t+3 \equiv 0 \mod 4$ and by the Chinese remainder theorem, we get that $9^t+3 \equiv 0 \mod 12$ and it then follows that $\lambda_3(2t)$ is even. Now observe that $\lambda_3(2t-1) = \frac{9^t+1}{2}$ and we will proceed by induction to show that it is odd. For the base case: $t=1$, $\lambda_3(1)=1$. Assume it is true for $t=k$, i.e. $\frac{3^{2k-2}+1}{2} = 2p+1$, for some positive integer p; then for $t=k+1$ we have

$$\lambda_3(2k+1) = \frac{3^{2k}+1}{2} = \frac{9(3^{2k-2})+1}{2},$$

and from the induction hypothesis, we get

$$\lambda_3(2k+1) = \frac{9(2(2p+1)-1)+1}{2} = 2j+1; j = 9p+2;$$

which is clearly odd. □

Lemma 1.10.

(a) $\ell_g(g^k m) = \ell_g(m)$ *for* $g \in \{2,3\}$, $k, m \in \mathbf{N}$.

(b) $\ell_g(g^s) = 1$ *for* $g \in \{2,3\}$ *and* $s \in \mathbf{N}$.

Proof. For a proof of part (a), let $\ell_g(m) = t$, where t is a positive integer. Then

$$m = \pm g^{\alpha_1} \pm g^{\alpha_2} \pm g^{\alpha_3} \pm \cdots \pm g^{\alpha_t},$$

where the $\alpha_i's \in \mathbf{N}$ for $i = 1, \ldots, t$, with t distinct terms that cannot be shortened, and since

$$g^k m = \pm g^{\alpha_1+k} \pm g^{\alpha_2+k} \pm g^{\alpha_3+k} \pm \cdots \pm g^{\alpha_t+k},$$

it follows that $g^k m$ has t distinct terms that cannot be shortened and as such $\ell_g(g^k m) = \ell_g(m)$. For a proof of part$(b)$, set $m=1$ in part (a) to get $\ell_g(g^s m) = \ell_g(1)$ and since $g^0 = 1$, the result follows. □

Remark: lemma 1.9(b) follows trivially from lemma 1.10(a). If we assume that $\lambda_2(h)$ is even we immediately get a contradiction since we can factor out the highest power of two to get a smaller number with the same length.

1.7 Explicit expressions for $\lambda_p(h)$, where p is any prime number.

We will now proceed to generalize Theorem 1.8 to write explicit expressions for λ_g, where $g \geq 3$ and it is an odd integer. To do so we will use Nathanson's algorithm for fixed odd integers $g \geq 3$. We first state Nathanson's theorem and then his shortening algorithm to generate special g- adic representations.

Theorem 1.11 (Nathanson [9]). *Let g be an odd integer, $g \geq 3$. Every nonzero integer n has a unique representation in the form*

$$n = \sum_{i=0}^{\infty} \varepsilon_i g^i$$

where $\varepsilon_i \in \{0, \pm 1, \pm 2, ..., (g-1)/2\}$ and $\varepsilon_i \neq 0$ for only finitely many non-negative integers i. Moreover, n has length

$$l_g(n) = \sum_{i=0}^{\infty} |\varepsilon_i|$$

in the metric space $(\mathbf{Z}, d_g)$ *associated with the generating set* $A_g = \{0\} \cup \{\pm g^j, j \in \mathbf{N}_0\}$.

A representation satisfying the conditions on the ε_i is a special g-adic representation of n for odd integers $g \geq 3$.

We now outline the shortening procedure that is used to represent any integer in the special g-adic form.

(a) Take any integer. For our purposes we will consider only positive integers n. Then write n in the standard base g representation. We will obtain $n = \sum_{i=0}^{\infty} \varepsilon_i g^i$ with $\varepsilon_i \in \{0, 1, 2, ..., (g-1)\}$.

(b) After applying the first operation, we obtain $n = \sum_{i=0}^{\infty} \varepsilon_i g^i$ with $\varepsilon_i \in \{0, 1, 2, ..., (g-1)\}$ for all i. If $(g+1)/2 \leq \varepsilon_i \leq g-1$ for some i, then choose the smallest such i and apply the identity

$$\varepsilon_i g^i = -(g - \varepsilon_i)g^i + g^{i+1}.$$

(c) If $-(g-1) \leq \varepsilon_i \leq -(g+1)/2$ for some i, then apply the identity

$$\varepsilon_i g^i = -(g + \varepsilon_i)g^i - g^{i+1}.$$

(d) Delete all occurrences of 0.

(e) If g^i and $-g^i$ occur, delete them.

(f) If g^i (resp $-g^i$) occurs g times, replace these g summands with the one summand g^{i+1} (resp $-g^{i+1}$).

By iterating the above process, we get a special g-adic representation of n. We will illustrate this procedure with the two integers, 303 and 39797 in bases 11 and 9 respectively.

$$\begin{aligned} 303 &= 2 \cdot 11^2 + 5 \cdot 11^1 + 6 \cdot 11^0 = 2 \cdot 11^2 + 5 \cdot 11^1 + (11^1 - 5 \cdot 11^0) \\ &= 2 \cdot 11^2 + 6 \cdot 11^1 - 5 \cdot 11^0 = 2 \cdot 11^2 + 1 \cdot 11^2 - 5 \cdot 11^1 - 5 \cdot 11^0 \\ &= 3 \cdot 11^2 - 5 \cdot 11^1 - 5 \cdot 11^0 \end{aligned}$$

It follows that $303 = 3 \cdot 11^2 - 5 \cdot 11^1 - 5 \cdot 11^0$ and $l_{11}(303) = 3 + |-5| + |-5| = 13$.

$$\begin{aligned} 39797 &= 6 \cdot 9^4 + 5 \cdot 9^2 + 2 \cdot 9^1 + 8 \cdot 9^0 = 6 \cdot 9^4 + 5 \cdot 9^2 + 2 \cdot 9^1 + 1 \cdot 9^1 - 1 \cdot 9^0 \\ &= 6 \cdot 9^4 + 5 \cdot 9^2 + 3 \cdot 9^1 - 1 \cdot 9^0 = 6 \cdot 9^4 + (1 \cdot 9^3 - 4 \cdot 9^2) + 3 \cdot 9^1 - 1 \cdot 9^0 \\ &= (1 \cdot 9^5 - 3 \cdot 9^4) + 1 \cdot 9^3 - 4 \cdot 9^2 + 3 \cdot 9^1 - 1 \cdot 9^0 \end{aligned}$$

It follows that $39797 = 1 \cdot 9^5 - 3 \cdot 9^4 + 1 \cdot 9^3 - 4 \cdot 9^2 + 3 \cdot 9^1 - 1 \cdot 9^0$ and $l_9(39797) = 1 + |-3| + 1 + |-4| + 3 + |-1| = 13.$

For any fixed odd integer $g \geq 3$ will now derive an explicit expression for $\lambda_g(h)$, where $h \in \mathbf{N}$.

Theorem 1.12. *For any fixed odd integer $g \geq 3$, $h \in \mathbf{N}$ and $j = 1, 2, 3, ...,$ $(g-1)/2$. we have that*

$$\lambda_g\left(\frac{(h-1)(g-1)+2j}{2}\right) = \frac{(2j-1)g^{h-1}+1}{2}.$$

Proof. From Theorem 1.11 above, every positive integer m can be written uniquely in the special g-adic form $m = \sum_{i=0}^{h-1} \varepsilon_i g^i$, for some $h \in \mathbf{N}$ and $\varepsilon_i \in \{0, \pm 1, \pm 2, ..., \pm (g-1)/2\}$. Observe that $\sum_{i=0}^{h-2} \varepsilon_i g^i \leq \frac{(g-1)}{2} \frac{g^{h-1}-1}{g-1} = \frac{g^{h-1}-1}{2} < g^{h-1} \leq j g^{h-1}$ where $j = 1, 2, 3, ..., (g-1)/2$. Now consider m_j, where

$$m_j = j g^{h-1} + \sum_{i=0}^{h-2} \varepsilon_i g^i,$$

and let $\varepsilon_i = -\frac{(g-1)}{2}$ for $i = 1, 2, ..., h-2$, from which it follows that

$$m_j = j g^{h-1} - \frac{(g-1)}{2} \sum_{i=0}^{h-2} g^i = \frac{(2j-1)g^{h-1}+1}{2}$$

and

$$l_g(m_j) = \left(\frac{(h-1)(g-1)+2j}{2}\right).$$

Using the fact that $\sum_{i=0}^{h-2} \varepsilon_i g^i < j g^{h-1}$, it follows that $\frac{(2j-1)g^{h-1}+1}{2}$ is the smallest integer with length $\left(\frac{(h-1)(g-1)+2j}{2}\right)$ and the claim follows. □

The following corollaries readily follows from the above results.

Corollary 1.13. *For $j = 1, 2, 3, ..., (g-1)/2$ we have that $\lambda_g(j) = j$.*

Proof. For $\lambda_g\left(\frac{(h-1)(g-1)+2j}{2}\right) = \frac{(2j-1)g^{h-1}+1}{2}$, let $h = 1$ and the claim follows. □

Corollary 1.14. *For any fixed prime $p \geq 3$, $h \in \mathbf{N}$ and $j = 1, 2, 3, ..., (g-1)/2$, we have that*

$$\lambda_p\left(\frac{(h-1)(p-1)+2j}{2}\right) = \frac{(2j-1)p^{h-1}+1}{2}.$$

Proof. Let $g = p$ be any prime number $p \geq 3$ in the expression for λ_g in Theorem 1.12 and the claim follows. □

We provide two Examples below for the λ sequences. In particular, example 3 gives $\lambda_5(h)$ explicitly; A question raised by Kevin O'Bryant at the CANT 2013 number theory conference at the CUNY Graduate Center during a presentation made by the author.

Example 1.15. *We give an explicit formula for $\lambda_5(h)$, where $h \in \mathbf{N}$ and $j = 1, 2$. It follows from Theorem 1.12 that*

$$\lambda_5(2h-1) = \frac{5^{h-1}+1}{2}, \ \ \lambda_5(2h) = \frac{3 \cdot 5^{h-1}+1}{2}$$

Example 1.16. *We give an explicit formula for $\lambda_{6991}(h)$, where $h \in \mathbf{N}$ and $j = 1, 2, 3, ..., 3495$. It follows from Theorem 1.12 that*

$$\lambda_{6991}(3495h - 3494) = \frac{6991^{(h-1)}+1}{2}, \ \ \lambda_5(3495h - 3493) = \frac{3 \cdot 6991^{(h-1)}+1}{2},$$

$$\cdots, \lambda_{6991}(3495h) = \frac{6989 \cdot 6991^{(h-1)}+1}{2}$$

1.8 Upper bounds for $\lambda_g(h)$, where g is any even number.

We will now proceed to give upper bounds for λ_g, where $g \geq 2$ and it is an even integer. To do so we will use Nathanson's algorithm for fixed even integers $g \geq 4$. We first state Nathanson's theorem and then his shortening algorithm to generate special g- adic representations.

Theorem 1.17 (Nathanson [9]). *Let g be an even integer, $g \geq 2$. Every nonzero integer n has a unique representation in the form*

$$n = \sum_{i=0}^{\infty} \varepsilon_i g^i$$

where $\varepsilon_i \in \{0, \pm 1, \pm 2, ..., g/2\}$, $\varepsilon_i \neq 0$ for only finitely many nonnegative integers i. and if $|\varepsilon_i| = g/2$ then $\varepsilon_i \varepsilon_{i+1} \geq 0$.

Moreover, n has length

$$l_g(n) = \sum_{i=0}^{\infty} |\varepsilon_i|$$

in the metric space $(\mathbf{Z}, d_g)$ associated with the generating set $A_g = \{0\} \cup \{\pm g^j, j \in \mathbf{N}_0\}$.

A representation satisfying the conditions on the ε_i is a special g-adic representation of n for even integers $g \geq 2$.

We now outline the shortening procedure that is used to represent any integer in the special g-adic form.

(a) Take any integer. For our purposes we will consider only positive integers n. Then write n in the standard base g representation. We will obtain $n = \sum_{i=0}^{\infty} \varepsilon_i g^i$ with $\varepsilon_i \in \{0, 1, 2, ..., (g-1)\}$.

(b) After applying the first operation, we obtain $n = \sum_{i=0}^{\infty} \varepsilon_i g^i$ with $\varepsilon_i \in \{0, 1, 2, ..., (g-1)\}$ for all i. If $g/2 < \varepsilon_i < g$ for some i then apply the identity

$$\varepsilon_i g^i = (\varepsilon_i - g)g^i + g^{i+1}.$$

(c) If $-g < \varepsilon_i < -g/2$ for some i, then apply the identity

$$\varepsilon_i g^i = (\varepsilon_i + g)g^i - g^{i+1}.$$

(d) Delete all occurrences of 0.

(e) If g^i and $-g^i$ occur, delete them.

(f) If g^i (resp $-g^i$) occurs g times, replace these g summands with the one summand g^{i+1} (resp $-g^{i+1}$).

(g) If $\varepsilon_i = -g/2$ and $\varepsilon_{i+1} \geq 1$ for some i use the identity

$$-(g/2)g^i + \varepsilon_{i+1}g^{i+1} = (g/2)g^i + (\varepsilon_{i+1} - 1)g^{i+1}.$$

(h) If $\varepsilon_i = g/2$ and $\varepsilon_{i+1} \leq -1$ for some i use the identity

$$(g/2)g^i + \varepsilon_{i+1}g^{i+1} = -(g/2)g^i + (\varepsilon_{i+1} + 1)g^{i+1}.$$

(i) If there are t consecutive sums of powers of g, where $\varepsilon_i = \varepsilon_{i+1} = \cdots = \varepsilon_{i+t-1} = g/2$ for $t \geq 2$, for some nonnegative integer i then choose the smallest power of i and for this i the largest integer t. Now use the identity

$$\sum_{j=i}^{i+t-1} \left(\frac{g}{2}\right) g^j = -\left(\frac{g}{2}\right) g^i + \sum_{j=i+1}^{i+t-1} \left(\frac{g}{2} - 1\right) g^j + g^{i+t}.$$

(j) If there are t consecutive sums of powers of g, where $\varepsilon_i = \varepsilon_{i+1} = \cdots = \varepsilon_{i+t-1} = -g/2$ for $t \geq 2$, for some nonnegative integer i then choose the smallest power of i and for this i the largest integer t. Now use the identity

$$\sum_{j=i}^{i+t-1}\left(-\frac{g}{2}\right)g^j=\left(\frac{g}{2}\right)g^i+\sum_{j=i+1}^{i+t-1}\left(\frac{g}{2}-1\right)g^j-g^{i+t}.$$

By iterating the above process, we get a special g-adic representation of n. Tables 1.3 and 1.4 lists the values for the first 92 positive integers, their corresponding special 4-adic representations and their corresponding word length's using the above algorithm .

We will use the following bracket notation:

$$[\varepsilon_0,\varepsilon_1,...,\varepsilon_\infty]_g=\sum_{i=0}^{\infty}\varepsilon_i g^i.$$

In our tables we will suppress the subscript $g=4$ for brevity in the bracketed expression.

