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A Result in the Theory of Twin Primes

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Abstract

This article determines a lower bound for the number of twin primes p and $p+2$ up to a large number x.

1 Introduction and the Main Result

The distribution of the sequence of twin primes $(3,5),(5,7),\ldots,(p,p+2),\ldots$, and the distributions of other sequences of prime pairs, and prime k-tuples are long standing topics of research in number theory. Discussions on the prime pairs problems appear in [9], [7], and many other references in the vast literature on this subject.

A century ago the expected quantitative forms of some of these problems were written down, and some numerical data were computed to back the claims. The predicted asymptotic form of the twin primes problem is given below.

Conjecture 1.1. ([4, Conjecture B, p. 43]) *There are infinitely many twin prime pairs. If $\pi_2(x)$ is the number of pairs less than x, then*

$$\pi_2(x)=2C_2\int_2^x \frac{1}{(\log t)^2}dt+O\left(\frac{x}{(\log x)^3}\right),$$

where $C_2 >$ is a constant defined by

$$C_2=\prod_{p\geq 3}\left(1-\frac{1}{(p-1)^2}\right)=0.660161860589840764676693891535 2060\ldots.$$

The modern theory of the distribution of twin primes are discussed in [8], [5] and by other authors.

Let $\Lambda(n)$ denotes the weighted prime power indicator function, (von Mangoldt function),

$$\Lambda(n)=\begin{cases}\log n & \text{if } n=p^k,\\ 0 & \text{if } n\neq p^k.\end{cases} \tag{1.1}$$

The conjecture is equivalent to the weighted sum

$$\sum_{2\leq n\leq x}\Lambda(n-1)\Lambda(n+1)=2C_2x+o(x). \tag{1.2}$$

This note proposes a partial and weaker asymptotic formula.

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Theorem 1.1. *If $x \geq 1$ is a large real number, then*

$$\sum_{2 \leq n \leq x} \Lambda(n-1)\Lambda(n+1) \gg \frac{x \log \log x}{(\log x)(\log \log \log x)}.$$

A short outline of this article is provided here. Theorem 1.1 is a simple corollary of Theorem 2.1 in Section 2. The basic materials required to prove the fundamental result in Theorem 2.1 are developed and proved in Section 5 to Section 4. Section 5 deals with several forms of the finite sum $\sum_{m,n \leq x} 1/[m,n]$, which are of independent interest in number theory. The proof Theorem 1.1 of appears in Section 6.

2 Fundamental Result

The classical weighted twin primes counting function has the form

$$\sum_{1 \leq n \leq x} \Lambda(n)\Lambda(n+2) = \sum_{2 \leq n \leq x+1} \Lambda(n-1)\Lambda(n+1). \tag{2.1}$$

The derivation of a lower bound for the number of twin primes is based on a new weighted twin primes counting function

$$\sum_{2 \leq n \leq x} \mu^2(n+1)\mu^2(n+1)\Lambda^2(n-1)\Lambda(n+1). \tag{2.2}$$

The extra "weight" factor $w(n) = \mu^2(n+1)\mu^2(n+1)\Lambda(n-1)$ provides very flexible and effective control over the error term at the cost of a smaller main term, by a factor of approximately $\log x$.

Theorem 2.1. *If $x \geq 1$ is a large real number, then*

$$\sum_{2 \leq n \leq x} \mu^2(n+1)\mu^2(n+1)\Lambda^2(n-1)\Lambda(n+1) \gg \frac{x \log \log x}{\log \log \log x}.$$

Proof. Substitute the identities $\mu^2(n) = \sum_{d^2 \mid n} \mu(d)$, see [1, p. 47], and $\Lambda(n) = -\sum_{d \mid n} \mu(d) \log d$, see [1, Theorem 2.11], respectively, then reverse the order of summations.

$$\begin{aligned}
\psi_T(x) &= \sum_{2 \leq n \leq x} \mu^2(n+1)\mu^2(n+1)\Lambda^2(n-1)\Lambda(n+1) \\
&= \sum_{n \leq x} \Lambda(n+1) \sum_{d_1^2 \mid n+1} \mu(d_1) \sum_{d_2^2 \mid n+1} \mu(d_2) \\
&\qquad \times \sum_{d_3 \mid n-1} \mu(d_3) \log d_3 \sum_{d_4 \mid n-1} \mu(d_4) \log d_4 \\
&= \sum_{d_1^2,\, d_2^2,\, d_3,\, d_4 \leq x} \mu(d_1)\mu(d_2)\mu(d_3)\log(d_3)\mu(d_4)\log(d_4) \\
&\qquad \times \sum_{\substack{n \leq x \\ d_1^2 \mid n+1,\, d_2^2 \mid n+1, \\ d_3 \mid n-1,\, d_4 \mid n-1}} \Lambda(n+1).
\end{aligned} \tag{2.3}$$

