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Discovering Geometric and Topological Properties of Ellipsoids by Curvatures

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Abstract

Aims/ Objectives: We are interested in discovering the geometric, topological and physical properties of ellipsoids by analyzing curvature properties on ellipsoids. We begin with studying ellipsoids as a starting point. Our aim is to find a way to study geometric, topological and physical properties from the analytic curvature properties for convex hyper-surfaces in the general setting.
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Methodology: Calculating curvatures of a surface is now at the threshold of a better understanding regarding geometric, topological and physical properties on a surface. In order to calculate various curvatures, we demonstrate the way to compute the second fundamental form associated with curvatures by extending the calculation method from spheres to ellipsoids.
Results: Just as curvatures of a sphere are determined by its radius, curvatures of an ellipsoid are determined by its longest axis and its shortest axis. On an ellipsoid, the value of the ratio of its longest axis to its shortest axis is also a critical index to characterize its geometric, topological and physical behaviors.
Conclusion: Our results on ellipsoids are extensions or generalizations of results of Lawson-Simons, Wei, and Simons on spheres, and Kobayashi-Ohnita-Takeuchi on an ellipsoid with "one variable". Methods and research findings in this paper can lead to future research on convex hyper-surfaces.

Keywords: Ellipsoids; second fundamental form; curvatures; p-harmonic maps; p-super-strongly unstable (p-SSU) manifolds; Yang-Mills instable manifolds.

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1 Introduction

For a space curve, the curvature is an amount of the bending degree to measure on how sharply a curve bends. A smaller circle bends more sharply with the bigger value of the curvature due to the fact the curvature of a circle is equal to the reciprocal of its radius. For a smooth surface, the curvature is an index to visualize the degree of the deviation from being flat. Calculating curvatures of surfaces is now at the threshold of a better understanding regarding geometric, topological, and physical properties for various shapes of surfaces.

For a convex hyper-surface in a Riemannian manifold, studying geometric, topological, and physical properties by analyzing its curvature properties is an impressive idea. It has a lasting and profound impact on mathematicians. In the direction of geometric properties such as stability or instability reflected by curvatures associated with the second fundamental form, Wei and Yau [1] defined a p-super-strongly unstable (p-SSU) manifold. Yang-Mills [2] defined a Yang-Mills instable manifold. Howard and Wei [3] studied the stability or instability of sub-manifolds of a surface. Wei, Wu and Zhang [4] proved the instability of minimal immersion on a manifold with non-positive scalar curvature and generalized Bernstein-type theorem. In the direction of topological properties reflected by curvatures, Synge [5] found the link between the topological vanishing properties of the first homotopy classes and the properties of the sectional curvature. Wei and Wu [6] proved the relationship between the topological vanishing properties of the higher-dimensional homotopy classes and the properties of the Ricci curvature. In the direction of the physical shape of a surface reflected by curvatures, Wente [7] found a non-spherical finite surface as a counter-example of a Hopf's conjecture. This counter-example is a constant mean curvature surface with the shape of a self-intersecting three-lobed torus.

In the study of a convex hyper-surface, most mathematicians began with a sphere as a starting point and have obtained numerous results on a sphere. For example, Wei [8] has obtained the topological vanishing theorem for minimal sub-manifolds of a sphere with the Ricci curvature bounded below. Lawson and Simons [9] have obtained the non-existence of stable sub-manifolds on a sphere.

In our paper, we are interested in an ellipsoid with $(n+1)$ variables where a sphere is a special case of an ellipsoid. We discover the topological vanishing properties and geometric properties of stability or instability on an ellipsoid, which are characterized by its curvature properties. The curvature values on an ellipsoid are determined by its longest axis and its shortest axis. The ratio of its longest axis to its shortest axis is considered as an indicator of an ellipsoid as a p-SSU manifold and a Yang-Mills instable manifold. In order to estimate curvature values, we demonstrate the way to compute the second fundamental form by extending the calculation method from a sphere to an ellipsoid. Our results on an ellipsoid are extensions or generalizations of results of Lawson and Simons [9], Wei [8], and Simons (announced in Tokyo in September of 1977, cf. [10], [11]) on spheres, and Kobayashi, Ohnita and Takeuchi [12] on an ellipsoid with "one variable".

2 Materials and Methods

Here and throughout this paper, we denote an ellipsoid by

$$E^n = \{(x_1, \cdots, x_{n+1}) \in R^{n+1} : \frac{x_1^2}{a_1^2} + \cdots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1, a_i > 0, \forall 1 \le i \le n+1\},$$

$$\max(a_i) = \max_{1\le i\le n+1}\{a_i\},\ \min(a_i) = \min_{1\le i\le n+1}\{a_i\},\ \text{and } \mu = \frac{\max(a_i)}{\min(a_i)}.$$

Note that an ellipsoid E^n defined in our notation contains a non-spherical ellipsoid (i.e. $\exists i \neq j$, $a_i \neq a_j$) and a sphere (i.e. $a_i = r > 0,\ \forall 1 \le i \le n+1$). We denote a sphere of radius r by

$$S^n(r) = \{(x_1, \cdots, x_{n+1}) \in R^{n+1} : \frac{x_1^2}{a_1^2} + \cdots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1, a_i = r > 0, \forall 1 \le i \le n+1\}.$$

We also denote a unit sphere by S^n, i.e. $r = 1$.