Table 1.3: Special 4-adic lengths

n	$l_4(n)$	$[\varepsilon_0, \varepsilon_1, ..., \varepsilon_\infty]$	n	$l_4(n)$	$[\varepsilon_0, \varepsilon_1, ..., \varepsilon_\infty]$
$\boxed{1}$	1	$[1]$	25	4	$[1, 2, 1]$
$\boxed{2}$	2	$[2]$	$\boxed{26}$	5	$[-2, -1, 2]$
3	2	$[-1, 1]$	27	4	$[-1, -1, 2]$
4	1	$[0, 1]$	28	3	$[0, -1, 2]$
5	2	$[1, 1]$	29	4	$[1, -1, 2]$
$\boxed{6}$	3	$[2, 1]$	30	4	$[-2, 0, 2]$
7	3	$[-1, 2]$	31	3	$[-1, 0, 2]$
8	2	$[0, 2]$	32	4	$[-2, 0, 2]$
9	3	$[1, 2]$	33	3	$[1, 0, 2]$
$\boxed{10}$	4	$[-2, -1, 1]$	34	4	$[2, 0, 2]$
11	3	$[-1, -1, 1]$	35	4	$[-1, 1, 2]$
12	2	$[0, -1, 1]$	36	3	$[0, 1, 2]$
13	3	$[1, -1, 1]$	37	4	$[1, 1, 2]$
14	3	$[1, -1, 1]$	38	5	$[2, 1, 2]$
15	2	$[-1, 0, 1]$	39	5	$[-1, -2, -1, 1]$
16	1	$[0, 0, 1]$	40	4	$[0, -2, -1, 1]$
17	2	$[1, 0, 1]$	41	5	$[1, -2, -1, 1]$
18	3	$[2, 0, 1]$	42	5	$[-2, -1, -1, 1]$
19	3	$[-1, 1, 1]$	43	4	$[-1, -1, -1, 1]$
20	2	$[0, 1, 1]$	44	4	$[1, -1, -1, 1]$
21	3	$[1, 1, 1]$	45	4	$[1, -1, -1, 1]$
22	4	$[2, 1, 1]$	46	4	$[-2, 0, -1, 1]$
23	4	$[-1, 2, 1]$	47	3	$[-1, 0, -1, 1]$
24	3	$[0, 2, 1]$	48	2	$[0, 0, -1, 1]$

Table 1.4: Special 4-adic lengths cont'd.

n	$l_4(n)$	$[\varepsilon_0, \varepsilon_1, ..., \varepsilon_\infty]$	n	$l_4(n)$	$[\varepsilon_0, \varepsilon_1, ..., \varepsilon_\infty]$
49	3	$[1, 0, -1, 1]$	71	4	$[-1, 2, 0, 1]$
50	4	$[2, 0, -1, 1]$	72	3	$[0, 2, 0, 1]$
51	4	$[-1, 1, -1, 1]$	73	4	$[1, 2, 0, 1]$
52	3	$[0, 1, -1, 1]$	74	5	$[-2, -1, 1, 1]$
53	4	$[1, 1, -1, 1]$	75	4	$[-1, -1, 1, 1]$
54	5	$[2, 1, -1, 1]$	76	3	$[0, -1, 1, 1]$
55	4	$[-1, -2, 0, 1]$	77	4	$[1, -1, 1, 1]$
56	3	$[0, -2, 0, 1]$	78	4	$[-2, 0, 1, 1]$
57	4	$[1, -2, 0, 1]$	79	3	$[-1, 0, 1, 1]$
58	4	$[-2, -1, 0, 1]$	80	2	$[0, 0, 1, 1]$
59	3	$[-1, -1, 0, 1]$	81	3	$[1, 0, 1, 1]$
60	2	$[0, -1, 0, 1]$	82	4	$[2, 0, 1, 1]$
61	3	$[1, -1, 0, 1]$	83	4	$[-1, 1, 1, 1]$
62	3	$[-2, 0, 0, 1]$	84	3	$[0, 1, 1, 1]$
63	2	$[-1, 0, 0, 1]$	85	4	$[1, 1, 1, 1]$
64	1	$[0, 0, 0, 1]$	86	5	$[2, 1, 1, 1]$
65	2	$[1, 0, 0, 1]$	87	5	$[-1, 2, 1, 1]$
66	3	$[2, 0, 0, 1]$	88	4	$[0, 2, 1, 1]$
67	3	$[-1, 1, 0, 1]$	89	5	$[1, 2, 1, 1]$
68	2	$[0, 1, 0, 1]$	$\boxed{90}$	6	$[-2, -1, 2, 1]$
69	3	$[1, 1, 0, 1]$	91	6	$[-1, 1, 2, 1, 1]$
70	4	$[2, 1, 0, 1]$	92	5	$[0, 1, 2, 1, 1]$

We see from the tables above that $\lambda_4(1) = 1, \lambda_4(2) = 2, \lambda_4(3) = 6, \lambda_4(4) = 10, \lambda_4(5) = 26$ and $\lambda_4(6) = 90$.

We will now give an upper bound for $\lambda_4(h)$.

Theorem 1.18. *For $g = 4$ and $h \geq 5$, where $h \in \mathbf{N}$, we have that*

$$\lambda_4(h) \leq \frac{5 \cdot 4^{h-3} - 2}{3}.$$

Proof. Let $h \geq 5$, where $h \in \mathbf{N}$. let $n = [-2, -1, -1, \cdots, -1, 2]_4$, where n consists of $h-4$ negative ones sandwiched between -2 at the beginning and 2 at the end. Clearly n consists of $h-2$ terms. Observe that $[-2, -1, -1, \cdots, -1, 2]_4$, is a special 4-adic representation of n since $\varepsilon_i \in \{-2, -1, 2\}$ for $i = 0, 1, ..., h-2$ and $\varepsilon_0 \cdot \varepsilon_1 = 2$ and $\varepsilon_{h-3} \cdot \varepsilon_{h-2} = 0$. In addition, we see that

$$n = -1 + 2 \cdot 4^{h-3} - \sum_{i=0}^{h-4} 4^i = \frac{5 \cdot 4^{h-3} - 2}{3},$$

and $l_4(n) = h$, from which it follows that $\lambda_4(h) \leq \frac{5 \cdot 4^{h-3} - 2}{3}$. □

We will now give exact values for $\lambda_g(h)$ when $h \in \{1, 2, 3, ..., g/2\}$ and then give an upper bound for all values of $\lambda_g(h)$.

Theorem 1.19. *For g an even integer such that $g \geq 4$ and $h \in \{1, 2, 3, ..., g/2\}$, we have that $\lambda_g(h) = h$.*

Proof. For $h \in \{1, 2, 3, ..., g/2\}$ let $n = [h]_g$. Clearly $n = h$ and $l_g(n) = h$. Furthermore $h \cdot g^0 < g^1$ for all $h \in \{1, 2, 3, ..., g/2\}$. So we have the smallest integer n with length h for each h and the claim follows. □

We will now give an explicit upper bound for $\lambda_g(h)$.

Theorem 1.20. *For even g, where $g \geq 4$ and $h \geq 5$, where $h \in \mathbf{N}$, we have that*

$$\lambda_g(h) \leq \frac{(2g-3) \cdot g^{h-3} - g + 2}{g-1}.$$

Proof. Let $h \geq 5$, where $h \in \mathbf{N}$. let $n = [-2, -1, -1, \cdots, -1, 2]_g$, where n consists of $h-4$ negative ones sandwiched between -2 at the beginning and 2 at the end. Clearly n consists of $h-2$ terms. Observe that $[-2, -1, -1, \cdots, -1, 2]_g$, is a special g-adic representation of n since $\varepsilon_i \in \{-2, -1, 2\}$ for $i = 0, 1, ..., h-2$ and $\varepsilon_0 \cdot \varepsilon_1 = 2$ and $\varepsilon_{h-3} \cdot \varepsilon_{h-2} = 0$. In addition, we see that

$$n = -1 + 2 \cdot g^{h-3} - \sum_{i=0}^{h-4} g^i = \frac{(2g-3) \cdot g^{h-3} - g + 2}{g-1},$$

and $l_g(n) = h$, from which it follows that $\lambda_g(h) \leq \frac{(2g-3) \cdot g^{h-3} - g + 2}{g-1}$. □

In the next chapter, we will consider results associated with the set P consisting of a finite set of primes. and their corresponding λ sequences.

Chapter 2

Metric diameter and such.

2.1 Introduction

In this chapter we begin with P some finite set of prime numbers and consider the additive group $\mathbf{Z}$ of integers with generating set

$$A_P = \{0\} \cup \{ \pm p^j : p \in P \text{ and } j \in \mathbf{N}_0 \}.$$

We will consider the diameter of the metric $(\mathbf{Z}, d_P)$. We will show that it is infinite for the case where $P = \{2, 3\}$ and discuss the pertinent background that led to this proof. We will discuss some consequences of this result. Nathanson did important work on the metric diameter and made a sweeping generalization to show the infinitude of the diameter of $(\mathbf{Z}, d_P)$ for all finite sets of primes P. Results derived by Prachar [11], Adleman, Pomerance and Rumely [1], and Bugeaud, Corvaja and Zannier [2] plays an important role in establishing that the diameter of $(\mathbf{Z}, d_P) = \infty$.

As in Chapter 1, we will let l_P and d_P denote, respectively, the corresponding word length function and metric induced on $\mathbf{Z}$. For every positive integer h, let $\lambda_P(h)$ denote the smallest positive integer of length h, that is, the smallest integer that can be represented as the sum of exactly h elements of A_P, but that cannot be represented as the sum of fewer than h elements of A_P. We will discuss the behavior of the function $\lambda_P(h)$. In particular we will look at the growth of $\lambda_P(h)$. We will also consider important results of Nathanson, Hajdu and Tijdeman and give an explicit lower bound for $\lambda_P(h)$ in the specific instance when $P = \{2, 3\}$.

We will begin by considering this deceptively simple case, namely when $P = \{2, 3\}$. This appears to be simple because positive integers K are literally written as sums and differences of terms of the form

$$K = \pm 2^{m_1} \pm 2^{m_2} \pm \cdots \pm 2^{n_j} \pm 3^{n_1} \pm 3^{n_2} \pm \cdots \pm 3^{n_t}$$

where $m_1, ..., m_j, n_1, ..., n_t$ are non negative integers. In the instances when the set P consists of any two primes greater than 3 we illustrate the difficulty encountered. Let $P = \{p_1, p_2\}$ where $\varepsilon_{v_i} \in \{1, 2, ..., \frac{p_1-1}{2}\}$ and $\varepsilon_{w_j} \in \{1, 2, ..., \frac{p_2-1}{2}\}$, then positive integers K are written in the form

$$K = \pm\varepsilon_{v_1} p_1^{v_1} \pm \varepsilon_{v_2} p_1^{v_2} \pm \cdots \pm \varepsilon_{v_g} p_1^{v_g} \pm \varepsilon_{w_1} p_2^{w_1} \pm \varepsilon_{w_2} p_2^{w_2} \pm \cdots \pm \varepsilon_{w_t} p_2^{w_t}$$

where g and t are non negative integers. Clearly the degrees of freedom of the coefficients of the powers of p_1 and p_2 are $\frac{p_1-1}{2}$ and $\frac{p_2-1}{2}$ respectively which increases the difficulty in the computations.

We will resolve a question raised by Conway and Nathanson to generate a new term of Nathanson's Lambda sequence by answering Problem 8 in [9]. Here is a statement of the problem listed as Problem CN below.

Problem CN.(Nathanson [9]) *Find all solutions in positive integers of the exponential Diophantine equations* $|2^a - 3^b| = 149$ *and* $|2^c - 3^d| = 151$. *These equations have no solutions if and only if* $\lambda_{2,3}(4) = 150$.

We will denote

$$A_{\{2,3\}} = A_{2,3} = \{0\} \cup \{\, \pm p^j : p \in \{2,3\} \,\text{and}\, j \in \mathbf{N}_0\}.$$

To generate this new term we will show the insolvability of the Diophantine equations $|2^a - 3^b| = 149$ and $|2^c - 3^d| = 151$ We will show this in three ways: The first by finding obstructions in $\mathbb{Z}/m\mathbb{Z}$, for specific values of m, the second by using Ellison's result that was derived from Baker's method and the final method by a theorem of Benne De Weger which extends Ellison's result. This in turn will help establish that $\lambda_{2,3}(4) = 150$, where $\lambda_{2,3}(h)$ plays an important role in the study of geometric diameter and additive number theory.

2.2 The diameter of $(\mathbf{Z}, d_{2,3})$.

We begin with the definition of the diameter of the metric space $(\mathbf{Z}, d_{2,3})$.

$$\text{diam}(\mathbf{Z}, A_{2,3}) = \sup\{d_{2,3}(m,n) : m,n \in \mathbf{Z}\}.$$

We will show that $\text{diam}(\mathbf{Z}, A_{2,3}) = \infty$. Before we proceed, we will state some important results from [1], [2] and [11] that can be used to prove that the $\text{diam}(\mathbf{Z}, d_{2,3}) = \infty$.

Prachar's result stated as Theorem P below was used by Nathanson in [10] to prove the sweeping generalization that the diameter of $(\mathbf{Z}, d_P)$ is infinite. Here is a statement of the result.

Theorem 2.1 (Prachar [11])**.** *Let $\delta(n)$ denote the number of prime numbers p such that $p-1$ divides n and $p-1$ is square free, then there exists infinitely many n such that $\delta(n) > \exp(c_0 \log n/(\log\log n)^2)$ and c_0 is an absolute constant.*

In Theorem 2.1 above, it is essential that $p-1$ be square free and this led to a delicate construction by Nathanson to show the infinitude of the metric diameter. Another result that we can use is the following improvement of the above Theorem which does not have a squared term in its denominator. This is stated as Theorem 2.2 below. This theorem of Adeleman, Pomerance and Rumley still depended on the square free property of $p-1$.

Theorem 2.2 (Adelman, Pomerance, Rumley, Proposition 10 [1])**.** *Let $\delta(n)$ denote the number of prime numbers p such that $p-1$ divides n and $p-1$ is square free, then there exists infinitely many n such that*

$$\delta(n) > \exp(c_1 \log n/\log\log n)$$

and c_1 is an absolute constant.

Finally we will give a third result, the one which we have elected to use. This result is stated as the theorem below. The improvements in this result with regards to the two theorems above are seen in the omission of the squared term in the denominator and the removal of the requirement that $p-1$ must be square free.

Theorem 2.3 (Beaugad, Corvaja, Zannier [2])**.** *For any two positive integers a and b, there exist infinitely many positive integers n for which* $\log(\gcd(a^n-1, b^n-1)) > \exp(c \log n / \log\log n)$*, where c is an absolute constant.*

Before we prove the infinitude of the diameter, we will define and prove some useful results.

Definition 2.4. *If X is an additive abelian group with identity 0, then for every subset B of X and every positive integer h, the h-fold sum and difference set is defined as $h^{\pm}B$, where*

$$h^{\pm}B = \left\{ \sum_{i=0}^{h} \varepsilon_i b_i : \varepsilon_i \in \{-1, 1\} \, and \, b_i \in B, \, for \, i \in \mathbf{N} \right\}$$

We now state the following combinatorial result:

Lemma 2.5. *For a finite set B, we will denote its order by $|B|$, and for all positive integers h, the cardinality of the h-fold sum and difference set, $h^{\pm}B$ has an upper bound that is given by*

$$|h^{\pm}B| \leq \binom{2|B| + h - 1}{h} \leq (2|B|)^h$$

Proof. We will prove this claim in two parts. We will first establish the first inequality. Note that each h combination of a set with order $2|B|$ and with repetition allowed is equivalent to considering a list of $2|B|-1$ bars and h stars. The $2|B|-1$ bars are used to denote $2|B|$ containers, with the nth container containing a star for each time the nth number is chosen in the combination. It follows that

$$|h^{\pm}B| \leq \binom{2|B|+h-1}{h}.$$

As an example, the diagram below shows a ten combination of a set with six elements which is depicted by using five bars and ten stars. The string below shows the combination with exactly five of the first number, one of the second, two of the third, none of the fourth, one of the fifth and one of the sixth.

$$\star\star\star\star\star|\star|\star\star|\;|\star|\star$$

For the second inequality, we make the following observations:

$$\binom{2|B|+h-1}{h} = \frac{(2|B|+h-1)!}{h!(2|B|-1)!} = \frac{(2|B|-1)(2|B|-2)\cdots(2|B|)}{h(h-1)\cdots 3\cdot 2\cdot 1},$$

$$2|B|(h-j+1) = 2|B| + 2|B|(h-j) \geq 2|B| + (h-j), \text{where } 1 < j \leq h.$$

It follows that $\frac{(2|B|+h-j)}{h-j+1} \leq 2|B|$ which in turn implies that

$$\binom{2|B|+h-1}{h} \leq (2|B|)^h.$$

□

We will now establish that the diameter of $(\mathbf{Z}, d_{2,3}) = \infty$.

Theorem 2.6. *The diameter of* $(\mathbf{Z}, d_{2,3}) = \infty$.

Proof. Assume that $\text{diam}(\mathbf{Z}, d_{2,3}) < \infty$, then consider the h-fold *sum and difference set* of A, where $A = A_{2,3} = \{0\} \cup \{ p^j : p \in \{2,3\} \text{ and } j \in \mathbf{N}_0\}$. From the assumption it follows that $h^{\pm}A = \mathbf{Z}$. Now for all $m < N$, where $\gcd(m, 2) = \gcd(m, 3) = 1$, define $\overline{A} = A \mod m$. It follows that $h^{\pm}\overline{A} = \mathbf{Z}/m\mathbf{Z}$. Now $2^k \equiv 1 \mod m$ and $3^l \equiv 1 \mod m$ for some k and l. It follows that $\overline{A} = \{-1, 0, 1\} \cup \{2^j : j = 1, 2, ..., k-1\} \cup \{ 3^r : r = 1, 2, ..., l-1\}$ and $|\overline{A}| = 3 + 2(k-1) + 2(l-1) < 2(k+l) \leq 2n$, where we choose $n = \text{lcm}[k, l]$. Clearly $2^n \equiv 1 \mod m$ and $3^n \equiv 1 \mod m$, which implies that $m|(2^n - 1)$ and $m|(3^n - 1)$ which in turn implies that $m|\gcd(2^n - 1, 3^n - 1)$, pick $m = \gcd(2^n - 1, 3^n - 1)$. We apply Lemma 2.1, to obtain $|h^{\pm}\overline{A}| \leq (2n)^h < m$. Now from Theorem BCZ, we get that $m = \gcd(a^n - 1, b^n - 1)) > \exp(\exp(c \log n / \log \log n))$ for infinitely many n which contradicts the assumption that $\text{diam}(\mathbf{Z}, d_{2,3})$ is finite. □

Nathanson showed in [10] that if P is a nonempty finite set of prime numbers and A is the set of positive integers whose prime factors all belong to P, then the metric space $(\mathbf{Z}, d_A)$ has infinite diameter and that the sphere: $S_A = \{n \in \mathbf{Z} : l_A(n) = h\}$ is infinite for every positive integer h. Some important consequences of this result are stated below.

