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Clear-Sighted Statistics: Appendix 1: Math Review

Edward Volchok

CUNY Queensborough Community College

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Clear-Sighted Statistics: An OER Textbook

Appendix 1: Math Review

I. Introduction

Statistics is not [mathematics](#). While the discipline of statistics deals with numbers, the mathematics necessary for [quantitative](#) analyses in an Introduction to Statistics course are basic calculations that most students can do by the time they complete middle school.

Perhaps, like me, you were one of those students who would have preferred a session in a dentist's chair to sitting through math class. Maybe the idea of looking at a table full of numbers fills you with dread and anxiety. Perhaps it has been a long time since you took a math class and you have not been using math in your daily life. It may be that you struggled to pass your math classes. You may have forgotten the math you learned or maybe there are serious gaps in your learning. If so, this review is for you. Please read it in its entirety or just examine those sections covering materials that you want to brush up on.

Experience teaching Introduction to Statistics courses tells me that with a little effort virtually all students can successfully complete a statistics class even if some may have to reacquaint themselves with basic mathematics.

Upon completing this review, you will understand:

1. Statistics is not mathematics.
2. The importance of the Hindu-Arabic numbers and positional or place-value notation.
3. Rounding Numbers.
4. Basic Arithmetic: Addition, Subtraction, Multiplication, and Division.
5. Fractions, Decimals, Percentages, and Ratios.

6. Dealing with negative numbers.
7. Exponents and Roots.
8. The Mathematical Order of Operations (PEMDAS).

II. Statistics is Not Math

Like accounting, computer science, economics, engineering, finance, and physics, the discipline of statistics depends on a mathematical or quantitative analysis of real-world [data](#). But statistics is not mathematics. As the great statistician John Tukey once said: “Statistics is a science in my opinion, and it is no more a branch of mathematics than are physics, chemistry, and economics; for if its methods fail the test of experience—not the test of logic—they are discarded.”ⁱ

Pure mathematics—academic mathematics—deals with idealized abstractions that nearly always have absolute solutions. It is about logical reasoning, patterns, and optimization. It is the science of deducing complex consequences from relatively simple [axioms](#), which are proposition that are assumed to be true for the purpose of studying the consequences that follow from them. Mathematics employs [deductive](#) reasoning and the deductions are based on previously known facts. Mathematics has been described as the deductive system that involves four things:

1. A set of primitive undefined terms.
2. Definitions evolved from these undefined terms.
3. Axioms or postulates.
4. Theorems and their proofs.ⁱⁱ

Statistics, on the other hand, is [inductive](#). Individual observations are viewed as evidence for the [veracity](#) of a conclusion. Statistics does not deal with idealized

abstractions. It can be defined as the *science of making educated guesses* about how observable phenomena work. Unlike a mathematicians' theorem, we are never 100 percent certain about the accuracy of our statistical [hypotheses](#). When experience shows that our hypothesis is likely wrong, we reject it.

The science of statistics (statistics used as a [singular](#) noun) is sometimes called mathematical statistics. This term—mathematical statistics—distinguishes the quantitative techniques used to analyze data from statistics used as a [plural](#) noun used or data derived from samples.” To be sure, the founders of mathematical statistics —Adolphe Quetelet, Florence Nightingale, Francis Galton, Karl Pearson, Ronald Fisher, William S. Gosset, Jerzy Neyman, Egon Pearson, and all the others—possessed superb mathematical skills. But to learn the *basic* techniques covered in an Introduction to Statistics course, you do not need advanced mathematics skills. All that is required is the basic mathematics you should have learned by the time you were thirteen years old. You need not understand the complex mathematics behind the Central Limit Theorem or any of the discrete or continuous probability distributions that we will cover. To learn basic statistics, you need:

1. A “numbers sense” or an understanding of basic concepts of quantities (more and less, larger and smaller), an understanding of the order of numbers, the ability to make numerical comparisons, and the knowledge of the symbols used to represent numbers). This “numbers sense” has been called an “‘at homeness’ with numbers and an ability to make use of mathematical skills which enables an individual to cope with the practical demands of everyday life.”ⁱⁱⁱ
2. An ability to round numbers.
3. A proficiency with the addition, subtraction, multiplication, and division.
4. An understanding of fractions, decimals, and percentages.
5. The ability to perform calculations involving negative numbers.