Let M^n be a compact Riemannian manifold with possibly non-empty boundary, and let N^k be isometrically immersed in R^q. $L_1^p(M^n, N^k)$ denotes the set of maps $u : M^n \to N^k (\hookrightarrow R^q)$ whose component functions have weak first derivatives in L^p and $u(x) \in N^k$ a.e. on M^n.

Definition 2.1. The p-energy ($p > 1$) for $u \in L_1^p(M^n, N^k)$ is given by

$$E_p(u) = \frac{1}{p}\int_{M^n} |du|^p dv$$

where du denotes the differential of u, and dv is the volume element of M^n.

Definition 2.2. A map $u \in L_1^p(M^n, N^k)$ is said to be weakly p-harmonic ($p > 1$) if it is a weak solution to the following Euler-Lagrange equation for E_p on $L_1^p(M^n, N^k)$:

$$div(|\nabla u|^{p-2}\nabla u) = 0$$

u is called p-stable (resp. p-minimizing) if u is a local (resp. global) minimum of p-energy functional E_p within a homotopy class of $L_1^p(M^n, N^k)$ having the same trace on ∂M^n.

Let's recall the definition of a p-strongly unstable (p-SU) manifold and the definition of a p-super-strongly unstable (p-SSU) manifold.

Definition 2.3. A Riemannian manifold is p-strongly unstable (p-SU) if it is neither the domain nor the target of any non-constant smooth p-stable map.

Definition 2.4. A Riemannian manifold N^k with a Riemannian metric $\langle , \rangle_{N^k}$ is said to be p-super-strongly unstable (p-SSU) for $p \ge 2$, if there exists an isometric immersion in R^q such that, for every unit tangent vector X to N^k at every point $x \in N^k$, the following function is always *negative*:

$$\begin{aligned} F_{p,x}(X) &= (p-2)\langle h(X,X), h(X,X)\rangle_{R^q} + \langle Q_x^{N^k}(X), X\rangle_{N^k} \\ &= (p-2)|h_{XX}|^2 + \langle Q_x^{N^k}(X), X\rangle_{N^k} \end{aligned}$$

where

$$\begin{aligned}\langle Q_x^{N^k}(X), X\rangle_{N^k} &= \sum_{i=1}^{k}(2\langle h(X,\alpha_i), h(X,\alpha_i)\rangle_{R^q} - \langle h(X,X), h(\alpha_i,\alpha_i)\rangle_{R^q}) \\ &= \sum_{i=1}^{k}(2|h_{X\alpha_i}|^2 - h_{XX}h_{\alpha_i\alpha_i}),\end{aligned}$$

h is the second fundamental form of N^k in R^q, and $\{\alpha_1, \cdots, \alpha_k\}$ is a local orthonormal frame on N^k near x.

In fact, every compact p-SSU manifold is p-SU (cf. Theorem 3.1 [1]).

Definition 2.5. A p-SSU index ω_p on a p-SSU manifold N^k is defined to be:

$$\omega_p = \inf_{x\in N^k} \frac{\phi_p(x)}{(k+p-2)S(x)}$$

where $\phi_p(x) = \inf_{\|X\|=1}(-F_{p,x}(X))$ for every unit tangent vector X to N^k at x and $S(x) > 0$ is a positive upper bound of the sectional curvature of N^k at x.

Here are the definitions of a Yang-Mills functional of connections, a Yang-Mills weakly stable connection, and a Yang-Mills instable manifold.

Definition 2.6. Let M^n be a compact Riemannian manifold and P be a principal G-bundle over M^n, where G is a compact Lie group. On the space $\mathcal{C}_P$ of the connections in P, the Yang-Mills functional $J : \mathcal{C}_P \to R$ is defined by

$$J(\omega) = \frac{1}{2}\int_{M^n} \|\Omega\|^2\, dv, \quad \omega \in \mathcal{C}_P$$

where Ω is the curvature of the connection ω in $\mathcal{C}_P$ and where the norm is defined in terms of the Riemannian metric on M and a fixed Ad_G-invariant scalar product on the Lie algebra $\mathcal{G}$ of G.

Definition 2.7. A critical point of J is called a Yang-Mills connection and its curvature a Yang-Mills field. A Yang-Mills connection ω is said to be weakly stable if the second variation of J at ω is non-negative, i.e.,

$$\frac{d^2}{dt^2}J(\omega_t)|_{t=0} \geq 0$$

for every smooth family of connections ω_t, $-\delta < t < \delta$, with $\omega_0 = \omega$. We say that a compact Riemannian manifold M^n is Yang-Mills instable if, for every choice of G and every principal G-bundle P over M^n, none of the non-flat Yang-Mills connections in P is weakly stable.