Theorem 2.7 (Nathanson [10]). *For every positive integer h there are infinitely many positive integers N such that, for some nonnegative integers j and k with $h = j + k$, the exponential Diophantine equation*

$$N = \pm 2^{v_1} \pm 2^{v_2} \pm \cdots \pm 2^{v_j} \pm 3^{w_1} \pm 3^{w_1} \pm \cdots \pm 3^{w_k}$$

has a solution in nonnegative integers $v_1, ..., v_j, w_1 ..., w_k$*, but the corresponding equations with* h *replaced by any positive integer* $h' < h$ *have no solution.*

An immediate consequence of Theorem 2.7 above is that for every positive integer h there is a smallest number $N = \lambda_{2,3}(h)$ that can be written as the sum or difference of exactly h but no fewer powers of 2 and 3. We need Lemma 1.10 from Chapter 1, for the example given below.

Example 2.8. *The equation* $\ell_g(x) = n$ *for* $g \in \{2, 3\}$*,* $n \in \mathbf{N}$ *has infinitely many solutions for each fixed* n*. We will do the cases for* $g = 2$ *and* $g = 3$ *separately by constructing sequences of* x_m*'s that satisfy* $\ell_g(x) = n$ *for some fixed* n *. For the case* $g = 2$*, let* $x_m = 2^m \left(\frac{2^{2n-1}+1}{3}\right)$ *and* $m \in \mathbf{N}$*; then by lemma 1.10(a),*

$$\ell_2(x_m) = \ell_2 \left(2^m \left(\frac{2^{2n-1}+1}{3}\right)\right) = \ell_2 \left(\frac{2^{2n-1}+1}{3}\right).$$

Now $\ell_2 \left(\frac{2^{2n-1}+1}{3}\right) = n$ *by the result of Theorem 1.7 of Chapter 1 and we are done. The case for* $g = 3$ *is proven in a similar manner by letting* $x_{m'} = 3^{m'} \left(\frac{3^{n-1}+1}{2}\right)$ *and proceeding as above and using the result of Theorem 1.8 of Chapter 1.*

We will now establish the first four terms of Nathanson's $\lambda_{2,3}(h)$ sequence, where $h = 1, 2, 3$ and 4.

2.3 The first four terms of Nathanson's $\lambda_{2,3}(h)$ sequence.

We begin with the additive abelian group of integers with the generating set $A_{2,3} = \{0\} \cup \{\pm g^j, j = 0, 1, 2, 3, ...\}$ and consider the special lambda sequence $\lambda_{2,3}(h)$. We will generate the first four terms in turn.

We begin by showing that $\lambda_{2,3}(1) = 1$.

Lemma 2.9. $\lambda_{2,3}(1) = 1$.

Proof. This lemma easily follows from the following observations: $1 = 2^0 = 3^0$ and $l_{2,3}(1) = 1$. □

We will now establish the second term, $\lambda_{2,3}(2) = 5$.

Lemma 2.10. $\lambda_{2,3}(2) = 5$.

Proof. We begin by observing that $1 = 3^0, 2 = 2^1, 3 = 3^1, 4 = 2^2$ and $l_{2,3}(j) = 1$ for $j = 1, 2, 3$ and 4. In addition $5 = 2^1 + 3^1$, which implies that $l_{2,3}(5) \leq 2$. It is clear that $2^a = 5$ and $3^b = 5$ are insoluble from which it follows that $\lambda_{2,3}(2) = 5$. □

We will now proceed to establish the third term $\lambda_{2,3}(3) = 21$.

Lemma 2.11. *If* $a \in \{2, 3\}$, $j = 1, 2, 3, 4$ *and* $a^k \geq 5$ *then* $l_{2,3}(a^k \pm j) \leq 2$.

Proof. Clearly $l_{2,3}(a^k) = 1$, for $a \in \{2, 3\}$ and it follows that $l_{2,3}(a^k \pm j) \leq 2$ since $a^k \pm j$ is the sum of two terms from $A_{2,3}$. □

Lemma 2.11 gives us a way of sieving out eight numbers centered around a^k with length $l_{2,3} \leq 2$. We are now ready to show that $\lambda_{2,3}(3) = 21$.

Lemma 2.12. $\lambda_{2,3}(3) = 21$.

Proof. We will do the proof in two parts. We will first establish that $l_{2,3}(21) \leq 3$. We need to check numbers that are greater than 5. From lemma 2.11 above, it is clear that each element in the sets below has length, $l_{2,3} \leq 2$. The boxed elements are used to sieve out the numbers listed in the sets below.

$$\{4, 5, 6, 7, \boxed{2^3}, 9, 10, 11, 12\}$$

$$\{12, 13, 14, 15, \boxed{2^4}, 17, 18, 19, 20\}.$$

It is also clear that $21 = 2^4 + 3 + 2$. These facts show us that $1 \leq l_{2,3}(21) \leq 3$.

We will now establish that $l_{2,3}(21) \neq 1$ and $l_{2,3}(21) \neq 2$. To establish $l_{2,3}(21) \neq 1$, we make the trivial observation that $2^a = 21$ and $3^b = 21$ are insoluble in positive integers. To establish that $l_{2,3}(21) \neq 2$, we need to establish that the following Diophantine equations listed below are insoluble:

$$\pm 2^{k_1} \pm 3^{k_2} = 21$$
$$\pm 2^{k_3} \pm 2^{k_4} = 21$$
$$\pm 3^{k_5} \pm 3^{k_6} = 21$$

In the instances when the $k_i \neq 0$, for $i = 1, 2, 3, 4, 5$ and 6. We find obstructions in the ring $\mathbb{Z}/3\mathbb{Z}$ for the equations $\pm 2^{k_1} \pm 3^{k_2} = 21$. In particular we get on the left hand side the set $\{1, 2\}$ and on the right side $\{0\}$ which clearly has an empty intersection. For the equations, $\pm 2^{k_3} \pm 2^{k_4} = 21$ we get obstructions in the ring $\mathbb{Z}/2\mathbb{Z}$, in fact we have that $\{0\} \cap \{1\} = \emptyset$. Now for the equations $\pm 3^{k_5} \pm 3^{k_6} = 21$, we get obstructions in $\mathbb{Z}/2\mathbb{Z}$, where $\{0\} \cap \{1\} = \emptyset$.

We will now consider the remaining cases. We call these the fringe cases, namely the cases where the $k_i's$ can take on the value 0. These instances are easily distilled to the set of equations listed below which are insoluble. We quickly observe their insolubility by assuming that they are soluble and derive a contradiction to the fundamental theorem of arithmetic.

$$\pm 2^{m_2} = 22 = 2 \times 11$$

$$\pm 3^{m_4} = 22 = 2 \times 11$$

$$\pm 2^{m_1} = 20 = 2^2 \times 5$$

$$\pm 3^{m_3} = 20 = 2^2 \times 5$$

We have now established our claim and it follows that $\lambda_{2,3}(3) = 21$.

□

We will now build upon the previous results to establish the fourth term in Nathanson's sequence. We will first establish that $l_{2,3}(150) \leq 4$. We do so by considering numbers greater than 21. We will establish that $l_{2,3}(n) \leq 3$, for $n \in \mathbf{N}$ and $21 < n < 150$. We can generate all integers from 1 through

149 by using the sets S_1, S_2 and S_3 below:

$$S_1 = \{|2^{n_0} - 3^{m_0} - 2^{k_0}| : 1 \leq n_0, m_0, k_0 \leq 7\}$$
$$S_2 = \{|2^{n_1} - 2^{m_1} - 2^{k_1}| : 1 \leq n_1, m_1, k_1 \leq 7\}$$
$$S_3 = \{|3^{n_2} - 3^{m_2} - 2^{k_2}| : 1 \leq n_2, m_2, k_2 \leq 7\}$$

If we define $S_4 = \{m : m \in \mathbf{N}, 1 \leq n \leq 149\}$, then we get that $(S_1 \cup\ S_2 \cup S_3) \cap\ S_4 = S_4$. We also provide a list of all positive integers from 1 to 149 to illustrate that they can be written as a sum or difference of at most three distinct elements of $A_{2,3}$. This shows that $l_{2,3}(n) \leq 3$, for $n \in \mathbf{N}$ and $1 \leq n \leq 149$. See Table II in Appendix A.

We give two representations of 150.

$$150 = 2^2 - 2^4 - 3^4 + 3^5 = -3^1 - 3^2 - 3^4 + 3^5.$$

From this it follows that $l_{2,3}(150) \leq 4$.

We will now show that we cannot represent 150 as a sum or difference of one, two or three distinct terms from A_2 or A_3 respectively. This will set the stage to consider the remaining cases. We will need some Theorems of Nathanson, which was stated in Chapter 1 and stated again below.

Theorem 1.1. (Nathanson, [9]) *Every integer n has a unique representation $n = \sum_{i=0}^{\infty} \varepsilon_i 2^i$, where $\varepsilon_i \in \{-1, 0, 1\}$ as a finite sum and difference of distinct powers of 2 providing $\varepsilon_i \varepsilon_{i+1} = 0$. We call such an enumeration a special 2-adic representation of shortest length.*

We will now establish the case for A_2. From the above theorem, we are able to decide on the insolubility of a class of exponential Diophantine equations of the form $M = \sum_{i=0}^{k} \varepsilon_i 2^i$, where $\varepsilon_i \in \{-1, 0, 1\}$ and $\varepsilon_i \varepsilon_{i+1} = 0$ for $M, k \in \mathbf{N}$. We need to apply Nathanson's algorithm to generate the special 2-adic representation for M. The resulting representation is unique and provides the only solution to $M = \sum_{i=0}^{k} \varepsilon_i 2^i$. We make use of this observation in Corollary 2.13 below.

Corollary 2.13. *The exponential Diophantine equations:* $\pm 2^{c_1} = 150$, $\pm 2^{a_1} \pm 2^{a_2} = 150$ *and* $\pm 2^{b_1} \pm 2^{b_2} \pm 2^{b_3} = 150$ *are insoluble.*

Proof. Since $-2^1 - 2^3 + 2^5 + 2^7 = 150$, which is the unique special 2-adic representation of 150, the Corollary follows from Theorem 1.1. □

We will now make use of another theorem of Nathanson to handle the case in A_3 which we state below.

Theorem 1.2. (Nathanson, [9]) *Every integer* n *has a unique representation* $n = \sum_{i=0}^{\infty} \varepsilon_i' 3^i$, *where* $\varepsilon_i' \in \{-1, 0, 1\}$ *as a finite sum and difference of distinct powers of* 3 *of minimal length providing the* 2 *digits in its standard ternary expansion gets replaced with* $(3 - 1)$. *We call such an enumeration a special* 3*-adic representation of shortest length.*

From the above Theorem, we are able to decide on the insolubility of a class of exponential Diophantine equations of the form $M' = \sum_{i=0}^{k'} \varepsilon_i' 3^i$, where $\varepsilon'_i \in \{-1, 0, 1\}$ for $M', k' \in \mathbf{N}$. We need to apply Nathanson's algorithm to generate the special 3-adic representation for M. The resulting representation is unique and provides the only solution to $M' = \sum_{i=0}^{k'} \varepsilon'_i 3^i$. We make use of this observation in the Corollary below.

Corollary 2.14. *The Diophantine equations:* $\pm 3^{c_1} = 150$*,* $\pm 3^{a_1} \pm 3^{a_2} = 150$ *and* $\pm 3^{b_1} \pm 3^{b_2} \pm 3^{b_3} = 150$ *are insoluble.*

Proof. Since $-3^1 - 3^2 - 3^4 + 3^5 = 150$, which is the unique special 3-adic representation of 150, the Corollary follows from Theorem 1.2. □

Lemma 2.15. *We cannot represent* 150 *as a sum or difference of two distinct terms from* $A_{2,3}$*. We will exclude the cases that were already covered in Corollaries 2.13 and 2.14 above.*

Proof. We need to show that the Diophantine equations: $\pm 2^p \pm 3^q = 150$ are insoluble. For $p = 0$, clearly $\pm 3^q \neq 149, 151$. Similarly when $q = 0$, $\pm 2^p \neq 149, 151$. We need to consider the remaining cases, namely when $p, q > 0$.

We will show that these exponential Diophantine equations are insoluble when $p, q > 0$.

$$-2^p - 3^q = 150$$

$$2^p = 150 - 3^q$$

$$2^p = 150 + 3^q$$

$$2^p = 3^q - 150$$

The first equation is trivially insoluble. Now $2^k \mod 3 \equiv \{1,2\}$ and $\{150-3^q\} \mod 3 \equiv \{150+3^q\} \mod 3 \equiv \{3^q-150\} \mod 3 \equiv \{0\}$ and it easily follows that the remaining three equations above have obstructions occurring in $\mathbb{Z}/3\mathbb{Z}$ which establishes their insolubility. □

We have established up to this point that $l_{2,3}(150) \neq 1$ and $l_{2,3}(150) \neq 2$. We will now establish that $l_{2,3}(150) \neq 3$. To prove this, we need to show that we cannot represent 150 as a sum or difference of three distinct terms from $A_{2,3}$, we will begin by considering the exponential Diophantine equations:

$$\pm 2^{k_1} \pm 2^{k_2} \pm 3^{k_3} = 150$$
$$\pm 2^{n_1} \pm 3^{n_2} \pm 3^{n_3} = 150$$

In the first eight equations, when $k_1, k_2, k_3 > 0$ we have that $\pm 2^{k_1} \pm 2^{k_2} \pm 3^{k_3} = 150$ is insoluble by observing that $\{\pm 2^{k_1} \pm 2^{k_2} \pm 3^{k_3}\} \mod 2 \equiv \{1\}$ and $\{150\} \mod 2 \equiv \{0\}$ and we get an obstruction in $\mathbb{Z}/2\mathbb{Z}$.

In the remaining eight equations, when $n_1, n_2, n_3 > 0$ we have that $\pm 2^{n_1} \pm 3^{n_2} \pm 3^{n_3} = 150$ is insoluble by observing that $\{\pm 2^{n_1} \pm 3^{n_2} \pm 3^{n_3}\} \mod 3 \equiv \{1,2\}$ and $\{150\} \mod 3 \equiv \{0\}$ and we get an obstruction in $\mathbb{Z}/3\mathbb{Z}$.

We will now consider the equations $\pm 2^{k_1} \pm 2^{k_2} \pm 3^{k_3} = 150$, and the fringe cases when one of the $k_i = 0$, for $i = 1, 2$ and 3. We see that these eight equations can be distilled into the following equations below, where $\delta \in \{149, 151\}$.

$$\pm 2^{k_1} \pm 2^{k_2} \pm 3^{k_3} = 150 \Rightarrow \begin{cases} \pm 2^{k_1} \pm 2^{k_2} = \delta & \text{when } k_3 = 0, \\ \pm 2^{k_4} \pm 3^{k_3} = \delta & \text{when } k_1 = 0 \text{ or } k_2 = 0. \end{cases}$$

We will now consider the other set of equations $\pm 2^{n_1} \pm 3^{n_2} \pm 3^{n_3} = 150$, and the fringe cases when one of the $n_i = 0$, for $i = 1, 2$ and 3. We see that these eight equations can be distilled into the following equations below, where $\delta \in \{149, 151\}$.

$$\pm 2^{n_1} \pm 3^{n_2} \pm 3^{n_3} = 150 \Rightarrow \begin{cases} \pm 3^{n_2} \pm 3^{n_3} = \delta & \text{when } n_1 = 0, \\ \pm 2^{n_1} \pm 3^{n_4} = \delta & \text{when } n_2 = 0 \text{ or } n_3 = 0. \end{cases}$$

The thirty two exponential Diophantine equations from above are now rewritten in the more compact list below:

$$\begin{aligned} 2^{k_a} + 3^{k_b} &= \delta \\ \pm 2^{k_c} \pm 2^{k_d} &= \delta \\ \pm 3^{k_5} \pm 3^{k_6} &= \delta \\ |2^{k_7} - 3^{k_8}| &= \delta \end{aligned}$$

We will now proceed to show the insolubility of the equations above for $k_a, k_b, k_c, k_d, k_5, k_6, k_7, k_8 \in \mathbf{N}$.

We first establish the insolubility of $2^{k_a} + 3^{k_b} = \delta$.