6. A grasp of exponents and roots.
7. A facility using the mathematical order of operations (PEMDAS) so your calculations are accurate.

III. The Importance of the Hindu-Arabic Numerals and Value-Place System

A) The Advantage of Hindu-Arabic Numbers and [Place-Value](#) Notation

One of the greatest advances of human knowledge was the development of Hindu-Arabic Numerals with its place-value notation. Our base-10 or decimal number system was built on the Hindu-Arabic Numeral system. In this system there are ten single digit numbers: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The value of each number or digit depends on its position or “place-value” in the number with the value of each value-place being 10 times the value-place to its right.

Here are the basic whole number groups. A [whole number](#) or [integer](#) is a number without fractions or decimals.

| Millions | Hundred Thousands | Ten Thousands | Thousands | Hundreds | Tens | Units or Ones |
|-----------|----------------------|------------------|-----------|----------|------|------------------|
| 1,000,000 | 100,000 | 10,000 | 1,000 | 100 | 10 | 1 |

Table 1: Whole Number Groups

Let’s look at the number 5,325. We pronounce this number as five thousand, three hundred, twenty-five. 5,325 is based on (5 times 1,000) + (3 times 100) + (two times 10) + (5 times 1).

| Thousands | Hundreds | Tens | Units |
|-----------|----------|------|-------|
| 5 | 3 | 2 | 5 |

Table 2: Whole Number Groups for 5,325

The big advantage of our numbers system is that it makes [arithmetic](#) much easier than a system like [Roman numerals](#) which does not use place-values. Roman numerals were developed in ancient Rome and spread with the Roman Empire. Roman numerals are based

on a system of seven symbols taken from the Latin alphabet. Here are the seven Roman numeral symbols and their values written in Hindu-Arabic numbers:

| | | | | | | | |
|---------------|----------|----------|----------|----------|----------|----------|----------|
| Symbol | I | V | X | L | C | D | M |
| Value | 1 | 5 | 10 | 50 | 100 | 500 | 1,000 |

Table 3: The seven Roman numeral symbols

Please note: There is no Roman numeral for zero. The lack of a zero along with the lack of place-values makes calculation difficult.

Here are the Roman Numerals for 1 through ten.

| | | | | | | | | | | |
|---------------|----------|-----------|------------|-----------|----------|-----------|------------|-------------|-----------|----------|
| Symbol | I | II | III | IV | V | VI | VII | VIII | IX | X |
| Value | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

Table 4: Roman numerals 1 through 10

Here are the symbols when we count by tens from 10 to 100.

| | | | | | | | | | | |
|---------------|----------|-----------|------------|-----------|----------|-----------|------------|-------------|-----------|----------|
| Symbol | X | XX | XXX | X: | L | LX | LXX | LXXX | XC | C |
| Value | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

Table 5: Roman numerals 10 to 100

Here are the Roman numerals counting by hundreds from 100 to 1,000.

| | | | | | | | | | | |
|---------------|----------|-----------|------------|-----------|----------|-----------|------------|-------------|-----------|----------|
| Symbol | X | XX | XXX | X: | L | DC | DCC | DCCC | CM | M |
| Value | C | CC | CCC | CD | 500 | 600 | 700 | 800 | 900 | 1,000 |

Table 6: Roman numerals counting by 100s to 1,000

The year 2019 is expressed as MMXIX and the year 1776 is written as MDCCLXXVI. How many years have passed between MMXIX and MDCCLXXVI? This is a simple subtraction problem using Hindu-Arabic numbers, but it is a very challenging calculation problem using Roman numerals. Try performing the arithmetic using Roman numerals. It can be done. It is not easy. But why bother? You have more important things to spend your time on than doing simple arithmetic with Roman numerals.