We are interested in exploring geometric, topological and physical properties of ellipsoids by analyzing their curvature properties. Therefore, calculating curvatures on ellipsoids is the first crucial step. Furthermore, we find that calculating the second fundamental form plays an important role in calculating curvatures. The difficulty of calculating curvatures comes from the difficulty of calculating the second fundamental form. In our paper, we demonstrate the way to compute the second fundamental form by extending the calculation method from spheres $S^n(r)$ to ellipsoids E^n.

Lemma 2.1. *(Estimate of the Second Fundamental Form of an Ellipsoid E^n Embedded in R^{n+1} [13]) Suppose ν is a unit normal vector field on an ellipsoid E^n imbedded in R^{n+1} and X is any unit tangent vector on E^n at a point x, then the second fundamental form h_{XX} defined by*

$$h_{XX} = \langle \bar{\nabla}_X \nu, X\rangle$$

satisfies

$$\frac{\min(a_i)}{(\max(a_i))^2} \leq h_{XX} \leq \frac{\max(a_i)}{(\min(a_i))^2}$$

where $\bar{\nabla}$ is the Riemannian connection on R^{n+1} and $\langle , \rangle$ is the Riemannian metric on R^{n+1}.

Proof. Suppose $\{U_1, U_2, \cdots, U_{n+1}\}$ is a standard orthonormal basis in R^{n+1}. Since E^n is a level set of $\frac{x_1^2}{a_1^2} + \cdots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1$, we can find a normal vector $Z = \sum_{i=1}^{n+1} \frac{x_i}{a_i^2} U_i$ and a unit normal vector $\nu = \frac{Z}{\|Z\|} = \frac{1}{(\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4})^{\frac{1}{2}}} \sum_{i=1}^{n+1} \frac{x_i}{a_i^2} U_i$. Suppose $X = \sum_{i=1}^{n+1} \beta_i U_i$ is a unit tangent vector field on E^n with $\sum_{i=1}^{n+1} \beta_i^2 = 1$, then we compute

$$\bar{\nabla}_X \nu = \frac{1}{\|Z\|} \bar{\nabla}_X Z + X(\frac{1}{\|Z\|})Z, \tag{2.1}$$

$$\begin{array}{rcl} \bar{\nabla}_X Z & = & \sum_{i=1}^{n+1} \frac{X[x_i]}{a_i^2} U_i \\ & = & \sum_{i=1}^{n+1} \frac{\beta_i}{a_i^2} U_i, \end{array} \tag{2.2}$$

where $X[x_i]$ is the directional derivative of x_i in the X direction and $X[x_i] = dx_i(X) = \beta_i$.

$$\begin{array}{rcl} h_{XX} & = & \langle \bar{\nabla}_X \nu, X \rangle \\ & = & \langle \frac{1}{\|Z\|} \bar{\nabla}_X Z, X \rangle + \langle X(\frac{1}{\|Z\|})Z, X \rangle \\ & = & \frac{1}{\|Z\|} \sum_{i=1}^{n+1} \frac{\beta_i^2}{a_i^2}, \end{array} \tag{2.3}$$

where $\langle X(\frac{1}{\|Z\|})Z, X \rangle = 0$ since Z is a normal vector to E^n and X is a tangent vector on E^n.
By observing $\sum_{i=1}^{n+1} \beta_i^2 = 1$ for a unit tangent vector X and $\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^2} = 1$ on E^n, we have:

$$\begin{array}{rcccl} \frac{1}{\max(a_i)} & \leq & \|Z\| & \leq & \frac{1}{\min(a_i)}, \\ \frac{1}{(\max(a_i))^2} & \leq & \|\bar{\nabla}_X Z\| & \leq & \frac{1}{(\min(a_i))^2}, \\ \frac{\min(a_i)}{(\max(a_i))^2} \leq \frac{1}{\|Z\|} \cdot \frac{1}{(\max(a_i))^2} & \leq & h_{XX} & \leq & \frac{1}{\|Z\|} \cdot \frac{1}{(\min(a_i))^2} \leq \frac{\max(a_i)}{(\min(a_i))^2}. \end{array} \tag{2.4}$$

□

Lemma 2.2. *(Estimates of Principal Curvatures and Sectional Curvatures of E^n Embedded in R^{n+1} [14]) Suppose $\{\lambda_i\}_{i=1}^n$ is a family of principal curvatures on E^n embedded in R^{n+1} satisfying $0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$ and Sec^{E^n} denotes any sectional curvature on E^n embedded in R^{n+1}. Then:*

$$\lambda_1 \geq \frac{\min(a_i)}{(\max(a_i))^2},$$

$$\lambda_n \leq \frac{\max(a_i)}{(\min(a_i))^2},$$

$$\frac{(\min(a_i))^2}{(\max(a_i))^4} \leq Sec^{E^n} \leq \frac{(\max(a_i))^2}{(\min(a_i))^4}.$$

Proof. cf. Lemma 2, Lemma 3, and Corollary 1 in [14]. □

3 Results and Discussion

We are interested in studying ellipsoids as one step to move on to the study of any convex hypersurface. Based on our calculation method for the second fundamental form (cf. Lemma 2.1) and curvatures (cf. Lemma 2.2) in the section 2, we will explore geometric, topological and physical properties on ellipsoids by estimating their curvature properties. Subjects on topological properties of vanishing theorems on ellipsoids, geometric properties of instability on ellipsoids, and applications of ellipsoids as different types of manifolds will be discussed in the following subsections.