Let $x_1 = (\log x)^{c_0}$, with $c_0 > 6$ constant, and partition the quintic finite sum.

$$\begin{aligned}\psi_T(x) &= \sum_{d_1^2,\, d_2^2,\, d_3,\, d_4 \le x_1} \mu(d_1)\mu(d_2)\mu(d_3)\log(d_3)\mu(d_4)\log(d_4) \\ &\qquad\qquad \times \sum_{\substack{n \le x \\ d_1^2 | n+1,\, d_2^2 | n+1, \\ d_3 | n-1,\, d_4 | n-1}} \Lambda(n+1) \\ &\quad + \sum_{x_1 < d_1^2,\, d_2^2,\, d_3,\, d_4 \le x} \mu(d_1)\mu(d_2)\mu(d_3)\log(d_3)\mu(d_4)\log(d_4) \\ &\qquad\qquad \times \sum_{\substack{n \le x \\ d_1^2 | n+1,\, d_2^2 | n+1, \\ d_3 | n-1,\, d_4 | n-1}} \Lambda(n+1) \\ &= M(x) + E(x). \end{aligned} \tag{2.4}$$

Summing the main term computed in Lemma 3.1 and the error term computed in Lemma 4.1, yields

$$\begin{aligned}\psi_T(x) &= M(x) + E(x) \\ &\gg \frac{x \log\log x}{\log\log\log x} + O\left(\frac{x}{(\log x)^c}\right) \\ &\gg \frac{x \log\log x}{\log\log\log x}, \end{aligned} \tag{2.5}$$

where $c = c_0 - 6 > 0$ is a constant. ■

The minimal constant $c_0 > 6$ arises on the calculations of the upper bound for the error term.

3 Lower Bound For The Main Term

Lemma 3.1. *If $x \ge 1$ is a large number, and $x_1 = (\log x)^{c_0}$, where $c_0 > 6$, then,*

$$\begin{aligned} M(x) &= \sum_{d_1^2,\, d_2^2,\, d_3,\, d_4 \le x_1} \mu(d_1)\mu(d_2)\mu(d_3)\log(d_3)\mu(d_4)\log(d_4) \\ &\qquad\qquad \times \sum_{\substack{n \le x \\ d_1^2 | n+1,\, d_2^2 | n+1, \\ d_3 | n-1,\, d_4 | n-1}} \Lambda(n+1) \\ &\gg \frac{x \log\log x}{\log\log\log x}. \end{aligned}$$

Proof. Let x be a large number and let $x_1 = (\log x)^{c_0} \le e^{c_1\sqrt{\log x}}$, where $c_0 > 6$ and $c_1 = c_1(c_0) > 0$ are constants. Applying the prime number theorem for prime in arithmetic progression, see [6, Corollary 11.19], yields

$$\begin{aligned}
M(x) &= \sum_{d_1^2,\, d_2^2,\, d_3,\, d_4 \le x_1} \mu(d_1)\mu(d_2)\mu(d_3)\log(d_3)\mu(d_4)\log(d_4) \\
&\qquad \times \sum_{\substack{n\le x \\ d_1^2|n+1,\, d_2^2|n+1, \\ d_3|n-1,\, d_4|n-1}} \Lambda(n+1) \\
&= \sum_{d_1^2,\, d_2^2,\, d_3,\, d_4 \le x_1} \mu(d_1)\mu(d_2)\mu(d_3)\log(d_3)\mu(d_4)\log(d_4) \\
&\qquad \times \left(\frac{x}{\varphi([d_1^2, d_2^2, d_3, d_4])} + O\left(x e^{-c_1\sqrt{\log x}} \right) \right) \\
&= x \sum_{d_1^2,\, d_2^2,\, d_3,\, d_4 \le x_1} \frac{\mu(d_1)\mu(d_2)\mu(d_3)\log(d_3)\mu(d_4)\log(d_4)}{\varphi([d_1^2, d_2^2, d_3, d_4])} \\
&\qquad + O\left(x e^{-c_1\sqrt{\log x}} \sum_{d_1^2,\, d_2^2,\, d_3,\, d_4 \le x_1} (\log d_3)(\log d_4) \right) \\
&= M_0(x) + M_1(x). \qquad (3.1)
\end{aligned}$$