| Roman Numerals | Hindu-Arabic Numerals |
|-----------------------|------------------------------|
| MMXIX | 2019 |
| - <u>MDCCLXXVI</u> | - <u>1776</u> |
| ? | 243 |

Table 7: MMXIX (2019) minus MDCCLXXVI (1776)

B) A Brief History of Hindu-Arabic Numerals

Hindu-Arabic numerals were developed around 500 CE by Indian mathematicians. By the 8th century this approach evolved into a decimal numeral system with a zero. This is when Hindu numerals were adopted by Arab mathematicians in Persia, today's Iran, and moved west throughout the Arab world. In 825, Al-Khwārizmī (c. 780 – c. 850), an Arab mathematician published a book in Arabic entitled *On the Calculation with Hindu Numbers*.

Europeans were beginning to express interest in Hindu-Arabic numbers by the late 10th century when Gerbert of Aurillac (c. 946 – 1003), who would become Pope Sylvester II, encouraged the adoption of Hindu-Arabic numerals.

At the start of the thirteenth century, Leonardo of Pisa who is also known as Fibonacci (c. 1170 – c. 1250) helped popularize Hindu-Arabic numbers in Italy. As a young boy in the late twelfth century, Fibonacci travelled with his merchant father to Algeria. There his Arab tutors introduced him to Hindu-Arabic numerals. In 1202, Fibonacci published a mathematics book entitled *Liber Abaci*. In this book, Fibonacci argued that the Hindu-Arabic numerals with its ten digits and place-value notation had important advantages for merchants because it simplified bookkeeping, converting weights and measures, exchanging currencies, and calculating interest. Fibonacci's book was well received.

The adoption of Hindu-Arabic numerals in Europe fostered the growth of banking and accounting and was accelerated in Europe with the invention of the printing press in the 15th century. By the mid 16th century, Hindu-Arabic numerals were used throughout most of Europe although it was not until the reign of Peter the Great in the early 18th century that

Hindu-Arabic numerals replaced Cyrillic numerals in Russia. During the Yuan Dynasty (1271– 1368), the Muslim Hui people brought Hindu-Arabic numerals to China.

The numbers we use today are actually Western Arabic numbers. There are also Eastern Arabic numbers, which are read right to left just like the Arabic language.

| | | | | | | | | | | |
|----------------|---|---|---|---|---|---|---|---|---|---|
| Western | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| Arabic | | | | | | | | | | |
| Eastern Arabic | ٠ | ١ | ٢ | ٣ | ٤ | ٥ | ٦ | ٧ | ٨ | ٩ |

Table 8: Western and Eastern Arabic numbers

Eastern Arabic numbers are still used alongside of Western Arabic numbers in Egypt, Sudan, Saudi Arabia, Arabic-speaking Asia, Iran, Pakistan, and Afghanistan.

IV. Rounding Numbers

We round numbers to make the number simpler while maintaining a close approximation to the actual number. For example, the U.S. Census Bureau’s estimate of the American population as of July 1, 2018 was 327,167,434.^{iv} In reality even the U.S. Census Bureau’s data are not that precise. For most occasions we would round off this number to 327 million.

To round numbers properly, we use the base-10 place-value system and the following rules.