3.1 Topological Vanishing Theorem on Minimal Sub-Manifolds of Ellipsoids

In this subsection, we study the topological vanishing properties on ellipsoids.

Theorem 3.1. *(Topological Vanishing Theorem on Minimal Sub-Manifolds of Ellipsoids) Let N^k be a minimal k-dimensional sub-manifold of an ellipsoid E^n such that the Ricci curvature Ric^{N^k} of N^k satisfies:*

$$Ric^{N^k} > k(1-\frac{1}{p})\frac{(\max(a_i))^2}{(\min(a_i))^4}$$

where $2 \le p < k$. Then N^k is p-SSU and

$$\pi_1(N^k) = \pi_2(N^k) = \cdots = \pi_{[p]}(N^k) = 0.$$

Proof. At any point $x \in N^k$, for any unit vector $X \in T_xN^k$, we choose an orthonormal frame $\{\alpha_1, \cdots, \alpha_k\}$ on N^k such that $X = \alpha_k$. Let $\{\beta_1, \cdots, \beta_{n-k}\}$ be an orthonormal frame in the normal space $(T_xN^k)^\perp$ on E^n and ν be a unit normal vector to E^n on $\mathbb{R}^{n+1}$ defined in Lemma 2.1. For $1 \le i, j \le k$, let R_{ij} (resp. $\bar{R}_{ij}$) denote the sectional curvature of N^k (resp. an ellipsoid E^n) for the section $\alpha_i \wedge \alpha_j$. For $1 \le i, j \le k$, let $\bar{h}_{ij} = (\bigtriangledown_{\alpha_i}\alpha_j)^\perp$ where $\bigtriangledown$ is the Riemannian connection on E^n and $()^\perp$ is the projection onto the normal space $(T_xN^k)^\perp$. The fact that N^k is minimal says:

$$\sum_{i=1}^{k} \bar{h}_{ii} = 0. \tag{3.1}$$

Let $h(Y, Z) = \langle \bar{\bigtriangledown}_Y \nu, Z\rangle$ where $\bar{\bigtriangledown}$ is the Riemannian connection on R^{n+1}, Y and Z are local vector fields on E^n, and ν is a unit normal vector to E^n. And the second fundamental form $\widetilde{h}$ of N^k in R^{n+1} splits into the sum of the second fundamental form $\bar{h}$ of N^k in E^n and the second fundamental form h of E^n in $\mathbb{R}^{n+1}$. That is,

$$|\widetilde{h}_{kk}|^2 = |\widetilde{h}(X, X)|^2 = |\bar{h}_{kk}|^2 + |h(X, X)|^2 = |\bar{h}_{kk}|^2 + |h_{kk}|^2. \tag{3.2}$$

Applying the Gauss-curvature equation, we have:

$$R_{ik} = \bar{R}_{ik} + \langle \bar{h}_{ii}, \bar{h}_{kk}\rangle_{E^n} - \bar{h}_{ik}^2 \tag{3.3}$$

for $1 \le i \le k$.

Summing (3.3) over $1 \le i \le k$ and via (3.1), we have:

$$Ric(X) = \sum_{i=1}^{k-1} \bar{R}_{ik} - \sum_{i=1}^{k} \bar{h}_{ik}^2. \tag{3.4}$$

Via (3.4) and the assumption $Ric^{N^k} > k(1-\frac{1}{p})\frac{(\max(a_i))^2}{(\min(a_i))^4}$, we obtain:

$$\begin{array}{rcl} \bar{h}_{kk}^2 & \le & \sum_{i=1}^{k} \bar{h}_{ik}^2 \\ & < & \sum_{i=1}^{k-1} \bar{R}_{ik} - k(1-\frac{1}{p})\frac{(max(a_i))^2}{(min(a_i))^4} \\ & \le & (k-1)\frac{(max(a_i))^2}{(min(a_i))^4} - k(1-\frac{1}{p})\frac{(max(a_i))^2}{(min(a_i))^4} \\ & = & (\frac{k}{p}-1)\frac{(\max(a_i))^2}{(\min(a_i))^4} \end{array} \tag{3.5}$$

where we apply Lemma 2.2 for $\bar{R}_{ij} = Sec^{E^n} \le \frac{(max(a_i))^2}{(min(a_i))^4}$.