Lemma 2.16. *The Diophantine equation: $2^{k_a} + 3^{k_b} = \delta$ is insoluble.*

Proof. Clearly $3^{k_b} > \delta$ for $k_b > 4$ and $2^{k_a} \notin \{\, \delta - 3^{k_b} \mid k_b = 0, 1, 2, 3, 4 \,\} = \{68, 70, 122, 124, 140, 142, 146, 148, 150\}$. The lemma follows. □

We will now establish the insolubility of $\pm 2^{k_3} \pm 2^{k_4} = \delta$.

Corollary 2.17. *The Diophantine equations: $\pm 2^{k_c} \pm 2^{k_d} = \delta$ are insoluble.*

Proof. Observe that $1 + 2^2 + 2^4 + 2^7 = 149$ and $-1 - 2^3 + 2^5 + 2^7 = 151$. Which represents the unique special 2-adic representations of 149 and 151, the Corollary follows from Theorem 1.1. □

The insolubility of $\pm 3^{k_5} \pm 3^{k_6} = \delta$ will now be established in a similar manner to that of Corollary 2.17.

Corollary 2.18. *The Diophantine equations: $\pm 3^{k_5} \pm 3^{k_6} = \delta$ are insoluble.*

Proof. Observe that $-1-3-3^2-3^4+3^5 = 149$ and $-1-3-3^2-3^4+3^5 = 151$. Which represents the unique special 3-adic representations of 149 and 151, the Corollary follows from Theorem 1.2. □

The remaining exponential Diophantine equations, $|2^{k_7} - 3^{k_8}| = \delta$ will be handled by three different methods. We will first establish some results to show their insolubility by obstructions in certain rings $\mathbb{Z}/m\mathbb{Z}$ for certain integral values of $m < 100$ in a general setting and then proceed to show their insolubility by using results derived from Baker's method.

Proposition 2.19. *The Diophantine equations $3^{k_7} - 2^{k_8} = d_1$ are insoluble when d_1 obeys the congruences below. We will illustrate this by showing* *obstructions in the rings* $\mathbb{Z}/m\mathbb{Z}$ *for* $m \in \{40, 56, 73, 80, 88\}$.

$$d_1 \equiv \begin{cases} 29 \mod 40, \\ 37 \mod 56, \\ 3 \mod 73, \\ 69 \mod 80, \\ 61 \mod 88. \end{cases}$$

Proof. We have tabulated the obstructions in table 2.1 below, which illustrates our claim.

Table 2.1: Obstructions

m	$S_1 = \{2^{k_7}\} \mod m$	$S_2 = \{3^{k_8} - 149\} \mod m$	$S_1 \cap S_2$ in $\mathbb{Z}/m\mathbb{Z}$
40	$\{1, 2, 4, 8, 16, 24, 32\}$	$\{12, 14, 20, 38\}$	$\emptyset$
56	$\{1, 2, 4, 8, 16, 32\}$	$\{20, 22, 28, 38, 44, 46\}$	$\emptyset$
73	$\{1, 2, 4, 8, 16, 32, 37, 55, 64\}$	$\{0, 5, 6, 21, 24, 43, 46, 61\}$	$\emptyset$
80	$\{1, 2, 4, 8, 16, 32, 48, 64\}$	$\{12, 14, 20, 38\}$	$\emptyset$
88	$\{1, 2, 4, 8, 16, 32\}\cup$	$\{6, 14, 20, 28, 30\}\cup$	
	$\{40, 48, 56, 64, 78, 80\}$	$\{36, 52, 54, 76, 86\}$	$\emptyset$

We will provide the details in the case when $m = 80$. The other cases can be handled similarly. For $k \geq 4$, and $k \in \mathbf{N}_0$ it is clear that $2^{4k} \equiv 1 \mod 5$ and $2^{4k} \equiv 16 \mod 16$ and it follows from the Chinese remainder theorem that $2^{4k} \equiv 16 \mod 80$. It is also clear that for $k \geq 4$, the residue class has a cycle of 4. In fact the residue class of $2^k \mod 80$ is $\{1, 2, 4, \overline{8, 32, 64, 48}\}$ and the overlined terms repeats in a cycle of four. Now observe that $3^{4t} \equiv 1 \mod 80$ and $t \in \mathbf{N}_0$ and the residue class of $3^t \mod 80 = \{1, 3, 9, 27\}$ with a cycle of order four. It now follows that there are obstructions for $2^{k_8} = 3^{k_7} - 149$ in $\mathbb{Z}/80\mathbb{Z}$. We see that $\{1, 2, 4, 8, 16, 32, 64, 48\} \cap \{12, 14, 20, 38\} = \emptyset$ and since $149 \equiv 69 \mod 80$, it follows that the Diophantine equations $3^{k_8} - 2^{k_7} = d_1$ are insoluble when $d_1 \equiv 69 \mod 80$.

□

We will now illustrate the insolubility of $3^{k_8} - 2^{k_7} = 151$.

Proposition 2.20. *The Diophantine equations $3^{k_8} - 2^{k_7} = d_2$ are insoluble when d_2 obeys the congruences below. We will illustrate this by showing obstructions in the rings $\mathbb{Z}/m\mathbb{Z}$ for $m \in \{40, 56, 80, 85\}$.*

$$d_2 \equiv \begin{cases} 31 \mod 40, \\ 39 \mod 56, \\ 71 \mod 80, \\ 66 \mod 85. \end{cases}$$

Proof. We have tabulated the obstructions in table 2.2 below, which illustrates our claim.

Table 2.2: Obstructions

m	$S_1 = \{2^{k_7}\} \mod m$	$S_2 = \{3^{k_8} - 151\} \mod m$	$S_1 \cap S_2$ in $\mathbb{Z}/m\mathbb{Z}$
40	$\{1, 2, 4, 8, 16, 24, 32\}$	$\{10, 12, 18, 36\}$	$\emptyset$
56	$\{1, 2, 4, 8, 16, 32\}$	$\{18, 20, 26, 28, 36, 42, 44\}$	$\emptyset$
80	$\{1, 2, 4, 8, 16, 32, 48, 64\}$	$\{10, 12, 18, 36\}$	$\emptyset$
85	$\{1, 2, 4, 8, 16, 32\}\cup$	$\{7, 15, 20, 22, 26, 28\}\cup$	
	$\{43, 64\}$	$\{35, 38, 40, 46, 67\}\cup$	
		$\{68, 76, 78, 81, 82\}$	$\emptyset$

We will provide the details in the case when $m = 80$. The other cases can be handled similarly. In the proof of Proposition 2.19 above, we saw that the residue class of $2^k \mod 80$ is $\{1, 2, 4, 8, 16, 32, 64, 48\}$ and the residue class of $3^t \mod 80 = \{1, 3, 9, 27\}$. It follows that there are obstructions for $2^{k_7} = 3^{k_8} - 151$ in $\mathbb{Z}/80\mathbb{Z}$. We see that $\{1, 2, 4, 8, 16, 32, 64, 48\} \cap \{10, 12, 18, 36\} = \emptyset$ and since $151 \equiv 71 \mod 80$, it follows that the Diophantine equations $3^{k_8} - 2^{k_7} = d_1$ are insoluble when $d_2 \equiv 71 \mod 80$.

□

We will now illustrate the insolubility of $2^{k_7} - 3^{k_8} = 149$.

Proposition 2.21. *The Diophantine equations* $2^{k_7} - 3^{k_7} = d_3$ *are insoluble when* d_3 *obeys the congruences below. We will illustrate this by showing obstructions in the rings* $\mathbb{Z}/m\mathbb{Z}$ *for* $m \in \{85, 91\}$*.*

$$d_3 \equiv \begin{cases} 64 \mod 85, \\ 58 \mod 91. \end{cases}$$

Proof. We have tabulated the obstructions in table 2.3 below, which illustrates our claim.

Table 2.3: Obstructions

m	$S_1 = \{2^{k_7}\} \mod m$	$S_2 = \{3^{k_8} + 149\} \mod m$	$S_1 \cap S_2$ in $\mathbb{Z}/m\mathbb{Z}$
85	$\{1, 2, 4, 8, 16, 32, 43, 64\}$	$\{0, 6, 27, 28, 36\}\cup$ $\{38, 41, 42, 52, 60, 65\}\cup$ $\{67, 71, 73, 80, 83\}$	$\emptyset$
91	$\{1, 2, 4, 8, 16, 23, 32, 37\}\cup$ $\{46, 57, 64, 74\}$	$\{28, 48, 59, 61, 67, 85\}$	$\emptyset$

We will provide the details in the case when $m = 91$. The other case can be handled similarly. Note that $2^{12} \equiv 1 \mod 91$, from which it follows that $2^l \mod 91 = \{1, 46, 23, 57, 74, 37, 64, 32, 16, 8, 4, 2\}$ for $l \in \mathbf{N}_0$. We also note that $3^6 \equiv 1 \mod 91$, and $3^r \mod 91 = \{1, 3, 9, 27, 81, 61\}$ for $r \in \mathbf{N}_0$. It follows that $3^r + 149 \mod 91 = \{28, 48, 59, 61, 67, 85\}$ and since $\{1, 46, 23, 57, 74, 37, 64, 32, 16, 8, 4, 2\} \cap \{1, 3, 9, 27, 81, 61\} = \emptyset$ we see that $2^{k_8} = 3^{k_7} + 149$ has obstructions in $\mathbb{Z}/91\mathbb{Z}$ and $d_3 \equiv 58 \mod 91$.

□

We will now illustrate the insolubility of $2^{k_7} - 3^{k_8} = 151$.

Proposition 2.22. *The Diophantine equations* $2^{k_7} - 3^{k_8} = d_4$ *are insoluble when* d_4 *obeys the congruences below. We will illustrate this by showing obstructions in the rings* $\mathbb{Z}/m\mathbb{Z}$ *for* $m \in \{80, 91\}$.

$$d_4 \equiv \begin{cases} 71 \mod 80, \\ 60 \mod 91. \end{cases}$$

Proof. We have tabulated the obstructions in table 2.4 below, which illustrates our claim.

Table 2.4: Obstructions

m	$S_1 = \{2^{k_7}\} \mod m$	$S_2 = \{3^{k_8} + 151\} \mod m$	$S_1 \cap S_2$ in $\mathbb{Z}/m\mathbb{Z}$
80	$\{1, 2, 4, 8, 16, 32, 48, 64\}$	$\{0, 18, 72, 74\}$	$\emptyset$
91	$\{1, 2, 4, 8, 16, 23, 32, 37\} \cup$ $\{46, 57, 64, 74\}$	$\{30, 50, 61, 63, 69, 87\}$	$\emptyset$

We will provide the details in the case when $m = 91$. The other case can be handled similarly. In the proof of Proposition 2.21 above, we saw that the residue class of $2^k \mod 91$ is $\{1, 46, 23, 57, 74, 37, 64, 32, 16, 8, 4, 2\}$ and the residue class of $3^t \mod 91 = \{1, 3, 9, 27, 81, 61\}$. It follows that there are obstructions for $2^{k_7} = 3^{k_8} + 151$ in $\mathbb{Z}/91\mathbb{Z}$. We see that $\{1, 2, 4, 8, 16, 32, 64, 48\} \cap \{30, 50, 61, 63, 69, 87\} = \emptyset$ and since $151 \equiv 60 \mod 91$, it follows that the Diophantine equations $2^{k_7} - 3^{k_8} = d_4$ are insoluble when $d_4 \equiv 60 \mod 91$. $\square$

Theorem 2.23. *The Diophantine equations* $|2^a - 3^b| = 149$ *and* $|2^c - 3^d| = 151$ *are insoluble.*

Proof. This theorem follows from Propositions 2.19 through 2.22. □

Theorem 2.24. $\lambda_{2,3}(4) = 150$.

Proof. This theorem follows from the fact that we have established above that $l_{2,3}(150) = 150$ and $l_{2,3}(n) \leq 3$ for all $n = 1, 2, ..., 149$. □

2.4 Proof of the insolubility of $|2^x - 3^y| = \delta$, where $\delta \in \{149, 151\}$ by Ellison's inequality.

Exponential Diophantine equations of the above type was studied by Pillai where he showed that $|2^x - 3^y| > 2^{(1-\varepsilon)x}$ for $\varepsilon > 0$. In particular we get immediately that $|2^x - 3^y| = k$ has a finite set of solutions for $x > x_0(\varepsilon)$; see [4] for example. It is instructive to get a more efficient method to establish the insolubility of such equations for certain values of k. This is particularly useful in establishing other terms of the $\lambda_{2,3}(h)$ sequence where we must consider a slew of Diophantine equations of this form.

Ellison improved upon Pillai's equation with the following beautiful result that was derived by applying Baker's method on logarithmic forms.

Theorem E. [13, 12] *The expression* $|2^x - 3^y| > 2^x e^{-x/10}$ *for* $x \geq 12$*, with* $x \neq 13, 14, 16, 19, 27$ *and all* y*.*

We will now proceed to prove Theorem 2.23 by using Ellison's result. We will first establish the following Proposition below. Instead of considering the exceptions to Ellison's Theorem, we will greedily establish Proposition 2.25 for $x \leq 27$, $x \in \mathbf{N}_0$

Proposition 2.25. *For $x \leq 27$, $x \in \mathbf{N}_0$ and all $y \in \mathbf{N}_0$ the equation $|2^x - 3^y| = \delta$ is insoluble.*

Proof. We need to show that $2^x \pm \delta$, is not of the form 3^y for some y, when $x \leq 27$, $x \in \mathbf{N}_0$. Let $\Lambda_\delta = \{\delta \pm 2^x \,|x \leq 27, x \in \mathbf{N}_0\}$. We list the set of positive elements of Λ_δ below.

{105, 107, 151, 153, 155, 157, 159, 165, 167, 181, 183, 213, 215, 277, 279, 361, 363, 405, 407, 661, 663, 873, 875, 1173, 1175, 1897, 1899, 2197, 2199, 3945, 3947, 4245, 4247, 8041, 8043, 8341, 8343, 16233, 16235, 16533, 16535, 32617, 32619, 32917, 32919, 65385, 65387, 65685, 65687, 130921, 130923, 131221, 131223, 261993, 261995, 262293, 262295, 524137, 524139, 524437, 524439, 1048425, 1048427, 1048725, 1048727, 2097001, 2097003, 2097301, 2097303, 4194153, 4194155, 4194453, 4194455, 8388457, 8388459, 8388757, 8388759, 16777065, 16777067, 16777365, 6777367, 33554281, 33554283, 33554581, 33554583, 67108713, 67108715, 67109013, 67109015, 134217577, 134217579, 134217877, 134217879}.

We need to only consider values of 3^y that are at most the biggest element of the set Λ_δ, i.e. $y \leq 17$. The Proposition now follows since $\Lambda_\delta \cap \{3^y|y = 0, 1, 2, ..., 17\} = \emptyset$.

□

We will now establish the insolvability of $|2^x - 3^y| = \delta$ for $x > 27$ and all y.

Proposition 2.26. *For $x > 27$ and all y the equation $|2^x - 3^y| = \delta$ is insoluble.*

Proof. From Theorem E, $|2^x - 3^y| > 2^x e^{-x/10}$ and $x \neq 13, 14, 16, 19, 27$. Define $f(x) = 2^x e^{-x/10}$, and observe that $f'(x) = 2^x e^{-x/10}(\log 2 - 0.1) > 0$ for all x. This implies that $f(x)$ is strictly increasing and since $f(28) > \delta$ the proposition follows. □

The two propositions above proves Theorem 2.23.

2.5 Proof of the insolubility of $|2^x - 3^y| = \delta$, where $\delta \in \{149, 151\}$ by Benne De Weger's method.

Benne de Weger proved a special case of a much more general result than that of Ellison. Here is a statement of his Theorem.

Theorem BDW[13, Theorem 5.5] *The Diophantine inequality*

$$|p_1^x - p_2^y| < \sqrt{\min\{p_1^x, p_2^y\}},$$

where $p_1, p_2 \in \{2, 3, 5, 7, 11, 13\}$, $x, y \in \mathbf{N}_0$ and $p_1 \neq p_2$, implies $x < a$ and $y < b$. See table 2.5 below for the values of a and b that corresponds to each p_1 and p_2.

Table 2.5: Benne De Weger Classifications

p_1	p_2	a	b	p_1	p_2	a	b
2	3	20	13	3	5	13	6
2	5	20	9	5	7	9	8
2	7	20	8	5	11	9	6
2	11	20	6	5	13	9	6
2	13	20	6	7	11	8	6
3	5	13	9	7	13	8	6
3	5	13	8	11	13	6	6
3	5	13	6				

We will now show that $|2^x - 3^y| = 149$ or 151 is insoluble with the help of the above theorem. Let $|2^x - 3^y| < 152$, and it follows from Theorem BDW (yellow highlight) that either $x < 20$ and $y < 13$ or $\min\{2^x, 3^y\} \leq 152^2$. Observe that $2^x \leq 152^2$, which implies that $x \leq \frac{2\ln 252}{\ln 2} \approx 14.5$ and similarly $3^y \leq 152^2$, which implies that $y \leq \frac{2\ln 252}{\ln 3} \approx 9.1$. Comparison of the two bounds gives us $x < 20, y < 13$. We will now proceed to check values in this interval to establish the insolubility. We provide the set of the values generated by computing $|2^x - 3^y|$ for $x < 20, y < 13$. We have omitted any duplications. We have a total of 271 values.