The first subsum $M_0(x)$ is estimated in Lemma 3.2 and the second subsum $M_1(x)$ is estimated in Lemma 3.3. Summing these estimates yields

$$\begin{aligned}
M(x) &= M_0(x) + M_1(x) \\
&\gg \frac{x \log\log x}{\log\log\log x} + O\left(x e^{-c_2\sqrt{\log x}} \right) \\
&\gg \frac{x \log\log x}{\log\log\log x}, \qquad (3.2)
\end{aligned}$$

as claimed. ∎

Lemma 3.2. *Assume that* $d_1^2 \mid n+1,\ d_2^2 \mid n+1,\ d_3 \mid n-1,\ d_4 \mid n-1$. *If* $x \ge 1$ *is a large number, and* $x_1 = (\log x)^{c_0}$, *where* $c_0 > 6$, *then,*

$$\begin{aligned}
M_0(x) &= x \sum_{d_1^2,\, d_2^2,\, d_3,\, d_4 \le x_1} \frac{\mu(d_1)\mu(d_2)\mu(d_3)\log(d_3)\mu(d_4)\log(d_4)}{\varphi([d_1^2, d_2^2, d_3, d_4])} \\
&\gg \frac{x \log\log x}{\log\log\log x}.
\end{aligned}$$

Proof. By Lemma 5.6 the quadruple finite sum

$$\begin{aligned}
F(x) &= \sum_{d_1^2,\, d_2^2,\, d_3,\, d_4 \le x_1} \frac{\mu(d_1)\mu(d_2)\mu(d_3)\log(d_3)\mu(d_4)\log(d_4)}{\varphi([d_1^2, d_2^2, d_3, d_4])} \\
&\gg \frac{\log\log x}{\log\log\log x}. \qquad (3.3)
\end{aligned}$$

Thus, the product $xF(x) \gg (x\log\log x)/(\log\log\log x)$ verifies the claim. ∎

Lemma 3.3. *If $x \geq 1$ is a large number, and $x_1 = (\log x)^{c_0}$, where $c_0 > 6$, then,*

$$\begin{aligned} M_1(x) &= O\left(xe^{-c_1\sqrt{\log x}} \sum_{d_1^2,\, d_2^2,\, d_3,\, d_4 \leq x_1} \log(d_3)\log(d_4)\right) \\ &= O\left(xe^{-c_2\sqrt{\log x}}\right), \end{aligned}$$

where $c_1 \geq c_2 > 0$ is a constant.

Proof. As previously stated $x_1 = (\log x)^{c_0} \leq e^{c_1\sqrt{\log x}}$, where $c_0 > 6$ and $c_1 = c_1(c_0) > 0$ are constants. Now, an estimate of the quadruple finite sum yields

$$\begin{aligned} M_1(x) &= O\left(xe^{-c_1\sqrt{\log x}} \sum_{d_1^2,\, d_2^2,\, d_3,\, d_4 \leq x_1} (\log d_3)(\log d_4)\right) \\ &= O\left(xe^{-c_1\sqrt{\log x}}\left((x_1 \log x_1)^2 \cdot (\sqrt{x_1})^2\right)\right) \\ &= O\left(xe^{-c_1\sqrt{\log x}}(\log x)^{3c_0+2}\right) \\ &= O\left(xe^{-c_2\sqrt{\log x}}\right), \end{aligned} \tag{3.4}$$

where $c_1 \geq c_2 > 0$ is a constant. ■

4 Upper Bound For The Error Term

Lemma 4.1. *Assume that $d_1^2 \mid +1$, $d_2^2 \mid n+1$, $d_3 \mid n-1$, $d_4 \mid n-1$. If $x \geq 1$ is a large number, and $x_1 = (\log x)^{c_0}$, with $c_0 > 6$, then,*

$$\begin{aligned} E(x) &= \sum_{x_1 < d_1^2,\, d_2^2,\, d_3,\, d_4 \leq x} \mu(d_1)\mu(d_2)\mu(d_3)\log(d_3)\mu(d_4)\log(d_4) \\ &\qquad\qquad \times \sum_{\substack{n \leq x \\ d_1^2 \mid n+1,\, d_2^2 \mid n+1, \\ d_3 \mid n-1,\, d_4 \mid n-1}} \Lambda(n+1) \\ &= O\left(\frac{x}{(\log x)^c}\right), \end{aligned}$$

where $c = c_0 - 6 > 0$ is a constant.