On the other hand, by the formula of $\langle Q_x^{N^k}(X), X\rangle_{N^k}$ for a minimal sub-manifold N^k of E^n (cf. [3] p.322), we have:

$$\begin{array}{rcl} \langle Q_x^{N^k}(X), X\rangle_{N^k} & = & \langle -2Ric(X), X\rangle_{N^k} + (\sum_{i=1}^{k}\langle \bar{\nabla}_{\alpha_i}\nu, \alpha_i\rangle)\langle \bar{\nabla}_X \nu, X\rangle \\ & = & \langle -2Ric(X), X\rangle_{N^k} + (\sum_{i=1}^{k} h_{ii})h_{XX} \end{array} \tag{3.6}$$

Therefore, via (3.2), (3.5), (3.6) and applying Lemma 2.1 for $h_{ii}(=h_{\alpha_i\alpha_i})$ and $h_{kk}(=h_{\alpha_k\alpha_k}=h_{XX})$, we can get:

$$\begin{array}{rcl} F_{p,x}(X) & = & (p-2)|\widetilde{h}_{kk}|^2+\langle Q_x^{N^k}(X),X\rangle \\ & = & (p-2)(|\bar{h}_{kk}|^2+|h_{kk}|^2)+\langle -2Ric(X),X\rangle+(\sum_{i=1}^{k}h_{ii})h_{kk} \\ & < & (p-2)\{(\frac{k}{p}-1)\frac{(max(a_i))^2}{(min(a_i))^4}+\frac{(max(a_i))^2}{(min(a_i))^4}\}-2k(1-\frac{1}{p})\frac{(max(a_i))^2}{(min(a_i))^4}+k\frac{(max(a_i))^2}{(min(a_i))^4} \\ & = & 0 \end{array} \tag{3.7}$$

Then, N^k is p-SSU. Furthermore, by the fact that every compact p-SSU manifold is $[p]$-connected [8], we know that N^k is $[p]$-connected, i.e. $\pi_1(N^k)=\pi_2(N^k)=\cdots=\pi_{[p]}(N^k)=0$. ☐

Remark 3.1. Applying our results of Theorem 3.1 to unit spheres S^n, we recapture Wei's results [8] to obtain the lower bound of Ricci curvature on unit spheres.

Remark 3.2. Our results of Theorem 3.1 on ellipsoids verify Wei and Wu's work on topological vanishing theorem for minimal sub-manifolds of compact convex hyper-surfaces in Euclidean spaces (cf. [6]).

Remark 3.3. When $E^n=S^n$ Theorem 3.1 is sharp, in the sense that the conclusion does not hold if Ricci curvature assumption is violated. For each $1\le i\le k$ one can find counter-examples for the non-vanishing i-th homotopy group $p_i(N)$ $\big(cf.$[15] also [6]$\big)$.

3.2 Non-existence of Stable Sub-Manifolds on Ellipsoids

In this subsection, by applying Trace Formulas, we discover the geometric properties of stability or instability on E^n.

Lemma 3.2. *(Trace Formulas [16]) Let $\mathcal{R}_k(\tilde{N}^n,G)$ be the group of rectifiable currents of degree k in $\tilde{N}^n$ over the finitely generated abelian group G. If $\mathcal{S}\in\mathcal{R}_k(\tilde{N}^n,G)$ and x is a point at which $\mathcal{S}$ has an approximate tangent space $T_x(\mathcal{S})$ then set*

$$\overrightarrow{\mathcal{S}_x}=e_1\wedge\cdots\wedge e_k$$

where $\{e_1,\cdots,e_k\}$ is an orthonormal basis of $T_x(\mathcal{S})$. Define $\mathcal{I}_{\mathcal{S}}$ to be the function on $\mathbb{R}^{n+m}$ given by

$$\mathcal{I}_{\mathcal{S}}(v)=\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{M}(\varphi_{t*}^{V^T}\,\mathcal{S})$$

where $\mathcal{M}$ is the mass and $\varphi_t^{V^T}$ is the flow (or one parameter pseudogroup) generated by V^T. Then

$$\mathrm{trace}(\mathcal{I}_{\mathcal{S}})=\int_M \mathrm{trace}(Q_{\vec{\mathcal{S}}_x})\,d||\mathcal{S}||(x)$$

(where $\|\mathcal{S}\|$ is a variational measure associated with $\mathcal{S}$) and if $\mathcal{S}$ is stable then $\mathrm{trace}(\mathcal{I}_{\mathcal{S}})\ge 0$. If $\{e_1,\cdots,e_{n+m}\}$ is an orthonormal basis of R^{n+m} such that $\{e_1,\cdots,e_n\}$ is an orthonormal basis of $T\tilde{N}^n_x,\{e_{n+1},\cdots,e_{n+m}\}$ is an orthonormal basis of the normal space of $T\tilde{N}^n_x$ in R^{n+m}, and $\xi=e_1\wedge\cdots\wedge e_k$, then

$$\begin{array}{rcl} trace(Q_\xi) & = & \displaystyle\sum_{i=1}^{k}\sum_{r=k+1}^{n}(2\|h(e_i,e_r)\|^2-h(e_i,e_i)h(e_r,e_r)) \\ & = & \displaystyle\sum_{i=1}^{k}\sum_{r=k+1}^{n}(2|h_{e_ie_r}|^2-h_{e_ie_i}h_{e_re_r}). \end{array}$$

And if for some k, $trace(Q_\xi)<0$ holds for all $\xi=e_1\wedge\cdots\wedge e_k$ tangent to $\tilde{N}^n$, then there are no stable currents of degree k in $\mathcal{R}_k(\tilde{N},G)$ for any finitely generated abelian group G. In particular there are no closed stable minimal sub-manifolds of dimension k or $n-k$ in $\tilde{N}^n$.