{0, 1, 2, 3, 5, 7, 8, 11, 13, 15, 17, 19, 23, 25, 26, 29, 31, 37, 47, 49, 55, 61, 63, 65, 73, 77, 79, 80, 101, 115, 119, 125, 127, 139, 175, 179, 211, 217, 227, 229, 235, 239, 241, 242, 247, 253, 255, 269, 295, 431, 473, 485, 503, 509, 511, 601, 665, 697, 713, 721, 725, 727, 728, 781, 943, 997, 1015, 1021, 1023, 1163, 1319, 1631, 1675, 1805, 1909, 1931, 1967, 2021, 2039, 2045, 2047, 2059, 2123, 2155, 2171, 2179, 2183, 2185, 2186, 2465, 3299, 3367, 3853, 4015, 4069, 4087, 4093, 4095, 4513, 5537, 6005, 6049, 6305, 6433, 6487, 6497, 6529, 6545, 6553, 6557, 6559, 6560, 7153, 7463, 7949, 8111, 8165, 8183, 8189, 8191, 9823, 11491, 13085, 14197, 15587, 15655, 16141, 16303, 16357, 16375, 16381, 16383, 17635, 18659, 19171, 19427, 19555, 19619, 19651, 19667, 19675, 19679, 19681, 19682, 26207, 26281, 30581, 32039, 32525, 32687, 32741, 32759, 32765, 32767, 42665, 45853, 46075, 50857, 54953, 57001, 58025, 58537, 58793, 58921, 58975, 58985, 59017, 59033, 59041, 59045, 59047, 59048, 63349, 64807, 65293, 65455, 65509, 65527, 65533, 65535, 72023, 84997, 111389, 111611, 124511, 128885, 130343, 130829, 130991, 131045, 131063, 131069, 131071, 144379, 160763, 168955, 173051, 175099, 176123, 176635, 176891, 177019, 177083, 177115, 177131, 177139, 177143, 177145, 177146, 203095, 242461, 255583, 259957, 261415, 261901, 262063, 262117, 262135, 262141, 262143, 269297, 347141, 400369, 465239, 465905, 498673, 504605, 515057, 517727, 522101, 523249, 523559, 524045, 524207, 524261, 524279, 524285, 524287, 527345, 529393, 530417, 530929, 531185, 531313, 531377, 531409, 531425, 531433, 531437, 531439, 531440, 1070035, 1332179, 1463251, 1528787, 1561555, 1577939, 1586131, 1590227, 1592275, 1593299, 1593811, 1594067, 1594195, 1594259, 1594291, 1594307, 1594315, 1594319, 1594321, 1594322}

Clearly 149 and 151 are not elements of the above set and our claim is proven.

The motivation for using the above technique is to create part of an arsenal of tools to assist in the computation of additional terms of various lambda sequences. It is not restricted to just the primes 2 and 3.

2.6 Additional results and open problems.

Hajdu and Tijdeman in [5] established the non-trivial bounds, $e^{c_0 h} < \lambda_{2,3}(h)$ $< e^{(h \ln 2)^{c_1}}$, where $c_0 = c_0(2,3)$ and c_1 is an absolute constant. They also showed that $\lambda_{2,3}(h) < e^{c_3 h \log(4h) \log\log h}$ for infinitely many h's, where c_3 are absolute constants.

We now give an explicit lower bound for $\lambda_{2,3}(h)$ by using our results from Chapter 1.

Theorem 2.27. $\lambda_{2,3}(h) \geq (2^{2h-1}+1)/3$.

Proof. In Chapter 1, it was shown for $p = 2$ and $p = 3$ that $\lambda_2(h) = (2^{2h-1}+1)/3$ and $\lambda_3(h) = (3^{h-1}+1)/2$ respectively. From this it readily follows that $\lambda_{2,3}(h) \geq \lambda_2(h) \geq \lambda_3(h)$ which gives us an explicit lower bound. □

It is also an open problem to generate terms of $\lambda_{2,5}(h)$, for $h \geq 1$. In the next chapter we will generate terms of the sequence $\lambda_{2,5}(h)$, for $h = 1, 2, 3$ and $h = 4$. We will generate all terms for $\lambda_{2,n}(h)$, where $h = 1, 2$ and 3 and n is any odd integer that is greater than 2. We will make extensive use of some of the tools used in Chapters 1 and 2.

Chapter 3

Odd Cases

3.1 Introduction

In this chapter we will generate terms of the sequence $\lambda_{2,5}(h)$, for $h = 1, 2, 3$ and $h = 4$. We will also generate all terms for $\lambda_{2,n}(h)$, where $h = 1, 2$ and 3 and n is any odd integer that is greater than 2. We make the startling discovery that $\lambda_{2,n}(h)$ for odd $n \geq 23$ attain the limit points 11 and 15 infinitely often and take on no other values. We will make extensive use of some of the tools used in Chapters 1 and 2.

3.2 New Definitions

We begin by redefining our notation from the previous chapters for brevity and to reflect the fact that other than 2 and 3, the sums of our representations have coefficients that are bigger than 1. We will write our generating set as A_n for any integer $n > 2$ in the following manner:

$$A_n = A_n^{\varepsilon_j(n)} = \{0\} \cup \{\pm\varepsilon_j(n) \cdot n^j : j = 0, 1, 2, ...\} \text{ where}$$

$$\varepsilon_j(n) \in \{0, 1, 2, ..., \lfloor n/2 \rfloor\}.$$

Note that $\lfloor x \rfloor$ denotes the greatest integer value $\leq x$ and refers to the floor function. Let P be a finite or infinite set of integers greater than 1, we extend the previous notion to consider the additive group $\mathbf{Z}$, of integers with generating set

$$A_P = A_P^{\varepsilon_j(P)} = \bigcup_{p \in P} A_p^{\varepsilon_j(p)}.$$

It is clear that A_g is symmetric since $a \in A_g$ if and only if $-a \in A_g$. The word length function $l_g : G \to \mathbf{N_0}$ is defined as follows: For $x \in G$ and $x \neq 0$, define $l_g(x) = r$, if r is the smallest positive integer such that there exist $a_i's \in A_g$, for $i = 1, 2, ..., r$ and $x = a_1 + a_2 + ... + a_r$. We will define $l_g(0) = 0$ trivially. The integer $l_g(x)$ is called the word length with respect to A_g.

Let $(\mathbf{Z}, d_g)$ be a metric space, where d_g is the metric induced on $\mathbf{Z}$. For $z \in \mathbf{Z}$ and $r \geq 0$, the sphere with center z and radius r is the set

$$S_z(r) = \{x \in \mathbf{Z} : d_g(x, z) = r\}.$$

Definition 3.1. *For every positive integer h, let $\lambda_P(h)$ denote the smallest positive integer m that can be represented as the sum of elements of A_P, where $h = \sum_{i=0}^{t} |\varepsilon_i|$, and $m = \sum_{i=0}^{t} \pm\varepsilon_i p^i$, but that cannot be represented as a sum of elements from A_P, where $\sum_{i=0}^{s} |\varepsilon_i| < h$ and $m = \sum_{i=0}^{s} \pm\varepsilon_i p^i$.*

3.3 $l_{2,n}(k)$ and $\lambda_{2,n}(h)$ for $h \in \{1, 2, 3\}$ and odd integers $n > 1$.

We will first tabulate in tables 3.1 and 3.2 pertinent representations for integers from 1 through 26. This will provide us with information on the length of positive integers in various special n-adic representations where n is odd. We will then proceed to compute the values of $\lambda_{2,n}(h)$ for $h \in \{1, 2, 3\}$ and any odd $n > 1$.

Table 3.1: Representations for k with elements from $A_{2,n}$.

k	Representations of k
1	2^0
2	2
3	$-1+2^2$
4	2^2
5	$1+2^2$
6	$-2+2^3$
7	$-1+2^3$
	7
8	2^3
9	$1+2^3$
	3^2
	9
10	$2+2^3$
11	$-1-2^2+2^4$
	2^3+3
	2^4-5
	2^2+7
	$2+9$
	11
	$-2+13$
	-2^2+15
	-2^3+19
	2^5-21
	-2^4+27

k	Representations of k
12	-2^2+2^4
	$1+11$
13	$1-2^2+2^4$
	2^3+5
	2^2+9
	$2+11$
	13
	$-2+15$
	2^5-19
	-2^3+21
14	$-2+2^4$
15	$-1+2^4$
16	2^4
17	$1+2^4$
18	$2+2^4$
19	$-1+2^2+2^4$
	2^3+11
	2^2+15
	2^5-13
	$-2+21$
	19
20	2^2+2^4
21	$1+2^2+2^4$
	2^5-11

Table 3.2: Representations for k with elements from $A_{2,n}$ cont'd.

k	Representations of k
21	$2^3 + 13$
	$2 + 19$
	21
22	$-2 - 2^3 + 2^5$
	$2 \cdot 11$
	$1 + 21$

k	Representations of k
23	$-1 - 2^3 + 2^5$
	$2 + 21$
24	$-2^3 + 2^5$
25	$1 - 2^3 + 2^5$
26	$2 - 2^3 + 2^5$
	$2^2 + 21$

3.4 $\lambda_{2,n}(h)$ for $h \in \{1, 2, 3\}$ and odd $n > 1$.

We will now prove some results that is essential to the establishment of $\lambda_{2,n}(h)$ for $h \in \{1, 2, 3\}$ and odd $n > 1$.

Lemma 3.2. *Let P be a finite or infinite set of integers > 1, then $l_P(k) = 1$ for all $k \in P$.*

Proof. Since $k \in P$, we can rewrite k as the special base k representation $1 \cdot k^0$ which has length 1. □

Lemma 3.3. *$l_{2,n}(t) \leq 2$ for $t \in Q$ where*

$$Q = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18, 20, 24\}.$$

Proof. For $t \in Q$, $l_2(t) \leq 2$ from which it follows that $l_{2,n}(t) \leq 2$. See tables 3.1 and 3.2 which contains the special base 2 representations of t. □

We will pay particular attention to the number 11 and consider its representations and length in the special n-adic form for odd n. We will classify all odd n's with $l_{2,n}(11) = 1$ and $l_{2,n}(11) = 2$. .

Lemma 3.4. *For odd $n > 1$, $l_{2,n}(11) = 1$ if and only if $n = 11$.*

Proof. From Lemma 3.2, for odd n it readily follows that $l_{2,n}(11) = 1$ for $n = 11$ and it is trivial to see that $n^t = 11$, for $n \neq 11$ and $2^a = 11$ is insoluble. □

Lemma 3.5. *$l_{2,n}(11) = 2$ for odd $n's$ if and only if $n \in \{3, 5, 7, 9, 13, 15, 19, 21\}$ and $n \in S$, where $S_1 = \{t : t = 2^\alpha + 11, \alpha \geq 4, \alpha \in \mathbf{N}\}$, $S_2 = \{s : s = 2^{\beta+2} - 11, \beta \geq 4, \beta \in \mathbf{N}\}$ and $S = S_1 \cup S_2$.*

Proof. Observe that the diophantine equation: $2 \cdot 11^b = 11$ is insoluble. Now from Table 3.1, we see that

$$11 = 2^3+3 = 2^4-5 = 2^2+7 = 2+9 = -2+13 = -2^2+15 = -2^3+19 = 2^5-21.$$

In addition, for $u \in S$ it readily follows that $l_{2,u}(11) = 2$, since 11 can be written as $|2^c - u|$ for odd $u \geq 3$ and from Lemma 3.4 it follows that $l_{2,u}(11) \neq 1$.

□

Lemma 3.6. *$l_{2,n}(13) = 2$ for any integer $n \in T$, where $T_1 = \{l : l = 2^\gamma + 13, \gamma \geq 4, \gamma \in \mathbf{N}\}$, $T_2 = \{k : k = 2^{\phi+2} - 13, \phi \geq 4, \phi \in \mathbf{N}\}$ and $T = T_1 \cup T_2$.*

Proof. For $v \in T$ it readily follows that $l_{2,v}(13) \leq 2$, since 13 can be written as $|2^c - v|$ for odd $v \geq 29$. Now consider the diophantine equations: $\pm 2^{m_1} = 13$ and $\pm v^{m_2} = 13$ which are clearly insoluble since 2 is even and $v \geq 29$ respectively. It follows that $l_{2,v}(13) \neq 1$ and therefore $l_{2,v}(13) = 2$.

□

Proposition 3.7. $l_{2,n}(11) = 3$ *for any* $n \notin \{3, 5, 7, 9, 11, 13, 15, 19, 21\} \cup S$.

Proof. For any $n \notin \{3, 5, 7, 9, 11, 13, 15, 19, 21\} \cup S$, by Lemmas 3.4, 3.5 and the fact that $l_{2,n}(11) = l_{2,n}(-1 - 2^2 + 2^4) = 3$ the claim follows.

□

Proposition 3.8. $l_{2,n}(13) = 3$ *for* $n \geq 27$, $n \in S$.

Proof. We first observe that $T \cap S = \emptyset$ since for $w \in T$, $l_{2,w}(13) = 2$ and this rules out the possibility for elements present in S of the form $|2^c - w|$ for odd $v \geq 29$ with length 2. This empty intersection follows from the insolubility of the following diophantine equations for $n \geq 27$.

$$n = 2^{c_1} + 13 = 2^{c_2} + 11$$
$$n = 2^{d_1} + 13 = 2^{d_2} - 11$$
$$n = 2^{e_1} - 13 = 2^{e_2} + 11$$
$$n = 2^{f_1} - 13 = 2^{f_2} - 11$$

It readily follows for the equations above that $(n, c_1, c_2) = (15, 1, 2)$, $(n, d_1, d_2) = (21, 3, 5)$, $(n, e_1, e_2) = (19, 5, 3)$ and $(n, f_1, f_2) = (-9, 2, 1)$, which we reject since $n \geq 27$. In addition, there are no other solutions. This is easily seen by observing obstructions in each case in the ring $\mathbb{Z}/2\mathbb{Z}$ when $c_1, c_2 > 1$, $d_1, d_2 > 3$, $e_1, e_2 > 3$ and $f_1, f_2 > 1$. We need to also check whether $2^{r_1} \pm 11 = 2 \cdot n^{r_2}$. This is insoluble and has obstructions in $\mathbb{Z}/2\mathbb{Z}$ for $r_1 > 0$. Since $T \cap S = \emptyset$ and the n's in the set S ($n \geq 27$) has $l_{2,n}(13) \neq 1, 2$. It follows that $n \in S$ will have $l_{2,n}(13) \geq 3$, but since $l_2(13) = 3$, ($13 = 1 - 2^2 + 2^4$) we get that $l_{2,n}(13) = 3$. □

We will now proceed to establish $\lambda_{2,n}(h)$ for $h \in \{1, 2, 3\}$ and odd $n > 1$. Recall that $\lambda_{2,3}(2) = 5$ from Chapter 2.

Proposition 3.9. *$\lambda_{2,n}(1) = 1$, and $\lambda_{2,n}(2) = 3$ for any odd integer $n \geq 5$, $n \in \mathbf{N}$.*

Proof. $\lambda_{2,n}(1) = 1$ is trivial and $\lambda_{2,n}(2) = 3$ follows from the fact that $l_2(1) = l_2(2) = 1$, $l_2(3) = 2$ and the Diophantine equations: $\pm 2^a = 3$, and $\pm n^b = 3$ are insoluble. □

Theorem 3.10. *For odd $n > 1$, where S is the set defined in Lemma 3.5 we have that*

$$\lambda_{2,n}(3) = \begin{cases} 11 \textit{ for } \; n \notin \{3, 5, 7, 9, 11, 13, 15, 19, 21\} \cup S, \\ 13 \textit{ for } \; n \in S. \end{cases}$$

Proof. From Lemma 3.3, we have that $l_{2,n}(t) \leq 2$ for $t \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12\}$. Using this fact and Propositions 3.7 and 3.8 our Theorem follows. □

We now provide in table 3.3 the values of $\lambda_{2,n}(3)$ for the first fourteen values of n obtained from Theorem 3.10.

Table 3.3: Values for $\lambda_{2,n}(3)$.

n	17	23	25	27	29	31	33	35	37	39	41	43	45	47
$\lambda_{2,n}(3)$	11	11	13	11	11	11	11	11	11	11	13	11	11	11

We will now proceed to generate the first nine odd values for $\lambda_{2,n}(3)$, where $n \geq 3$. We display the values in table 3.4 below before proceeding to verify them.