Proof. Take the absolute value and estimate the inner finite sums in succession.

$$\begin{aligned}
|E(x)| &\leq \sum_{x_1<d_1^2,\, d_2^2,\, d_3,\, d_4 \leq x} \log(d_3)\log(d_4) \sum_{\substack{n\leq x \\ d_1^2|n+1,\, d_2^2|n+1, \\ d_3|n-1,\, d_4|n-1}} \Lambda(n+1) \\
&\leq x(\log x) \sum_{x_1<d_1^2,\, d_2^2,\, d_3,\, d_4 \leq x} \frac{\log(d_3)\log(d_4)}{[d_1^2, d_2^2, d_3, d_4]} \\
&\leq x(\log x)^3 \sum_{x_1<d_1^2,\, d_2^2,\, d_3,\, d_4 \leq x} \frac{1}{[d_1^2, d_2^2, d_3, d_4]} \\
&\leq x(\log x)^3 \sum_{x_1<d_1^2,\, d_2^2 \leq x} \frac{1}{[d_1^2, d_2^2]} \sum_{x_1<d_3,\, d_4 \leq x} \frac{1}{[d_3, d_4]}. \qquad (4.1)
\end{aligned}$$

The factorization of the finite sum follows from $[d_1^2, d_2^2, d_3, d_4] = [d_1^2, d_2^2][d_3, d_4]$ since

$$d_1^2 \mid n+1,\ d_2^2 \mid n+1,\ d_3 \mid n-1,\ d_4 \mid n-1 \qquad (4.2)$$

and $(d_1^2 d_2^2, d_3 d_4) = 1$. Applying Lemma 5.4 to the first factor and Lemma 5.5 to the other factor lead to

$$\begin{aligned}
|E(x)| &\ll x(\log x)^3 \left((\log x)^3\right) \left(\frac{1}{(\log x)^{c_0}}\right) \\
&\ll \frac{x(\log x)^6}{(\log x)^{c_0}} \\
&\ll \frac{x}{(\log x)^c}, \qquad (4.3)
\end{aligned}$$

where $c = c_0 - 6 > 0$ is a constant. In particular, $c_0 > 6$ implies a nontrivial error term. ∎

5 Foundational Results

The expressions $(m, n) = \gcd(m, n)$ and $[m, n] = \operatorname{lcm}(m, n)$ denote the greatest common divisor and the lowest common multiple respectively. The totient function is defined by

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right), \qquad (5.1)$$

and the Mobius function is defined by

$$\mu(n) = \begin{cases} (-1)^w & \text{if } n = p_1 p_2 \cdots p_w, \\ 0 & \text{if } n \neq p_1 p_2 \cdots p_w. \end{cases} \qquad (5.2)$$

The nonnegativity of the finite sum

$$\sum_{m,n\leq x} \frac{\mu(m)\mu(n)}{[m, n]} > 0 \qquad (5.3)$$

and the convergence of the associated series as $x \to \infty$ is the subject a study in [2], and in sieve theory. Similar techniques are used here to derive several estimates and verify the nonnegativity of some related finite sums. These finite sums arise in the analysis of the main term and error term of Theorem 2.1.