Theorem 3.3. *(Non-existence of Stable Sub-Manifolds of Ellipsoids) An ellipsoid E^n has no closed stable k-dimensional sub-manifolds if $\mu^4 \leq \frac{n-k+2}{2}$. In particular, there are no closed stable k-dimensional sub-manifolds in a unit sphere S^n for any $1 \leq k < n$.*

Proof. Let N^k be a k-dimensional sub-manifold of E^n. Suppose $\{e_1, \cdots, e_k\}$ is an orthonormal basis in TN^k_x and $\{e_1, \cdots, e_n\}$ is an orthonormal basis in TE^n_x. Then $\{e_1, \cdots, e_n, \nu(= \frac{Z}{\|Z\|})\}$ is an orthonormal basis in R^{n+1} where Z and ν are defined in Lemma 2.1. For any fixed i $(1 \leq i \leq k)$, we have

$$\begin{aligned} \frac{1}{\|Z\|}\bar{\nabla}_{e_i}Z &= \sum_{l=1}^n \langle \frac{1}{\|Z\|}\bar{\nabla}_{e_i}Z, e_l\rangle e_l + \langle \frac{1}{\|Z\|}\bar{\nabla}_{e_i}Z, \nu\rangle\nu \\ &= \sum_{l=1}^n \langle \bar{\nabla}_{e_i}\nu - e_i(\frac{1}{\|Z\|})Z, e_l\rangle e_l + \langle \frac{1}{\|Z\|}\bar{\nabla}_{e_i}Z, \nu\rangle\nu \\ &= \sum_{l=1}^n h_{e_ie_l}e_l + \langle \frac{1}{\|Z\|}\bar{\nabla}_{e_i}Z, \nu\rangle\nu \\ &= \sum_{j=1}^k h_{e_ie_j}e_j + \sum_{r=k+1}^n h_{e_ie_r}e_r + \langle \frac{1}{\|Z\|}\bar{\nabla}_{e_i}Z, \nu\rangle\nu. \end{aligned} \tag{3.8}$$

By a direct calculation on the norm of $\frac{1}{\|Z\|}\bar{\nabla}_{e_i}Z$, we have the following inequality

$$\sum_{r=k+1}^n |h_{e_ie_r}|^2| < |\frac{1}{\|Z\|}\bar{\nabla}_{e_i}Z|^2 - \sum_{j=1}^k |h_{e_ie_j}|^2. \tag{3.9}$$

Applying Lemma 2.1 for $h_{XX} = h_{e_ie_i} = h_{e_re_r}$, $\|\bar{\nabla}_{e_i}Z\|$ and $\|Z\|$, Lemma 3.2, and (3.9) we get

$$\begin{aligned} trace(Q_\xi) &= 2\sum_{i=1}^k \left(\sum_{r=k+1}^n |h_{e_ie_r}|^2\right) - \sum_{i=1}^k\sum_{r=k+1}^n h_{e_ie_i}h_{e_re_r} \\ &< 2\sum_{i=1}^k \left(\frac{1}{\|Z\|^2}\frac{1}{(\min(a_i))^4} - \sum_{j=1}^k |h_{e_ie_j}|^2\right) - \frac{1}{\|Z\|^2}\frac{1}{(\max(a_i)))^4}k(n-k) \\ &= \frac{1}{\|Z\|^2}\left(\frac{2k}{(\min(a_i))^4} - \frac{k(n-k)}{(\max(a_i))^4}\right) - 2\sum_{i,j=1}^k |h_{e_ie_j}|^2 \\ &= \frac{1}{\|Z\|^2}\left(\frac{2k}{(\min(a_i))^4} - \frac{k(n-k)}{(\max(a_i))^4}\right) - 2\sum_{i=1}^k |h_{e_ie_i}|^2 - 2\sum_{i,j=1,\, i\neq j}^k |h_{e_ie_j}|^2 \\ &< \frac{1}{\|Z\|^2}\left(\frac{2k}{(\min(a_i))^4} - \frac{k(n-k)}{(\max(a_i))^4}\right) - 2k\frac{1}{\|Z\|^2}\frac{1}{(\max(a_i))^4} \\ &= \frac{k}{\|Z\|^2}\left\{\frac{2}{(\min(a_i))^4} - \frac{n-k+2}{(\max(a_i))^4}\right\} \end{aligned} \tag{3.10}$$

Hence $trace(Q_\xi) < 0$ if $\mu^4 \leq \frac{n-k+2}{2}$. In particular, $trace(Q_\xi) < 0$ on S^n for any $1 \leq k < n$. ☐

Remark 3.4. Our results of Theorem 3.3 extend Lawson and Simons' result [9] from spheres to ellipsoids.