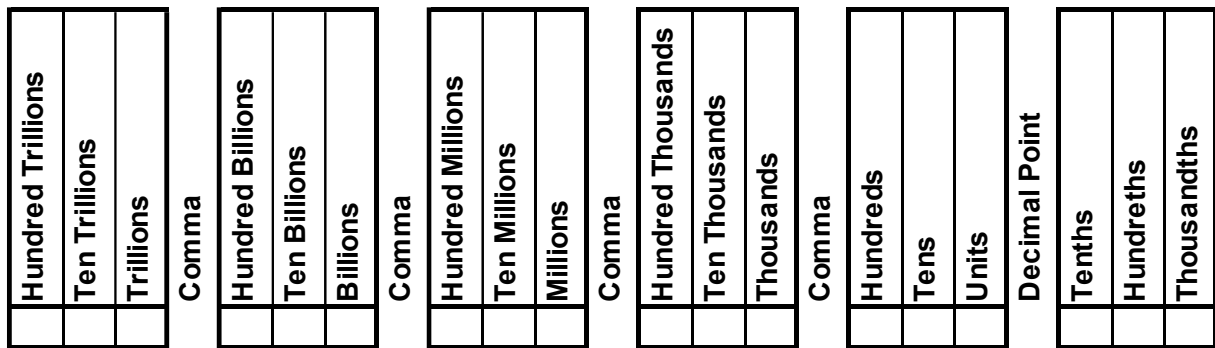


Figure 1: The Base-10 Place-Value System

Rounding Up or Rounding Down?

When rounding numbers, the first thing we do is to identify the rounding digit. Then there are two basic rules for rounding [whole numbers](#):

1. **Rounding Down:** When the number to the right of the rounding digit is less than 5 (0, 1, 2, 3, or 4), **round down** to the next lower number.
2. **Rounding Up:** When the number to the right of the rounding digit is 5 or more (5, 6, 7, 8, or 9), **round up** by adding 1 to the rounding digit and change the digits to the right of the rounding digit to zero.

There are also two basic rules for rounding decimal numbers. Decimal numbers are used to represent a number smaller than one unit. Determine the rounding unit then:

1. **Rounding Down:** If the digit is less than 5 (0, 1, 2, 3, or 4), all digits to the right of the rounding unit become zero.
2. **Rounding Up:** When the number to the right of the rounding digit is 5 or more (5, 6, 7, 8, or 9), round up the rounding unit's digit by one.

Rounding Problem 1: Round 1,420 to the nearest hundreds (Note: The rounding digit is the third digit to the left of the decimal point. While the decimal point is not shown, it is just to the right of the zero.)

Rounding Problem 2: Round 1,452 to the nearest tens. (Note: The rounding digit is the second digit to the left of the decimal point. While the decimal point is not shown, it is just to the right of the zero.)

Rounding Problem 3: Round \$13.66666 to the nearest penny, which is the hundredths column. (Note: The rounding digit is the second digit to the right of the decimal point.)

Rounding Problem 4: Round \$39.3333 to the nearest penny, which is the hundredths column. (Note: The rounding digit is the second digit to the right of the decimal point.)

Answers to the rounding problems:

1. The rounding digit is the 4 in the hundreds column. Answer is 1,400.

2. The rounding digit is the 5 in the tens column. 1,450.
3. The rounding digit is the second 6 to the right of the decimal point. \$13.67.
4. The rounding digit is the second 3 to the right of the decimal point. \$39.33.

V. Basic Arithmetic: Addition, Subtraction, Multiplication, and Division

A) Common Arithmetic Symbols

Here is a list of the most common arithmetic symbols.

| Symbol | Meaning |
|----------------------|-----------------------------|
| = | Equals |
| ≠ | Not equal |
| ≈ | Approximately |
| +, ∑ | Operation of Addition |
| - | Operation of Subtraction |
| x, •, * | Operation of Multiplication |
| /, ÷, $\overline{)}$ | Operation of Division |

Table 9: Common Arithmetic Symbols

B) Addition

Addition is one of the four basic arithmetic operations along with subtraction, multiplication, and division. When we add numbers, we are seeking to find the *total* or [sum](#) of these numbers. The individual values we are adding are called [addends](#). Our answer is called the *sum*, *total*, or *amount*. The operation of addition is signified by the plus symbol (+). In statistics, we often use the capital [Greek letter Sigma, Σ](#), to indicate the operation of summation.

The operation of addition has two important properties: 1) [Commutativity](#) and 2) [Associativity](#). **Commutativity:** When we say that addition is commutative, we mean that

the order of terms in an addition problem does not matter: $a + b$ is the same as $b + a$; or, for example, $2 + 3 = 3 + 2$. **Associativity:** The associative property means that if we are adding three or more *addends*, the order of the *addends* does not matter: $a + b + c = (a + b) + c = a + (b + c)$; or, using numbers, $1 + 2 + 3 = (1 + 2) + 3 = 1 + (2 + 3)$.