Table 3.4: Additional values for $\lambda_{2,n}(3)$.

n	3	5	7	9	11	13	15	19	21
$\lambda_{2,n}(3)$	21	19	13	19	23	22	21	22	26

Theorem 3.11. *We will show that* $\lambda_{2,3}(3) = 21$*,* $\lambda_{2,5}(3) = 19$*,* $\lambda_{2,7}(3) = 13$*,* $\lambda_{2,9}(3) = 19$*,* $\lambda_{2,11}(3) = 23$*,* $\lambda_{2,13}(3) = 22$*,* $\lambda_{2,15}(3) = 21$*,* $\lambda_{2,19}(3) = 22$*,* $\lambda_{2,21}(3) = 26$.

Proof. $\lambda_{2,3}(3) = 21$, was shown in Chapter 2. Consider the pairs $(g, k_{2,g})$, where $(g, k_{2,g}) \in \{(5, 19), (7, 13), (9, 19), (11, 23), (13, 22), (15, 21), (19, 22), (21, 26)\}$.

For integer valued x, where $1 \leq x < k_{2,g}$; $l_{2,g}(x) \leq 2$, which follows from the representations presented in Table 3.1. We need to now establish that the Diophantine equations listed below are insoluble in non-negative integers for each pair $(g, k_{2,g})$. This will show that the lengths of the representations of $l_{2,g}(k_{2,g}) \geq 3$.

$$\pm 2^{a_1} = k_{2,g} \tag{3.1}$$

$$\pm g^{a_2} = k_{2,g} \tag{3.2}$$

$$\pm 2 \cdot g^{a_3} = k_{2,g} \tag{3.3}$$

$$\pm 2^{a_4} \pm 2^{a_5} = k_{2,g} \tag{3.4}$$

$$\pm g^{a_6} \pm g^{a_7} = k_{2,g} \tag{3.5}$$

$$\pm 2^{a_8} \pm g^{a_9} = k_{2,g} \tag{3.6}$$

We now compute $l_2(k_{2,g})$ and $l_g(k_{2,g})$ and display in table 3.5 below.

Table 3.5: $l_2(k_{2,g})$ and $l_g(k_{2,g})$.

g	$k_{2,g}$	$k_{2,g} \in A_g$	$k_{2,g} \in A_2$	$l_2(k_{2,g})$	$l_g(k_{2,g})$
5	19	$-1-5+5^2$	$-1+2^2+2^4$	3	3
7	13	$-1+2\cdot 7$	$1-2^2+2^4$	3	3
9	19	$1+2\cdot 9$	$-1+2^2+2^4$	3	3
11	23	$1+2\cdot 11$	$-1-2^3+2^5$	3	3
13	22	$-4+2\cdot 13$	$-2-2^3+2^5$	3	6
15	21	$6+15$	$1+2^2+2^4$	3	7
19	22	$3+19$	$-2-2^3+2^5$	3	4
21	26	$5+21$	$2-2^3+2^5$	3	6

Observe that $l_2(k_{2,g}) > 2$ and $l_g(k_{2,g}) > 2$, and as a consequence of Theorems 1.1 and 1.11 in Chapter 1, equations represented by (3.1) through (3.5) above are insoluble since the equations(representations) have length less than 3. We will now consider the eight cases for $g \in \{5, 7, 9, 11, 13, 15, 19, 21\}$ individually to establish insolubility of equations represented by (3.6) above.

1. (a) For $2^{c_1} + 5^{c_2} = 19$, rewrite as $5^{c_2} = 19 - 2^{c_1}$, we get insolubility by substituting for $c_1 = 0, 1, 2, 3, 4$. We see that there are no corresponding values for c_2.

   (b) For $2^{c_3} - 5^{c_4} = 19$, rewrite as $2^{c_3} = 19 + 5^{c_4}$ and we get obstructions in the ring $\mathbb{Z}/15\mathbb{Z}$. Note that $2^{c_3} \mod 15 \equiv \{1, 2, 4, 8\}$. Similarly $5^{c_4} \mod 15 \equiv \{1, 5, 10\}$ which implies that $(19 + 5^{c_4}) \mod 15 \equiv \{5, 9, 14\}$. It follows that $\{1, 2, 4, 8\} \cap \{5, 9, 14\} = \emptyset$ and the equation is insoluble.

   (c) For $-2^{c_5} + 5^{c_6} = 19$, rewrite as $5^{c_6} = 19 + 2^{c_5}$. Clearly for $c_5 = 0, 1$ the equation is insoluble. For $c_5 > 1$ we get obstructions in the ring $\mathbb{Z}/4\mathbb{Z}$. We see that $\{1\} \cap \{3\} = \emptyset$ and the equation is insoluble.

2. (a) For $2^{d_1} + 7^{d_2} = 13$, rewrite as $7^{d_2} = 19 - 2^{d_1}$, we get insolubility by substituting for $d_1 = 0, 1, 2, 3$. We see that there are no corresponding values for d_2.

   (b) For $2^{d_3} - 7^{d_4} = 13$, rewrite as $2^{d_3} = 13 + 7^{d_4}$. For $d_3 = 0, 1, 2$ there are no solutions and for $d_3 > 2$, we get obstructions in the ring $\mathbb{Z}/8\mathbb{Z}$. Note that $2^{d_3} \mod 8 \equiv \{0\}$. Similarly $7^{d_4} \mod 8 \equiv \{1, 7\}$ which implies that $13 + 7^{d_4} \mod 8 \equiv \{4, 6\}$. It follows that $\{0\} \cap \{4, 6\} = \emptyset$ and the equation is insoluble.

   (c) For $-2^{d_5} + 7^{d_6} = 13$, rewrite as $7^{d_6} = 13 + 2^{d_5}$. Clearly for $d_6 = 0$ the equation is insoluble. For $d_6 > 0$ we get obstructions in the ring $\mathbb{Z}/3\mathbb{Z}$. We see that $\{1\} \cap \{0, 2\} = \emptyset$ and the equation is insoluble.

3. (a) For $2^{e_1} + 9^{e_2} = 19$, rewrite as $9^{e_2} = 19 - 2^{e_1}$, we get insolubility by

substituting for $e_1 = 0, 1, 2, 3, 4$. We see that there are no corresponding values for e_2.

(b) For $2^{e_3} - 9^{e_4} = 19$, rewrite as $2^{e_3} = 19 + 9^{d_4}$. For $e_3 = 0, 1$ or for $e_4 = 0$ there are no solutions and for $e_3 > 2$ and $e_4 > 0$, we get obstructions in the ring $\mathbb{Z}/8\mathbb{Z}$. Note that $2^{e_3} \mod 8 \equiv \{0\}$. Similarly $9^{e_4} \mod 8 \equiv \{1\}$ which implies that $(19 + 9^{e_4}) \mod 8 \equiv \{4\}$. It follows that $\{0\} \cap \{4\} = \emptyset$ and the equation is insoluble.

(c) For $-2^{e_5} + 9^{e_6} = 19$, rewrite as $9^{e_6} = 19 + 2^{e_5}$. Clearly for $e_6 = 0, 1$ the equation is insoluble. For $d_6 > 0$ we get obstructions in the ring $\mathbb{Z}/4\mathbb{Z}$. We see that $\{1\} \cap \{3\} = \emptyset$ and the equation is insoluble.

4. (a) For $2^{f_1} + 11^{f_2} = 23$, rewrite as $11^{f_2} = 23 - 2^{f_1}$, we get insolubility by substituting for $f_1 = 0, 1, 2, 3, 4$. We see that there are no corresponding values for f_2.

   (b),(c) For $|2^{f_3} - 9^{f_4}| = 23$, we will apply Theorem BDW from Chapter 2 to establish insolubility. Let $|2^{f_3} - 9^{f_4}| < 24$, it follows that $f_3 < 20$, and $f_4 < 6$, or $\min(2^{f_3}, 9^{f_4}) \leq 24^2$ ($f_3 \leq \frac{\ln 24^2}{\ln 2} < 10, f_4 \leq \frac{\ln 24^2}{\ln 9} < 3$). Comparing bounds we get that $f_3 < 20$ and $f_4 < 6$ and $|2^{f_3} - 9^{f_4}| \in \{0, 1, 3, 5, 7, 9, 10, 15, 21, 31, 53, ..., 524287\}$ which does not contain 23 and hence the equations are insoluble.

5. (a) For $2^{g_1} + 13^{g_2} = 22$, rewrite as $13^{g_2} = 22 - 2^{g_1}$, we get insolubility by substituting for $g_1 = 0, 1, 2, 3, 4$. We see that there are no corresponding values for g_2.

(b) For $2^{g_3} - 13^{g_4} = 22$, rewrite as $2^{g_3} = 22 + 13^{g_4}$ and we see that there is no solution for $g_3 = 0$ or $g_4 = 0$. We get obstructions in the ring $\mathbb{Z}/2\mathbb{Z}$ for $g_3 > 0$ or $g_4 > 0$. We see that $\{0\} \cap \{1\} = \emptyset$ and the equation is insoluble.

(c) For $-2^{g_5} + 13^{g_6} = 22$, rewrite as $13^{g_6} = 22 + 2^{g_5}$. Clearly for $g_5 = 0$ or for $g_6 = 0$, the equation is insoluble. For $g_5 > 0$ and $g_6 > 0$ we get obstructions in the ring $\mathbb{Z}/2\mathbb{Z}$. We see that $\{1\} \cap \{0\} = \emptyset$ and the equation is insoluble.

6. (a) For $2^{m_1} + 15^{m_2} = 21$, rewrite as $15^{m_2} = 21 - 2^{m_1}$, we get insolubility by substituting for $m_1 = 0, 1, 2, 3, 4$. We see that there are no corresponding values for k_2.

   (b) For $2^{m_3} - 15^{m_4} = 21$, rewrite as $2^{m_3} = 21 + 15^{k_4}$ and we see that there is no solution for $k_4 = 0$. We get obstructions in the ring $\mathbb{Z}/15\mathbb{Z}$ for $k_4 > 0$. We see that $\{1, 2, 4, 8\} \cap \{6\} = \emptyset$ and the equation is insoluble.

   (c) For $-2^{k_5} + 15^{k_6} = 21$, rewrite as $15^{k_6} = 21 + 2^{k_5}$. Clearly for for $k_6 = 0$, the equation is insoluble. For $k_6 > 0$ we get obstructions in the ring $\mathbb{Z}/15\mathbb{Z}$. We see that $\{0\} \cap \{7, 8, 10, 14\} = \emptyset$ and the equation is insoluble.

7. (a) For $2^{u_1} + 19^{u_2} = 22$, rewrite as $19^{u_2} = 22 - 2^{u_1}$, we get insolubility by substituting for $u_1 = 0, 1, 2, 3, 4$. We see that there are no corresponding values for u_2.

(b) For $2^{u_3} - 19^{u_4} = 22$, rewrite as $2^{u_3} = 22 + 19^{u_4}$ and we see that there is no solution for $u_3 = 0$ or $u_4 = 0$. We get obstructions in the ring $\mathbb{Z}/2\mathbb{Z}$ for $u_3 > 0$ or $u_4 > 0$. We see that $\{0\} \cap \{1\} = \emptyset$ and the equation is insoluble.

(c) For $-2^{u_5} + 19^{u_6} = 22$, rewrite as $19^{u_6} = 22 + 2^{u_5}$. Clearly for $u_5 = 0$ or for $u_6 = 0$, the equation is insoluble. For $u_5 > 0$ and $u_6 > 0$ we get obstructions in the ring $\mathbb{Z}/2\mathbb{Z}$. We see that $\{1\} \cap \{0\} = \emptyset$ and the equation is insoluble.

8. (a) For $2^{z_1} + 21^{z_2} = 26$, rewrite as $21^{z_2} = 26 - 2^{z_1}$, we get insolubility by substituting for $z_1 = 0, 1, 2, 3, 4$. We see that there are no corresponding values for z_2.

(b) For $2^{z_3} - 21^{z_4} = 26$, rewrite as $2^{z_3} = 26 + 21^{z_4}$ and we see that there is no solution for $z_3 = 0$ or $z_4 = 0$. We get obstructions in the ring $\mathbb{Z}/2\mathbb{Z}$ for $z_3 > 0$ or $z_4 > 0$. We see that $\{0\} \cap \{1\} = \emptyset$ and the equation is insoluble.

(c) For $-2^{z_5} + 21^{z_6} = 26$, rewrite as $21^{z_6} = 26 + 2^{z_5}$. Clearly for $z_5 = 0$ or for $z_6 = 0$, the equation is insoluble. For $z_5 > 0$ and $z_6 > 0$ we get obstructions in the ring $\mathbb{Z}/2\mathbb{Z}$. We see that $\{1\} \cap \{0\} = \emptyset$ and the equation is insoluble.

We have shown that $l_{2,g}(k_{2,g}) \geq 3$, but since $l_2(k_{2,g}) = 3$, it follows that $l_{2,g}(k_{2,g}) = 3$. This is the first instance (i.e. $x = k_{2,g}$) when the length $l_{2,g}(x) = 3$ for each g. Therefore we have that $\lambda_{2,3}(3) = 21$, $\lambda_{2,5}(3) = 19$,

$\lambda_{2,7}(3) = 13$, $\lambda_{2,9}(3) = 19$, $\lambda_{2,11}(3) = 23$, $\lambda_{2,13}(3) = 22$, $\lambda_{2,15}(3) = 21$, $\lambda_{2,19}(3) = 22$, $\lambda_{2,21}(3) = 26$.

□

3.5 The generation of the $\lambda_{2,5}(4)$ term.

We will now generate the $\lambda_{2,5}(4)$ term. Recall that

$$A_2 = \{0\} \cup \{\pm 2^j, j = 0, 1, 2, 3, ...\}.$$

$$A_5 = \{0\} \cup \{\pm 5^j, j = 0, 1, 2, 3, ...\} \cup \{\pm 2 \cdot 5^j, j = 0, 1, 2, 3, ...\}.$$

We saw in the previous section that $\lambda_{2,5}(1) = 1$, $\lambda_{2,5}(2) = 3$, $\lambda_{2,5}(3) = 19$. We will now show that $\lambda_{2,5}(4) = 83$. We begin by referring to Table III in appendix A.3. The numbers from 1 through 83 are written in the special partition form with elements from $A_{2,5}$. Since the special partitions use at most three elements from $A_{2,5}$ it follows that $l_{2,5}(x) \leq 3$ for integer valued x, where $1 \leq x \leq 82$. We will now proceed to establish that $l_{2,5}(83) = 4$.

We need to now establish that the Diophantine equations listed below are insoluble in non-negative integers. This will show that $l_{2,5}(83) \geq 4$.

$$\pm 2^{a_{11}} = 83,\ \pm 2^{a_{12}} \pm 2^{a_{13}} = 83,\ \pm 2^{a_{14}} \pm 2^{a_{15}} \pm 2^{a_{16}} = 83, \tag{3.7}$$

$$\pm 5^{a_{21}} = 83,\ \pm 2 \cdot 5^{a_{22}} = 83,\ \pm 5^{a_{23}} \pm 5^{a_{24}} = 83, \tag{3.8}$$

$$\pm 5^{a_{25}} \pm 2 \cdot 5^{a_{26}} = 83,\ \pm 5^{a_{27}} \pm 5^{a_{28}} \pm 5^{a_{29}} = 83, \tag{3.9}$$

$$\pm 2^{b_{11}} \pm 5^{b_{12}} = 83 \tag{3.10}$$

$$\pm 2^{b_{13}} \pm 2^{b_{14}} \pm 5^{a_{15}} = 83 \tag{3.11}$$

$$\pm 2^{b_{16}} \pm 5^{b_{17}} \pm 5^{a_{18}} = 83 \tag{3.12}$$

$$\pm 2^{b_{19}} \pm 2 \cdot 5^{b_{20}} = 83 \tag{3.13}$$

We begin by proving the following Lemma which will be useful in showing the insolubility of some of the equations above.

Lemma 3.12. *The Diophantine equations* $|2^a - 5^b| = t$ *for* $t \in \{82, 83, 84\}$ *are insoluble.*

Proof. For $|2^a - 5^b| = t$, for $t \in \{82, 83, 84\}$, we will apply Theorem BDW from Chapter 2 to establish insolubility. Let $|2^a - 5^b| < 85$, it follows that $a < 20$, and $b < 6$, or $\min(2^a, 5^b) \leq 85^2$ ($a \leq \frac{\ln 85^2}{\ln 2} < 13, b \leq \frac{\ln 85^2}{\ln 5} < 6$). Comparing bounds we get that $a < 20$ and $b < 6$ and $|2^a - 5^b| \in G = \{0, 1, 3, 7, 15, 31, 63, 127, ..., 521163\}$ and $t \notin G$ and hence the equations are insoluble.

□

We will now proceed to show that the Diophantine equations presented by (3.7) above are insoluble.