Lemma 5.1. *If* $m, n \geq 1$ *are a pair of integers, then,*

$$\gcd(m,n) = \sum_{d|(m,n)} \varphi(d).$$

Proof. The claim follows from the additive to multiplicative relation

$$\begin{aligned} \sum_{d|(m,n)} \varphi(d) &= \prod_{p^v||(m,n)} \left(1 + \varphi(p) + \varphi(p^2) + \cdots + \varphi(p^v)\right) \\ &= \prod_{p^v||(m,n)} p^v \\ &= \gcd(m,n), \end{aligned} \tag{5.4}$$

where $p^v \mid\mid (m,n)$ is the maximal prime power divisor. ■

Lemma 5.2. *If* $m, n \geq 1$ *are a pair of integers, then,*

$$\frac{1}{[m,n]} = \frac{1}{mn} \sum_{d|(m,n)} \varphi(d).$$

Proof. Use Lemma 5.1, to transform the denominator as follows.

$$\frac{1}{[m,n]} = \frac{\gcd(m,n)}{mn} = \frac{1}{mn} \sum_{d|(m,n)} \varphi(d). \tag{5.5}$$

■

Lemma 5.3. *If* $m, n \geq 1$ *are a pair of integers, then,*

$$\frac{1}{\varphi([m,n])} = \frac{1}{\varphi(mn)} \sum_{d|(m,n)} \varphi(d).$$

Proof. Substitute the identity (5.1) to transform the denominator as follows.

$$\begin{aligned} \frac{1}{\varphi([m,n])} &= \frac{1}{[m,n]} \prod_{p|[m,n]} \left(1 - \frac{1}{p}\right)^{-1} \\ &= \frac{\gcd(m,n)}{mn} \prod_{p|mn} \left(1 - \frac{1}{p}\right)^{-1} \\ &= \frac{1}{\varphi(mn)} \sum_{d|(m,n)} \varphi(d). \end{aligned} \tag{5.6}$$

The reverse the identity (5.1) is used on the penultimate line of equation (5.6), and Lemma 5.1 is used to obtain the last line. ■

Lemma 5.4. *If* $x \geq 1$ *is a large number, then,*

$$\sum_{m,\, n \leq x} \frac{1}{[m,n]} \ll (\log x)^3$$

as $x \to \infty$.

Proof. Use Lemma 5.2 and switch the order of summation to obtain

$$\begin{aligned}\sum_{m,\,n\le x}\frac{1}{[m,n]} &= \sum_{m,n\le x}\frac{1}{mn}\sum_{d|(m,n)}\varphi(d)\\ &= \sum_{d\le x}\varphi(d)\sum_{\substack{m,n\le x\\ d|(m,n)}}\frac{1}{mn}. \end{aligned} \tag{5.7}$$

Estimating the double finite sum and simplifying yield the upper bound

$$\begin{aligned}\sum_{d\le x}\varphi(d)\sum_{\substack{m,n\le x\\ d|(m,n)}}\frac{1}{mn} &= \sum_{d\le x}\frac{\varphi(d)}{d^2}\sum_{m,n\le x/d}\frac{1}{mn}\\ &\ll (\log x)^2\sum_{d\le x}\frac{\varphi(d)}{d^2}\\ &\ll (\log x)^3, \end{aligned} \tag{5.8}$$

since $\sum_{d\le x}\varphi(d)d^{-2}\ll\log x$. ∎

Lemma 5.5. *If* $x\ge 1$ *is a large number, and* $x_1=(\log x)^{c_0}$, *with* $c_0>0$ *constant, then,*

$$\sum_{x_1<m^2,\,n^2\le x}\frac{1}{[m^2,n^2]}=O\left(\frac{1}{(\log x)^{c_0}}\right).$$

Proof. Apply Lemma 5.2 to rewrite the finite sum as

$$\begin{aligned}\sum_{x_1<m^2,\,n^2\le x}\frac{1}{[m^2,n^2]} &= \sum_{x_1<m^2,n^2\le x}\frac{1}{m^2n^2}\sum_{d|(m^2,n^2)}\varphi(d)\\ &= \sum_{d\le x}\varphi(d)\sum_{\substack{x_1<m^2,n^2\le x\\ d|(m^2,n^2)}}\frac{1}{m^2n^2}\\ &\ll \sum_{d\le x}\frac{\varphi(d)}{d^4}\sum_{x_1/d<m^2\le x/d}\frac{1}{m^2}\sum_{x_1/d<n^2\le x/d}\frac{1}{n^2}. \end{aligned} \tag{5.9}$$