Here is a basic approach to adding numbers.

1. Align the *addends* in a column according to their place-value.
2. Add the “units” column. If the sum is greater than 9, write the units digit and carry the tens digit.
3. Moving left, repeat step two for until all place-values have been added.

Let’s work through an example step by step. Add the following values: 5, 86, and 199; stated another way, what is the sum of 5, 86, and 199.

Step 1: Align the *addends* in a column

$$\begin{array}{r} 5 \\ 86 \\ + 199 \\ \hline \end{array}$$

Table 10: Step 1 Addition

Step 2: Sum the “units” column

$$\begin{array}{r} 2 \\ 5 \\ 86 \\ + 199 \\ \hline 0 \end{array}$$

Table 11: Step 2 Addition. Note the 2 above the number 5 is found by $5 + 6 + 9 = 20$.

Step 3: Sum the “tens” column

$$\begin{array}{r} 12 \\ 5 \\ 86 \\ + 199 \\ \hline 90 \end{array}$$

Table 12: Step 3 Addition

Step 4: Complete the calculation

$$\begin{array}{r} 12 \\ 5 \\ 86 \\ + 199 \\ \hline =290 \end{array}$$

Table 13: Step 4 Addition

A Word Problem

Old McDonald had a farm. On his farm he had 12 dairy cows, 36 hens that lay eggs, 1 rooster, along with 96 chicks, 36 ducks, and 25 turkeys that he raises for their meat. How many farm animals are on McDonald's farm?

Step 1: Align the *addends* in a column

$$\begin{array}{r} 12 \text{ Dairy Cows} \\ 36 \text{ Hen that lay eggs} \\ 1 \text{ Rooster} \\ 96 \text{ Chicks} \\ 36 \text{ Ducks} \\ + 25 \text{ Turkey} \\ \hline \end{array}$$

Table 14: Step 1 Addition

Step 2: Sum the "units" column

$$\begin{array}{r} 2 \\ 12 \text{ Dairy Cows} \\ 36 \text{ Hen that lay eggs} \\ 1 \text{ Rooster} \\ 96 \text{ Chicks} \\ 36 \text{ Ducks} \\ + 25 \text{ Turkey} \\ \hline 6 \end{array}$$

Table 15: Step 2 Addition

Step 3: Sum the "tens" column

$$\begin{array}{r} 22 \\ 12 \text{ Dairy Cows} \end{array}$$

| | |
|------|-------------------|
| 36 | Hen that lay eggs |
| 1 | Rooster |
| 96 | Chicks |
| 36 | Ducks |
| + 25 | Turkey |
| 06 | |

Table 16: Step 3 Addition

Step 4: Complete the calculation

| | |
|-------|-------------------|
| 22 | |
| 12 | Dairy Cows |
| 36 | Hen that lay eggs |
| 1 | Rooster |
| 96 | Chicks |
| 36 | Ducks |
| + 25 | Turkey |
| = 206 | Total |

Table 17: Step 4 Addition

Addition Practice Problems: Complete these addition problems using a pencil and paper.

Check your answers using a hand-held calculator. One conclusion you may reach is that using a hand-held calculator is faster and more convenient than performing this calculation using paper and pencil. That is a fair conclusion and it is true for all arithmetic operations. But you must be adept at estimating your answer without the aid of a calculator. Doing so will develop your “numbers sense” and help you determine whether you may have an erroneous answer.

1. The registrar of Bayside State College reports that the college has 3,250 freshman, 2,975 sophomores, 2,783 juniors, and 2,761 seniors enrolled during the spring semester. How many students are enrolled at the college?
2. Dewey, Cheatem, and Howe is a leading international law firm that just hired five attorneys who graduated from top law schools at the end of May. The starting monthly salaries for their new hires: \$12,000, \$11,575, \$12,750, \$9,975, and \$12,500. How much is the total monthly salary for the five newly hired attorneys?
3. In December 2018 General Motors sold 179,212 Chevrolets, 54,203 GMC trucks, 17,305 Buicks, and 14,001 Cadillacs in the United States. How many vehicles did GM sell in the American market during December 2018?