3.3 Application of Ellipsoids as p-Super-Strongly Unstable (p-SSU) Manifolds

In this subsection, we discuss the criteria for ellipsoids as p-SSU manifolds. Furthermore, we calculate p-SSU index ω_p when ellipsoids are p-SSU manifolds.

Theorem 3.4. *(Criteria for Ellipsoids as p-SSU Manifolds) An ellipsoid E^n is p-SSU if $\mu^3 < \frac{n}{p}$. In particular, a unit n-sphere S^n is p-SSU for $p < n$.*

Proof. A complete hyper-surface in R^{n+1} is p-SSU if and only if its principal curvatures satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < \frac{H}{p},$$

where $H = \lambda_1 + \cdots + \lambda_n$ (cf.Theorem 3.3 [1]). We only need to prove that any principal curvature is less than one p-th of the sum of all principal curvatures. In fact, via Lemma 2.2 for λ_j on E^n and the assumption of $\mu^3 < \frac{n}{p}$, we have

$$\lambda_j \leq \frac{max(a_i)}{(min(a_i))^2} < \frac{n}{p}\frac{min(a_i)}{(max(a_i))^2} \leq \frac{H}{p}, \quad \forall j = 1, \cdots, n.$$

So we know that E^n is p-SSU when $\mu^3 < \frac{n}{p}$. ☐

Theorem 3.5. *(p-SSU Index ω_p on p-SSU Ellipsoids) Suppose an ellipsoid E^n is a p-SSU manifold, then:*

$$\omega_p \geq \frac{\frac{n}{\mu^6} - p}{n+p-2}.$$

In particular, $\omega_p = \frac{n-p}{n+p-2}$ on a unit sphere S^n.

Proof.

$$\omega_p = inf_{x \in E^n} \frac{\phi_p(x)}{(n+p-2)S(x)}$$

where by applying Lemma 2.2 we know $S(x) = sup_{x \in E^n} Sec^{E^n}(x) = \frac{(max(a_i))^2}{(min(a_i))^4}$ and $\phi_p(x) = inf_{\|X\|=1}(-F_{p,x}(X)), E^n \hookrightarrow R^{n+1}$. Note that, for any point $x \in E^n$, we can choose an orthonormal frame $\{\alpha_1, \cdots, \alpha_n\}$ on E^n. Observing that $\{\alpha_1, \alpha_2, \cdots, \alpha_n, \nu(= \frac{Z}{\|Z\|})\}$ is an orthonormal basis in R^{n+1} where $\nu = \frac{Z}{\|Z\|}$ is defined in Lemma 2.1, we have

$$-F_{p,x}(X) = -(p-2)|h_{XX}|^2 + h_{XX}(\sum_{i=1}^{n} h_{\alpha_i\alpha_i}) - 2\sum_{j=1}^{n} |h_{X\alpha_j}|^2$$

and

$$\begin{array}{rcl} \sum_{j=1}^n |h_{X\alpha_j}|^2 & = & \sum_{j=1}^n |\langle \bar{\nabla}_X \nu, \alpha_j \rangle|^2 \\ & = & \sum_{j=1}^n |\langle \frac{1}{\|Z\|}\bar{\nabla}_X Z + X(\frac{1}{\|Z\|})Z, \alpha_j \rangle|^2 \\ & = & \sum_{j=1}^n |\langle \frac{1}{\|Z\|}\bar{\nabla}_X Z, \alpha_j \rangle|^2 \\ & < & \frac{1}{\|Z\|^2}\|\bar{\nabla}_X Z\|^2 \\ & \leq & \frac{(\max(a_i))^2}{(\min(a_i))^4} \end{array} \tag{3.11}$$

where in the last step we apply Lemma 2.1 for $\|\bar{\nabla}_X Z\|$ and $\|Z\|$.
By applying Lemma 2.1 for $\sum_{i=1}^n h_{\alpha_i\alpha_i}$ and plugging in (3.11), we have

$$\begin{array}{rcl} -F_{p,x}(X) & = & -(p-2)|h_{XX}|^2 + h_{XX}(\sum_{i=1}^n h_{\alpha_i\alpha_i}) - 2\sum_{j=1}^n |h_{X\alpha_j}|^2 \\ & \geq & -(p-2)\frac{(max(a_i))^2}{(min(a_i))^4} + n\frac{(min(a_i))^2}{(max(a_i))^4} - 2\frac{(max(a_i))^2}{(min(a_i))^4} \\ & = & -p\frac{(max(a_i))^2}{(min(a_i))^4} + n\frac{(min(a_i))^2}{(max(a_i))^4}. \end{array} \tag{3.12}$$

Therefore, we have

$$\begin{array}{rcl} \omega_p & \geq & \frac{-p\frac{(max(a_i))^2}{(min(a_i))^4} + n\frac{(min(a_i))^2}{(max(a_i))^4}}{(n+p-2)\frac{(max(a_i))^2}{(min(a_i))^4}} \\ & = & \frac{n\frac{1}{\mu^6} - p}{n+p-2} \end{array} \tag{3.13}$$

In particular, $\omega_p = \frac{n-p}{n+p-2}$ on a unit sphere S^n. □

Remark 3.5. Wei and Yau [1] calculated p-SSU index ω_p on p-SSU spheres. We extend their results of p-SSU index ω_p to p-SSU ellipsoids.