Now estimating the inner sums yields

$$\begin{aligned}\sum_{x_1<m^2,\,n^2\le x}\frac{1}{[m^2,n^2]} &\ll \sum_{d\le x}\frac{\varphi(d)}{d^4}\left(\sqrt{\frac{d}{x_1}}\right)^2\\ &\ll \frac{1}{(\log x)^{c_0}}\sum_{d\le x}\frac{\varphi(d)}{d^3}\\ &\ll \frac{1}{(\log x)^{c_0}}, \end{aligned} \tag{5.10}$$

since $\sum_{d\le x}\varphi(d)d^{-3}\ll 1$ and $x_1=(\log x)^{c_0}$. ∎

Lemma 5.6. *Assume that* $d_1^2 \mid n+1,\ d_2^2 \mid n+1,\ d_3 \mid n-1,\ d_4 \mid n-1$. *If* $x \geq 1$ *is a large number, and* $x_1 = (\log x)^{c_0}$, *with* $c_0 > 0$, *then,*

$$\sum_{d_1^2,\, d_2^2,\, d_3,\, d_4 \leq x_1} \frac{\mu(d_1)\mu(d_2)\mu(d_3)\log(d_3)\mu(d_4)\log(d_4)}{\varphi([d_1^2, d_2^2, d_3, d_4])} \gg \frac{\log\log x}{\log\log\log x}.$$

Proof. The hypothesis $d_1^2 \mid n+1,\ d_2^2 \mid n+1,\ d_3 \mid n-1,\ d_4 \mid n-1$ implies that $\gcd(d_1^2 d_2^2, d_3 d_4) = 1$. This condition implies the decomposition

$$\begin{aligned} F(x) &= \sum_{d_1^2,\, d_2^2,\, d_3,\, d_4 \leq x_1} \frac{\mu(d_1)\mu(d_2)\mu(d_3)\log(d_3)\mu(d_4)\log(d_4)}{\varphi([d_1^2, d_2^2, d_3, d_4])} \\ &= \sum_{d_1^2,\, d_2^2 \leq x_1} \frac{\mu(d_1)\mu(d_2)}{\varphi([d_1^2, d_2^2])} \sum_{d_3,\, d_4 \leq x_1} \frac{\mu(d_3)\mu(d_4)\log(d_3)\log(d_4)}{\varphi([d_3, d_4])}. \end{aligned} \tag{5.11}$$

By Lemma 5.7 and Lemma 5.8, the product

$$\begin{aligned} F(x) &\gg \frac{\log x_1}{\log\log x_1} \\ &\gg \frac{\log\log x}{\log\log\log x} \end{aligned} \tag{5.12}$$

since $x_1 = (\log x)^{c_0}$, with $c_0 > 0$. ■

Lemma 5.7. *If* $x \geq 1$ *is a large number, then,*

$$\sum_{m^2,\, n^2 \leq x} \frac{\mu(m)\mu(n)}{\varphi([m^2, n^2])} > 0$$

converges to a constant $a_0 > 0$ *as* $x \to \infty$.

Proof. Applying Lemma 5.3 and switching the order of summation yield

$$\begin{aligned} A(x) &= \sum_{m,\, n \leq x} \frac{\mu(m)\mu(n)}{\varphi([m^2, n^2])} \\ &= \sum_{m,n \leq x} \frac{\mu(m)\mu(n)}{\varphi(m^2 n^2)} \sum_{d \mid (m^2, n^2)} \varphi(d) \\ &= \sum_{d \leq x} \varphi(d) \sum_{\substack{m,n \leq x \\ d \mid (m^2, n^2)}} \frac{\mu(m)\mu(n)}{\varphi(m^2 n^2)}. \end{aligned} \tag{5.13}$$

Replacing the change of variables $m = dr$ and $n = ds$, where $r, s \geq 1$ are squarefree integers, and $\gcd(r, s) = 1$. Simplifying yield, see [6, p. 83], and [2, p. 55] for similar calculations,

$$\begin{aligned} A(x) &= \sum_{d \leq x} \frac{\varphi(d)\mu^2(d)}{\varphi(d^4)} \sum_{\substack{r,s \leq x/d \\ (d, r^2 s^2) = 1 \\ \gcd(r,s) = 1}} \frac{\mu(r)\mu(s)}{\varphi(r^2 s^2)} \\ &= \sum_{d \leq x} \frac{\mu^2(d)}{d^3} \left(\sum_{\substack{r \leq x/d \\ (d,r) = 1}} \frac{\mu(r)}{r^2} \right)^2 \\ &> 0. \end{aligned} \tag{5.14}$$

Clearly, this proves that the finite sum is nonnegative, and converges to a constant $a_0 > 0$ as $x \to \infty$. ∎

Lemma 5.8. *If $x \geq 1$ is a large number, then,*

$$\sum_{m,\, n \leq x} \frac{\mu(m)\mu(n) \log m \log n}{\varphi([m,n])} \gg \frac{\log x}{\log \log x}$$

is an increasing nonnegative function as $x \to \infty$, and bounded by $\log x$.