Lemma 3.13. *The Diophantine equations represented by (3.7) above are insoluble.*

Proof. Observe that $83 = -1 + 2^2 + 2^4 + 2^6$, and $l_2(83) = 4$. It now follows as a consequence of Theorem 1.1 in Chapter 1 that these representations(equations) are insoluble since their lengths are less than 4.

□

We will now proceed to show that the Diophantine equations presented by (3.8) and (3.9) above are insoluble.

Lemma 3.14. *The Diophantine equations represented by (3.8) and (3.9) above are insoluble.*

Proof. Observe that $83 = -1 + 2 \cdot 5 - 2 \cdot 5^2 + 5^3$, and $l_5(83) = 6$. It now follows as a consequence of Theorem 1.11 in Chapter 1 that these representations(equations) are insoluble since their lengths are less than 6. □

We will now show that the Diophantine equations presented by (3.10) above are insoluble.

Lemma 3.15. *The Diophantine equations* $\pm 2^{b_{11}} \pm 5^{b_{12}} = 83$ *are insoluble.*

Proof. To show the insolubility of $\pm 2^{b_{11}} \pm 5^{b_{12}} = 83$ is equivalent to showing the insolubility of $2^{b_{11}} + 5^{b_{12}} = 83$ and $|2^{b_{11}} - 5^{b_{12}}| = 83$. For the equation $2^{b_{11}} + 5^{b_{12}} = 83$, we will first rewrite it as $5^{b_{12}} = 83 - 2^{b_{11}}$, and check for solutions when $b_{11} \in \{0, 1, 2, 3, 4, 5, 6\}$. We see there are no solutions. For the equations: $|2^{b_{11}} - 5^{b_{12}}| = 83$, we have insolubility by Lemma 3.12. □

We will now establish the insolubility of the equations $\pm 2^{b_{13}} \pm 2^{b_{14}} \pm 5^{a_{15}} = 83$ which is listed as (3.11) above. First we distill these equations into the following cases and consider each separately:

$$\pm 2^a \pm 2^b - 5^c = 83 \tag{3.14}$$

$$\pm 2^d \pm 2^e + 5^f = 83 \tag{3.15}$$

Lemma 3.16. *The Diophantine equations* $\pm 2^{b_{13}} \pm 2^{b_{14}} \pm 5^{a_{15}} = 83$ *are insoluble.*

Proof. 1. (a) We will first establish the insolubility of the set of equations in (3.14), $\pm 2^a \pm 2^b - 5^c = 83$. These equations can be distilled further as the two expressions below. We will then show insolubility for each case by showing appropriate obstructions.

$$2^m + 2^n = 83 + 5^k \text{ and } 2^u - 2^v = 83 + 5^w$$

(i) For $2^m + 2^n = 83 + 5^k$, since $2^4 \equiv 1 \mod 31$, we have that $(2^m + 2^n) \mod 31 \equiv \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 17, 18, 20, 24\} = S_1$. In addition, $5^3 \equiv 1 \mod 31$, from which it follows that $(83 + 5^w) \mod 31 \equiv \{15, 22, 26\} = S_2$ and we get an obstruction in the ring $\mathbb{Z}/31\mathbb{Z}$ since $S_1 \cap S_2 = \emptyset$.

(ii) For $2^u - 2^v = 83 + 5^w$, When $w = 0$, the resulting equation $2^u - 2^v = 84$ is insoluble since $84 = 2^2 + 2^4 + 2^6$, and $l_2(84) = 3$. Now for $w > 0$, observe that $5^w \mod 195 \equiv \{5, 25, 125, 40\}$ and $2^{12} \mod 195 \equiv 1$. It follows that $(2^u - 2^v) \mod 195 \equiv S_3$, where S_3={0,1,2,3,4,6,7,8,12,14,15,16,17,24,28,29,30,31,32,33,34,37,41,45,47,

48,49,53,56,57,58,59,60,61,62,63,64,66,67,68,69,71,73,74,75,77,79,81

82, 83, 89,90,94,96,97,98,99,101,105,106,112,113,114,116,118,120,121,

122,124,126,127,128,129,131,132,133,134,135,136,137,138,139,142,146,

147,148,150,154,158,161,162,163,164,165,166,167,171,178,179,180,181,

183,187,188,189,191,192,193,194}.

In addition, $(83 + 5^w) \mod 195 \equiv \{13, 88, 108, 123\} = S_4$ and we see that $S_3 \cap S_4 = \emptyset$ to get an obstruction in the ring $\mathbb{Z}/195\mathbb{Z}$.

(b) We will now establish the insolubility of the equations $\pm 2^d \pm 2^e + 5^f = 83$ which is listed as (3.15) above. First we distill these equations into the following three cases and consider each separately.

$$2^x + 2^y + 5^z = 83,\ 2^m - 2^n + 5^k = 83,\ -2^a - 2^b + 5^c = 83.$$

(1) For $2^x + 2^y + 5^z = 83$, we will illustrate the fringe cases in table 3.6.

Table 3.6: Fringe cases.

(x, y)	$2^x + 2^y + 5^z = 83$	soluble
$(0, 0)$	$5^z = 81$	No
$(0, 1)$	$5^z = 80$	No
$(1, 0)$	$5^z = 80$	No
$(1, 1)$	$5^z = 79$	No

For $z = 0$, we have that $2^x + 2^y = 82$, which is insoluble since $82 = 2 + 2^4 + 2^6$, and $l_2(81) = 3$. Finally, when $x, y > 1$, and $z > 0$, we have an obstruction in the ring $\mathbb{Z}/4\mathbb{Z}$, i.e. $\{1\} \cap \{0\} = \emptyset$.

(2) For $2^m - 2^n + 5^k = 83$, we will illustrate the fringe cases in table 3.7.

Table 3.7: Fringe cases.

(m,n)	$2^m - 2^n + 5^k = 83$	soluble
$(0,0)$	$5^k = 83$	No
$(0,1)$	$5^k = 84$	No
$(1,0)$	$5^k = 82$	No
$(1,1)$	$5^k = 83$	No

For $k = 0$, we have that $2^m - 2^n = 82$, which is insoluble since $82 = 2 + 2^4 + 2^6$, and $l_2(81) = 3$. Finally, when $m, n > 1$, and $k > 0$, we have an obstruction in the ring $\mathbb{Z}/4\mathbb{Z}$, i.e. $\{1\} \cap \{0\} = \emptyset$.

(3) For $-2^a - 2^b + 5^c = 83$, we will illustrate the fringe cases in table 3.8.

Table 3.8: Fringe cases.

(a,b)	$-2^a - 2^b + 5^c = 83$	soluble
$(0,0)$	$5^z = 85$	No
$(0,1)$	$5^z = 86$	No
$(1,0)$	$5^z = 86$	No
$(1,1)$	$5^z = 87$	No

For $c = 0$, we have that $2^a + 2^b = -82$, which is clearly insoluble. Finally, when $a, b > 1$, and $c > 0$, we have an obstruction in the ring $\mathbb{Z}/4\mathbb{Z}$, i.e. $\{1\} \cap \{0\} = \emptyset$.

□

We will now show that the Diophantine equations presented by (3.12) above are insoluble.

Lemma 3.17. *The Diophantine equations* $\pm 2^{b_{16}} \pm 5^{b_{17}} \pm 5^{a_{18}} = 83$ *are insoluble.*

Proof. We will first dispense of the fringe cases. For $b_{16} = 0$, we have that $\pm 5^{b_{17}} \pm 5^{a_{18}} = 82, 84$, which is insoluble since $82 = 2 + 5^1 - 2 \cdot 5^2 + 5^3$ and $84 = -1 + 2 \cdot 5^1 - 2 \cdot 5^2 + 5^3$, and we get that $l_5(82) = l_5(84) = 6$.

For $5^{b_{17}} = 0$ or $5^{a_{18}} = 0$, we have that $|2^{b_{16}} - 5^{b_{17}}| = 82, 84$, which is insoluble by Lemma 3.12. We also get the equations $2^{b_{16}} + 5^{b_{17}} = 82, 84$, which is insoluble, since substituting $b_{16} \in \{0, 1, 2, 3, 4, 5, 6\}$ results in no valid solution for b_{17}.

For the case when $b_{16} > 0$, $b_{17} > 0$ and $a_{18} > 0$, we get an obstruction in the ring $\mathbb{Z}/2\mathbb{Z}$, i.e. $\{0\} \cap \{1\} = \emptyset$. □

We will now show that the Diophantine equations presented by (3.13) above are insoluble.

Lemma 3.18. *The Diophantine equations* $\pm 2^{b_{19}} \pm 2 \cdot 5^{b_{20}} = 83$ *are insoluble.*

Proof. We will first dispense of the fringe cases. For $b_{19} = 0$, we have that $\pm 2 \cdot 5^{b_{20}} = 82, 84$, which is insoluble since $l_5(82) = l_5(84) = 6$. For $b_{20} = 0$, we have that $\pm 2^{b_{19}} = 81, 85$, which is clearly insoluble. Finally, for $b_{19} > 0$ and $b_{20} > 0$, we have obstructions in $\mathbb{Z}/2\mathbb{Z}$, i.e. $\{0\} \cap \{1\} = \emptyset$.

□

We will now proceed to establish that $\lambda_{2,5}(4) = 83$.

Theorem 3.19. $\lambda_{2,5}(4) = 83$

Proof. This readily follows from Lemmas 3.13 through 3.18 and the fact that $l_2(83) = 4$, since $83 = -1 + 2^2 + 2^4 + 2^6$. □

3.6 Projects in progress

We are pursuing the generation of additional terms of $\lambda_{2,3}(h)$, where $h \geq 5$. We were able to show that $l_{2,3}(3003) = 5$, by using appropriate obstructions to certain Diophantine equations and Theorem BDW. We were also able to show that $\lambda_{2,3}(5) \in \{2581, 2613, 2798, 2865, 2870, 3003\}$. We plan to find $\lambda_{2,3}(5)$ precisely, generate additional terms and study sequences of the form $\lambda_{g_1,g_2,g_3}(h)$. A short calculation shows that $\lambda_{2,3,5}(1) = 1, \lambda_{2,3,5}(2) = 6, \lambda_{2,3,5}(3) = 38$. We are also working on sharper bounds for the $\lambda_P(h)$ terms for certain finite sets of positive integers P.

Appendix A

A.1 Table I

For $g = 5$, $m = 2$ and $k = 3$, the unique representations of the integers J_3 from the interval $[-62, 62]$

n	Partitions of $n \in J_3$
-62	$-2 \cdot 5^2 - 2 \cdot 5^1 - 2 \cdot 5^0$
-61	$-2 \cdot 5^2 - 2 \cdot 5^1 - 1 \cdot 5^0$
-60	$-2 \cdot 5^2 - 2 \cdot 5^1 + 0 \cdot 5^0$
-59	$-2 \cdot 5^2 - 2 \cdot 5^1 + 1 \cdot 5^0$
-58	$-2 \cdot 5^2 - 2 \cdot 5^1 + 2 \cdot 5^0$
-57	$-2 \cdot 5^2 - 1 \cdot 5^1 - 2 \cdot 5^0$
-56	$-2 \cdot 5^2 - 1 \cdot 5^1 - 1 \cdot 5^0$
-55	$-2 \cdot 5^2 - 1 \cdot 5^1 + 0 \cdot 5^0$
-54	$-2 \cdot 5^2 - 1 \cdot 5^1 + 1 \cdot 5^0$
-53	$-2 \cdot 5^2 - 1 \cdot 5^1 + 2 \cdot 5^0$
-52	$-2 \cdot 5^2 + 0 \cdot 5^1 - 2 \cdot 5^0$
-51	$-2 \cdot 5^2 + 0 \cdot 5^1 - 1 \cdot 5^0$
-50	$-2 \cdot 5^2 + 0 \cdot 5^1 + 0 \cdot 5^0$

n	representations of $n \in J_3$
-49	$-2 \cdot 5^2 + 0 \cdot 5^1 + 1 \cdot 5^0$
-48	$-2 \cdot 5^2 + 0 \cdot 5^1 + 2 \cdot 5^0$
-47	$-2 \cdot 5^2 + 1 \cdot 5^1 - 2 \cdot 5^0$
-46	$-2 \cdot 5^2 + 1 \cdot 5^1 - 1 \cdot 5^0$
-45	$-2 \cdot 5^2 + 1 \cdot 5^1 + 0 \cdot 5^0$
-44	$-2 \cdot 5^2 + 1 \cdot 5^1 + 1 \cdot 5^0$
-43	$-2 \cdot 5^2 + 1 \cdot 5^1 + 2 \cdot 5^0$
-42	$-2 \cdot 5^2 + 2 \cdot 5^1 - 2 \cdot 5^0$
-41	$-2 \cdot 5^2 + 2 \cdot 5^1 - 1 \cdot 5^0$
-40	$-2 \cdot 5^2 + 2 \cdot 5^1 + 0 \cdot 5^0$
-39	$-2 \cdot 5^2 + 2 \cdot 5^1 + 1 \cdot 5^0$
-38	$-2 \cdot 5^2 + 2 \cdot 5^1 + 2 \cdot 5^0$
-37	$-1 \cdot 5^2 - 2 \cdot 5^1 - 2 \cdot 5^0$

n	representations of $n \in J_3$
-36	$-1 \cdot 5^2 - 2 \cdot 5^1 - 1 \cdot 5^0$
-35	$-1 \cdot 5^2 - 2 \cdot 5^1 + 0 \cdot 5^0$
-34	$-1 \cdot 5^2 - 2 \cdot 5^1 + 1 \cdot 5^0$
-33	$-1 \cdot 5^2 - 2 \cdot 5^1 + 2 \cdot 5^0$
-32	$-1 \cdot 5^2 - 1 \cdot 5^1 - 2 \cdot 5^0$
-31	$-1 \cdot 5^2 - 1 \cdot 5^1 - 1 \cdot 5^0$
-30	$-1 \cdot 5^2 - 1 \cdot 5^1 + 0 \cdot 5^0$
-29	$-1 \cdot 5^2 - 1 \cdot 5^1 + 1 \cdot 5^0$
-28	$-1 \cdot 5^2 - 1 \cdot 5^1 + 2 \cdot 5^0$
-27	$-1 \cdot 5^2 + 0 \cdot 5^1 - 2 \cdot 5^0$
-26	$-1 \cdot 5^2 + 0 \cdot 5^1 - 1 \cdot 5^0$
-25	$-1 \cdot 5^2 + 0 \cdot 5^1 + 0 \cdot 5^0$
-24	$-1 \cdot 5^2 + 0 \cdot 5^1 + 1 \cdot 5^0$
-23	$-1 \cdot 5^2 + 0 \cdot 5^1 + 2 \cdot 5^0$
-22	$-1 \cdot 5^2 + 1 \cdot 5^1 - 2 \cdot 5^0$
-21	$-1 \cdot 5^2 + 1 \cdot 5^1 - 1 \cdot 5^0$
-20	$-1 \cdot 5^2 + 1 \cdot 5^1 + 0 \cdot 5^0$
-19	$-1 \cdot 5^2 + 1 \cdot 5^1 + 1 \cdot 5^0$
-18	$-1 \cdot 5^2 + 1 \cdot 5^1 + 2 \cdot 5^0$
-17	$-1 \cdot 5^2 + 2 \cdot 5^1 - 2 \cdot 5^0$
-16	$-1 \cdot 5^2 + 2 \cdot 5^1 - 1 \cdot 5^0$
-15	$-1 \cdot 5^2 + 2 \cdot 5^1 + 0 \cdot 5^0$
-14	$-1 \cdot 5^2 + 2 \cdot 5^1 + 1 \cdot 5^0$
-13	$-1 \cdot 5^2 + 2 \cdot 5^1 + 2 \cdot 5^0$
-12	$0 \cdot 5^2 - 2 \cdot 5^1 - 2 \cdot 5^0$