3.4 Application of Ellipsoids as Yang-Mills Instable Manifolds

In this subsection, we find the criteria for ellipsoids as Yang-Mills instable manifolds.

Theorem 3.6. *(Criteria for Ellipsoids as Yang-Mills Instable Manifolds) An ellipsoid E^n is Yang-Mills instable if $\mu^3 < \frac{n-2}{2}$. In particular, a unit n-sphere S^n is Yang-Mills instable for $n > 4$.*

Proof. M^n is Yang-Mills instable if $\lambda_i + \lambda_j < \sum_{s=1,s\neq i,j}^{n} \lambda_s$ where $\{\lambda_s\}$ is a family of principal curvatures on M^n (cf. Theorem (5.3) [12]). On an ellipsoid E^n, via Lemma 2.2 and the assumption of $\mu^3 < \frac{n-2}{2}$, we have:

$$\lambda_i + \lambda_j \leq 2\frac{\max(a_i)}{(\min(a_i))^2} < (n-2)\frac{\min(a_i)}{(\max(a_i))^2} \leq \sum_{s=1,s\neq i,j}^{n} \lambda_s$$

for all pair $i \neq j$. Hence, we know that E^n is Yang-Mills instable if $\mu^3 < \frac{n-2}{2}$. □

Remark 3.6. The study on n-spheres as Yang-Mills instable manifolds where $n \geq 5$ was due to James Simons, announced in Japan in September of 1977 (cf. [10],[11]). Here we obtain the same results for spheres with a different approach and extend the results from spheres to ellipsoids. The study on the product of spheres as Yang-Mills instable manifolds can be found in [17], generalizing Simons' results on shperes.

Remark 3.7. Kobayashi, Ohnita and Takeuchi (cf. Example (2) [12]) studied the Yang-Mills instable manifolds for ellipsoids with only one variable $E^n = E^n_a$, (i.e. $E^n_a = \{(x_1, \cdots, x_{n+1}) \in R^{n+1}; \frac{x_1^2}{a^2} + x_2^2 + \cdots + x_{n+1}^2 = 1, a > 0\}$) when $1 \leq a(= \mu) < \sqrt[3]{\frac{n-2}{2}}$. Our results of Theorem 3.6 recapture Kobayashi, Ohnita and Takeuchi's results in a different approach.

4 Conclusion

In the study of a convex hyper-surface in a Riemannian manifold, most mathematicians began with a sphere as a starting point and have obtained numerous results on a sphere. In our paper, we study an ellipsoid while a sphere is regarded as a special case of an ellipsoid. We discover the geometric, topological and physical properties reflected by the curvature properties on an ellipsoid. Just as the curvature values of a sphere are determined by its radius, the curvature values of an ellipsoid are determined by its longest axis and its shortest axis. The ratio of its longest axis to its shortest axis is a critical index to characterize geometric, topological and physical behaviors on an ellipsoid. The ratio is also an indicator of an ellipsoid as different types of manifolds.

a In the section (2), we demonstrate the way to compute the second fundamental form (cf. Lemma 2.1), which plays an important role to estimate curvature values. We extend the calculation method from spheres to ellipsoids. Curvatures estimated by our method match Shen and Pan's results with a different approach.

b In the subsection (3.1), we obtain the topological vanishing properties for minimal sub-manifolds of ellipsoids (cf. Theorem 3.1). Our results recapture Wei's results on unit spheres.

c In the subsection (3.2), we obtain the non-existence of stable sub-manifolds on ellipsoids (cf. Theorem 3.3), which extends Lawson and Simons' result about non-existence of stable sub-manifolds of spheres.

d In the subsection (3.3), we find the criteria for ellipsoids as p-SSU manifolds (cf. Theorem 3.4) and calculate p-SSU index ω_p for p-SSU Ellipsoids (cf. Theorem 3.5). Our results recapture Wei's results on spheres.

e In the subsection (3.4), we find the criteria for ellipsoids as Yang-Mills Instable manifolds (cf. Theorem 3.6). We recapture Simons' results on spheres and Kobayashi, Ohnita, and Takeuchi's results on spheres and on ellipsoids with only one variable.

On a convex hyper-surface, how can we link the curvatures properties to the geometric, topological, and physical properties? Answering this question will be very challenging. Methods and research findings from the point of view on ellipsoids in this paper provide a clue to the future research on convex hyper-surfaces.

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Competing Interests

The authors declare that no competing interests exist.

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