Proof. By Lemma 5.3, the finite sum transforms as

$$\begin{aligned} B(x) &= \sum_{m,\, n \leq x} \frac{\mu(m)\mu(n) \log m \log n}{\varphi([m,n])} \\ &= \sum_{m,n \leq x} \frac{\mu(m)\mu(n) \log m \log n}{\varphi(mn)} \sum_{d \mid (m,n)} \varphi(d) \\ &= \sum_{d \leq x} \varphi(d) \sum_{\substack{m,n \leq x \\ d \mid (m,n)}} \frac{\mu(m)\mu(n) \log m \log n}{\varphi(mn)}. \end{aligned} \tag{5.15}$$

Replacing the change of variables $m = dr$ and $n = ds$, where $r, s \geq 1$ are squarefree integers, and $\gcd(r,s) = 1$. Simplifying yield,

$$\begin{aligned} B(x) &= \sum_{d \leq x} \frac{\varphi(d)}{\varphi(d^2)} \sum_{\substack{r,s \leq x/d \\ (d,rs)=1 \\ \gcd(r,s)=1}} \frac{\mu^2(d)\mu(r)\mu(s) \log dr \log ds}{\varphi(rs)} \\ &= \sum_{d \leq x} \frac{\mu^2(d)}{d} \left(\sum_{\substack{r \leq x/d \\ (d,r)=1}} \frac{\mu(r) \log dr}{\varphi(r)} \right)^2 \\ &> 0. \end{aligned} \tag{5.16}$$

The first factor has the asymptotic

$$\frac{\log x}{\log \log x} \ll \sum_{d \leq x} \frac{\mu^2(d)}{d} = \frac{6}{\pi^2} \log x + O\left(\frac{1}{\log x} \right), \tag{5.17}$$

which is monotonically increasing as $x \to \infty$, and the second factor

$$\sum_{\substack{n \leq x/d \\ (d,n)=1}} \frac{\mu(n) \log dn}{\varphi(n)} \approx c_d \sum_{\substack{n \leq x \\ \text{odd } n}} \frac{\mu(n) \log n}{\varphi(n)}, \tag{5.18}$$

where $c_d \neq 0$ is a constant depending on $d \geq 1$, converges to a constant $a_1 \neq 0$ as $x \to \infty$, see (5.20) below. Hence, the product $B(x)$ satisfies the inequality

$$\frac{\log x}{\log \log x} \ll \sum_{m,\, n \leq x} \frac{\mu(m)\mu(n) \log m \log n}{\varphi([m,n])} \ll \log x, \tag{5.19}$$

which is monotonically increasing as $x \to \infty$. ∎

The expression (5.18) is nearly the same as the finite sum

$$-\sum_{n\leq x,\ \text{odd}\ n}\frac{\mu(n)\log n}{\varphi(n)}=\mathfrak{S}(2)+O\left(e^{-c_3\sqrt{\log x}}\right), \tag{5.20}$$

where $\mathfrak{S}(2)=2C_2>0$, and $c_3>0$ is a constant. More generally, for $m\geq 2$, the singular series is given by

$$\mathfrak{S}(2m)=2C_2\prod_{2<p|2m}\left(\frac{p-1}{p-2}\right)>0 \tag{5.21}$$

see [3, Lemma 2.1] for more details.

6 The Main Result

Proof. (Theorem 1.1) Partial summation, and an application of Theorem 2.1 yield

$$\begin{aligned}\psi_2(x) &= \sum_{2\leq n\leq x}\Lambda(n-1)\Lambda(n+1)\\ &\gg \sum_{2\leq n\leq x}\frac{\mu^2(n+1)\mu^2(n+1)\Lambda^2(n-1)\Lambda(n+1)}{\log n}\\ &\gg \int_2^x\frac{1}{\log z}d\psi_T(z)\\ &\gg \frac{x\log\log x}{(\log x)(\log\log\log x)}.\end{aligned} \tag{6.1}$$

Quod erat inveniendum. ■

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