n	representations of $n \in J_3$
-11	$0 \cdot 5^2 - 2 \cdot 5^1 - 1 \cdot 5^0$
-10	$0 \cdot 5^2 - 2 \cdot 5^1 + 0 \cdot 5^0$
-9	$0 \cdot 5^2 - 2 \cdot 5^1 + 1 \cdot 5^0$
-8	$0 \cdot 5^2 - 2 \cdot 5^1 + 2 \cdot 5^0$
-7	$0 \cdot 5^2 - 1 \cdot 5^1 - 2 \cdot 5^0$
-6	$0 \cdot 5^2 - 1 \cdot 5^1 - 1 \cdot 5^0$
-5	$0 \cdot 5^2 - 1 \cdot 5^1 + 0 \cdot 5^0$
-4	$0 \cdot 5^2 - 1 \cdot 5^1 + 1 \cdot 5^0$
-3	$0 \cdot 5^2 - 1 \cdot 5^1 + 2 \cdot 5^0$
-2	$0 \cdot 5^2 + 0 \cdot 5^1 - 2 \cdot 5^0$
-1	$0 \cdot 5^2 + 0 \cdot 5^1 - 1 \cdot 5^0$
0	$0 \cdot 5^2 + 0 \cdot 5^1 + 0 \cdot 5^0$
1	$0 \cdot 5^2 + 0 \cdot 5^1 + 1 \cdot 5^0$
2	$0 \cdot 5^2 + 0 \cdot 5^1 + 2 \cdot 5^0$
3	$0 \cdot 5^2 + 1 \cdot 5^1 - 2 \cdot 5^0$
4	$0 \cdot 5^2 + 1 \cdot 5^1 - 1 \cdot 5^0$
5	$0 \cdot 5^2 + 1 \cdot 5^1 + 0 \cdot 5^0$
6	$0 \cdot 5^2 + 1 \cdot 5^1 + 1 \cdot 5^0$
7	$0 \cdot 5^2 + 1 \cdot 5^1 + 2 \cdot 5^0$
8	$0 \cdot 5^2 + 2 \cdot 5^1 - 2 \cdot 5^0$
9	$0 \cdot 5^2 + 2 \cdot 5^1 - 1 \cdot 5^0$
10	$0 \cdot 5^2 + 2 \cdot 5^1 + 0 \cdot 5^0$
11	$0 \cdot 5^2 + 2 \cdot 5^1 + 1 \cdot 5^0$
12	$0 \cdot 5^2 + 2 \cdot 5^1 + 2 \cdot 5^0$
13	$1 \cdot 5^2 - 2 \cdot 5^1 - 2 \cdot 5^0$

n	representations of $n \in J_3$
14	$1 \cdot 5^2 - 2 \cdot 5^1 - 1 \cdot 5^0$
15	$1 \cdot 5^2 - 2 \cdot 5^1 + 0 \cdot 5^0$
16	$1 \cdot 5^2 - 2 \cdot 5^1 + 1 \cdot 5^0$
17	$1 \cdot 5^2 - 2 \cdot 5^1 + 2 \cdot 5^0$
18	$1 \cdot 5^2 - 1 \cdot 5^1 - 2 \cdot 5^0$
19	$1 \cdot 5^2 - 1 \cdot 5^1 - 1 \cdot 5^0$
20	$1 \cdot 5^2 - 1 \cdot 5^1 + 0 \cdot 5^0$
21	$1 \cdot 5^2 - 1 \cdot 5^1 + 1 \cdot 5^0$
22	$1 \cdot 5^2 - 1 \cdot 5^1 + 2 \cdot 5^0$
23	$1 \cdot 5^2 + 0 \cdot 5^1 - 2 \cdot 5^0$
24	$1 \cdot 5^2 + 0 \cdot 5^1 - 1 \cdot 5^0$
25	$1 \cdot 5^2 + 0 \cdot 5^1 + 0 \cdot 5^0$
26	$1 \cdot 5^2 + 0 \cdot 5^1 + 1 \cdot 5^0$
27	$1 \cdot 5^2 + 0 \cdot 5^1 + 2 \cdot 5^0$
28	$1 \cdot 5^2 + 1 \cdot 5^1 - 2 \cdot 5^0$
29	$1 \cdot 5^2 + 1 \cdot 5^1 - 1 \cdot 5^0$
30	$1 \cdot 5^2 + 1 \cdot 5^1 + 0 \cdot 5^0$
31	$1 \cdot 5^2 + 1 \cdot 5^1 + 1 \cdot 5^0$
32	$1 \cdot 5^2 + 1 \cdot 5^1 + 2 \cdot 5^0$
33	$1 \cdot 5^2 + 2 \cdot 5^1 - 2 \cdot 5^0$
34	$1 \cdot 5^2 + 2 \cdot 5^1 - 1 \cdot 5^0$
35	$1 \cdot 5^2 + 2 \cdot 5^1 + 0 \cdot 5^0$
36	$1 \cdot 5^2 + 2 \cdot 5^1 + 1 \cdot 5^0$
37	$1 \cdot 5^2 + 2 \cdot 5^1 + 2 \cdot 5^0$
38	$2 \cdot 5^2 - 2 \cdot 5^1 - 2 \cdot 5^0$

n	representations of $n \in J_3$
39	$2 \cdot 5^2 - 2 \cdot 5^1 - 1 \cdot 5^0$
40	$2 \cdot 5^2 - 2 \cdot 5^1 + 0 \cdot 5^0$
41	$2 \cdot 5^2 - 2 \cdot 5^1 + 1 \cdot 5^0$
42	$2 \cdot 5^2 - 2 \cdot 5^1 + 2 \cdot 5^0$
43	$2 \cdot 5^2 - 1 \cdot 5^1 - 2 \cdot 5^0$
44	$2 \cdot 5^2 - 1 \cdot 5^1 - 1 \cdot 5^0$
45	$2 \cdot 5^2 - 1 \cdot 5^1 + 0 \cdot 5^0$
46	$2 \cdot 5^2 - 1 \cdot 5^1 + 1 \cdot 5^0$
47	$2 \cdot 5^2 - 1 \cdot 5^1 + 2 \cdot 5^0$
48	$2 \cdot 5^2 + 0 \cdot 5^1 - 2 \cdot 5^0$
49	$2 \cdot 5^2 + 0 \cdot 5^1 - 1 \cdot 5^0$
50	$2 \cdot 5^2 + 0 \cdot 5^1 + 0 \cdot 5^0$
51	$2 \cdot 5^2 + 0 \cdot 5^1 + 1 \cdot 5^0$
52	$2 \cdot 5^2 + 0 \cdot 5^1 + 2 \cdot 5^0$
53	$2 \cdot 5^2 + 1 \cdot 5^1 - 2 \cdot 5^0$
54	$2 \cdot 5^2 + 1 \cdot 5^1 - 1 \cdot 5^0$
55	$2 \cdot 5^2 + 1 \cdot 5^1 + 0 \cdot 5^0$
56	$2 \cdot 5^2 + 1 \cdot 5^1 + 1 \cdot 5^0$
57	$2 \cdot 5^2 + 1 \cdot 5^1 + 2 \cdot 5^0$
58	$2 \cdot 5^2 + 2 \cdot 5^1 - 2 \cdot 5^0$
59	$2 \cdot 5^2 + 2 \cdot 5^1 - 1 \cdot 5^0$
60	$2 \cdot 5^2 + 2 \cdot 5^1 + 0 \cdot 5^0$
61	$2 \cdot 5^2 + 2 \cdot 5^1 + 1 \cdot 5^0$
62	$2 \cdot 5^2 + 2 \cdot 5^1 + 2 \cdot 5^0$

A.2 Table II

Representations of 1 through 149 with elements from $A_{2,3}$.

$\mathbf{1 = 2^0}$	$2 = 2^1$	$3 = 3^1$
$4 = 2^2$	$\mathbf{5 = 2^1 + 3^1}$	$6 = 2^1 + 2^2$
$7 = 2^2 + 3^1$	$8 = 2^3$	$9 = 3^2$
$10 = 3^0 + 3^2$	$11 = 2^3 + 3$	$12 = 3^2 + 3^1$
$13 = 2^4 - 3^1$	$14 = 2^4 - 2^1$	$15 = 2^4 - 2^0$
$16 = 2^4$	$17 = 2^4 + 3^0$	$18 = 2^4 + 2^1$
$19 = 2^4 + 3^1$	$20 = 2^4 + 2^2$	$\mathbf{21 = 2^4 + 2^1 + 3^1}$
$22 = 2^4 + 3^2 - 3^1$	$23 = 3^3 - 2^2$	$24 = 3^3 - 3^1$
$25 = 3^3 - 2^1$	$26 = 3^3 - 3^0$	$27 = 3^3$
$28 = 3^3 + 2^0$	$29 = 3^3 + 2^1$	$30 = 3^3 + 3^1$
$31 = 3^3 + 2^2$	$32 = 2^5$	$33 = 2^5 + 2^0$
$34 = 2^5 + 2$	$35 = 2^5 + 3^1$	$36 = 2^5 + 2^2$
$37 = 2^5 + 2^1 + 3^1$	$38 = 2^5 + 2^2 + 2^1$	$39 = 2^5 + 2^2 + 3$
$40 = 2^5 + 3^2 - 2^0$	$41 = 2^5 + 3^2$	$42 = 2^5 + 3^2 + 3^0$
$43 = 2^5 + 3^2 + 2^1$	$44 = 2^5 + 3^2 + 3$	$45 = 2^5 + 3^2 + 2^2$
$46 = 2^4 + 3^2 + 3^1$	$47 = 2^4 + 3^3 + 2^2$	$48 = 2^6 - 2^4$
$49 = 2^6 - 2^4 + 3^0$	$50 = 2^6 - 2^4 + 2^1$	$51 = 3^4 - 2^5 + 2$
$52 = 3^4 - 2^5 + 3^1$	$53 = 3^4 - 2^5 + 2^2$	$54 = 2^6 - 2^3 - 2^1$
$55 = 2^6 - 2^3 - 2^0$	$56 = 2^6 - 2^3$	$57 = 2^6 - 2^3 + 3^0$
$58 = 2^6 - 2^3 + 2^1$	$59 = 2^6 - 2^3 + 3^1$	$60 = 2^6 - 2^3 + 2^2$
$61 = 2^6 - 3^1$	$62 = 2^6 - 2^1$	$63 = 2^6 - 2^0$
$64 = 2^6$	$65 = 2^6 + 2^0$	$66 = 2^6 + 2^1$
$67 = 2^6 + 3^1$	$68 = 2^6 + 2^2$	$69 = 2^6 + 2^1 + 3^1$
$70 = 2^6 + 2^3 - 2^1$	$71 = 2^6 + 2^3 - 2^0$	$72 = 2^6 + 2^3$

$73 = 2^6 + 3^2$	$74 = 3^4 - 2^3 + 2^1$	$75 = 3^4 - 2^3 + 2$
$76 = 3^4 - 2^3 + 3^1$	$77 = 3^4 - 2^3 + 2^2$	$78 = 3^4 - 3^1$
$79 = 3^4 - 2^2$	$80 = 3^4 - 3^0$	$81 = 3^4$
$82 = 3^4 + 3^0$	$83 = 3^4 + 2^1$	$84 = 3^4 + 3^1$
$85 = 3^4 + 2^2$	$86 = 3^4 + 2^2 + 2^0$	$87 = 3^4 + 2^2 + 2^1$
$88 = 3^4 + 2^2 + 3^1$	$89 = 3^4 + 2^3$	$90 = 3^4 + 3^2$
$91 = 3^4 + 3^2 + 3^0$	$92 = 3^4 + 3^2 + 2^1$	$93 = 3^4 + 3^2 + 2^1$
$94 = 3^4 + 3^2 + 2^2$	$95 = 3^4 + 2^4 - 2^1$	$96 = 3^4 + 2^4 - 3^0$
$97 = 3^4 + 2^4$	$98 = 3^4 + 2^4 + 2^0$	$99 = 3^4 + 2^4 + 2^1$
$100 = 3^4 + 2^4 + 3^1$	$101 = 2^7 - 3^3$	$102 = 2^7 - 3^3 + 2^0$
$103 = 2^7 - 3^3 + 2$	$104 = 3^4 + 3^3 - 2^2$	$105 = 3^4 + 3^3 - 3^1$
$106 = 3^4 + 3^3 - 2^1$	$107 = 3^4 + 3^3 - 2^0$	$108 = 3^4 + 3^3$
$109 = 3^4 + 3^3 + 2^0$	$110 = 3^4 + 3^3 + 2^1$	$111 = 3^4 + 3^3 + 3^1$
$112 = 3^4 + 3^3 + 3^2$	$113 = 2^5 + 3^4$	$114 = 2^5 + 3^4 + 3^0$
$115 = 2^5 + 3^4 + 2$	$116 = 2^5 + 3^4 + 3^1$	$117 = 2^5 + 3^4 + 2^2$
$118 = 2^7 - 2^3 - 2^1$	$119 = 2^7 - 2^3 - 2^0$	$120 = 2^7 - 2^3$
$121 = 2^7 - 2^3 + 2^0$	$122 = 2^7 - 2^3 + 2^1$	$123 = 2^7 - 2^3 + 3^1$
$124 = 2^7 - 2^3 + 2^2$	$125 = 2^7 - 3^1$	$126 = 2^7 - 2^1$
$127 = 2^7 - 3^0$	$128 = 2^7$	$129 = 2^7 + 2^0$
$130 = 2^7 + 2^1$	$131 = 2^7 + 3^1$	$132 = 2^7 + 2^2$
$133 = 2^7 + 2^3 - 3^1$	$134 = 2^7 + 2^3 - 2^1$	$135 = 2^7 + 2^3 - 1$
$136 = 2^7 + 2^3$	$137 = 2^7 + 2^3 + 3^0$	$138 = 2^7 + 2^3 + 2^1$
$139 = 2^7 + 2^3 + 3^1$	$140 = 2^7 + 2^3 + 2^2$	$141 = 2^7 + 2^4 - 3^1$
$142 = 2^7 + 2^4 - 2^1$	$143 = 2^7 + 2^4 - 3^0$	$144 = 2^7 + 2^4$
$145 = 2^7 + 2^4 + 2^0$	$146 = 2^7 + 2^4 + 2^1$	$147 = 2^7 + 2^4 + 3^1$
$148 = 2^7 + 2^4 + 2^2$	$149 = 2^6 + 2^2 + 3^4$	

A.3 Table III

Representations of 1 through 83 with elements from $A_{2,5}$.

$\mathbf{1 = 2^0}$	$2 = 2^1$	$\mathbf{3 = -2 + 5}$
$4 = 2^2$	$5 = 5$	$6 = 2^1 + 2^2$
$7 = 2 + 5$	$8 = 2^3$	$9 = 1 + 2^3$
$10 = 2 \cdot 5$	$11 = 2^4 - 5$	$12 = 2^4 - 2^2$
$13 = 2^3 + 5$	$14 = 2^4 - 2^1$	$15 = 2^4 - 2^0$
$16 = 2^4$	$17 = 2^4 + 2^0$	$18 = 2^4 + 2^1$
$\mathbf{19 = -1 - 5 + 5^2}$	$20 = 2^4 + 2^2$	$21 = 2^4 + 5$
$22 = 1 + 2^4 + 5$	$23 = 2 + 2^4 + 5$	$24 = -1 + 5^2$
$25 = 5^2$	$26 = 1 + 5^2$	$27 = 2^1 + 5^2$
$28 = 1 + 2 + 5^2$	$29 = 2^2 + 5^2$	$30 = 1 + 2^2 + 5^2$
$31 = 2 + 2^2 + 5^2$	$32 = 2^5$	$33 = 2^5 + 2^0$
$34 = 2 + 2^5$	$35 = 1 + 2 + 2^5$	$36 = 2^2 + 2^5$
$37 = 1 + 2^2 + 2^5$	$38 = 2 + 2^2 + 2^5$	$39 = -1 + 2^3 + 2^5$
$40 = 2^3 + 2^5$	$41 = 1 + 2^3 + 2^5$	$42 = 2 + 2^3 + 2^5$
$43 = 2 + 2^4 + 5$	$44 = 2^2 + 2^3 + 2^5$	$45 = 2^2 + 2^4 + 5^2$
$46 = -2^2 + 2 \cdot 5^2$	$47 = 2^6 - 2^4 - 1$	$48 = 2^6 - 2^4$
$49 = 2^6 - 2^4 + 5^0$	$50 = 2^6 - 2^4 + 2^1$	$51 = 1 + 2 \cdot 5^2$
$52 = 2^2 + 2^4 + 2^5$	$53 = 2^4 + 2^5 + 5$	$54 = 2^6 - 2^3 - 2^1$
$55 = 2^6 - 2^3 - 2^0$	$56 = 2^6 - 2^3$	$57 = 2^6 - 2^3 + 5^0$
$58 = 2^6 - 2^3 + 2^1$	$59 = 2^6 - 2^2 - 1$	$60 = 2^6 - 2^2$
$61 = 2^6 - 2^2 + 1$	$62 = 2^6 - 2^1$	$63 = 2^6 - 2^0$
$64 = 2^6$	$65 = 2^6 + 2^0$	$66 = 2^6 + 2^1$
$67 = 2^6 + 2 + 1$	$68 = 2^6 + 2^2$	$69 = 2^6 + 2^2 + 1$
$70 = 2^6 + 2^3 - 2^1$	$71 = 2 + 2^6 + 5$	$72 = 2^6 + 2^3$

$73 = 2^6 + 2^3 + 1$	$74 = 2^6 + 2^3 + 2$	$75 = 2^6 + 2^4 - 5$
$76 = 2^6 + 2^4 - 2^2$	$77 = 2^3 + 2^6 + 5$	$78 = 2^6 + 2^4 - 2$
$79 = 2^6 + 2^4 - 1$	$80 = 2^6 + 2^4$	$81 = 2^6 + 2^4 + 1$
$82 = 2^6 + 2^4 + 2$		

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