4. Tesla sold the following numbers of cars worldwide: 1st Quarter 2018; 29,980; 2nd Quarter 2018; 40,470; 3rd Quarter 83,500; and 4th Quarter 90,700. How many cars did Tesla sell in 2018?
5. Total revenue for the top five American Wireless Carriers for 2018 in billions of dollars were: AT&T (\$170.72), Verizon (\$130.86), T-Mobile (\$43.31), Sprint (\$33.60), and U.S. Cellular (\$5.20). What was the total revenue in billions of dollars for the top five U. S. Wireless Telecommunications companies for 2018?

Answers to the addition problems:

1. 11,769
2. \$58,800
3. 264,721
4. 244,650
5. \$383.690 billion (\$383,690,000,000)

B) Subtraction

Subtraction is the [inverse](#) or opposite operation of addition. Where addition seeks to find the total number in a collection, subtraction removes numbers to find the *difference*. The starting number of our collection is called the [minuend](#). The number we are subtracting from the minuend is called the [subtrahend](#). The operation of subtraction is signified by the minus symbol (-).

Here is a basic approach to subtracting numbers.

1. Align the minuend and subtrahend in a column according to their place-value. Place the minuend on the first row and the subtrahend on the second row.
2. In the third row, start the subtraction with the “units” digits. When the units in the minuend are smaller than the units in the subtrahend, “borrow” 1 from the “tens” digit. Remember: One “tens” digit is 10.
3. Move to the left until all place-values in the subtrahend are subtracted.
4. Check your answer by adding the difference to the subtrahend. The sum of these numbers must equal the minuend. If they are not equal, your calculation is wrong.

A Word Problem

Subtraction Example: In 2018, the number of vehicles the Ford Motor Company sold in the United States was 2,497,318. In 2019, Ford sold 2,586,714 vehicles. Ford's sales in 2019 increased by how many vehicles?

Step 1: Align the minuend in the first row of a column, place the subtrahend in the second row, and in the third row leave room for the difference, which you will calculate by subtraction. in a column.

$$\begin{array}{r} 2,586,714 \\ - 2,497,318 \\ \hline \end{array} \begin{array}{l} \text{Minuend} \\ \text{Subtrahend} \\ \text{Difference} \end{array}$$

Table 18: Step 1 Subtraction

Step 2: Starting in the "units" column subtract the subtrahend from the minuend. When the units in the minuend are smaller than the units in the subtrahend, "borrow" 1 from the "tens" digit. Remember: One "tens" digit is 10.

$$\begin{array}{r} 2,586,714 \\ - 2,497,318 \\ \hline 6 \end{array} \begin{array}{l} \text{Minuend} \\ \text{Subtrahend} \\ \text{Difference} \end{array}$$

Table 19: Step 2 Subtraction

Step 3: In the "tens" column subtract the subtrahend from the minuend. Remember that you already borrowed one unit from the "tens" column so the minuend is not smaller than the subtrahend. Borrow one unit from the "hundreds" column.

$$\begin{array}{r} 2,586,714 \\ - 2,497,318 \\ \hline 96 \end{array} \begin{array}{l} \text{Minuend} \\ \text{Subtrahend} \\ \text{Difference} \end{array}$$

Table 20: Step 3 Subtraction

Step 4: In the “hundreds” column subtract the subtrahend from the minuend.

Remember that you already borrowed one unit from the “hundreds” column so the minuend is not smaller than the subtrahend. Borrow one unit from the “hundreds” column.

$$\begin{array}{r} 2,586,714 \quad \text{Minuend} \\ - \underline{2,497,318} \quad \text{Subtrahend} \\ \hline 396 \quad \text{Difference} \end{array}$$

Table 21: Step 4 Subtraction

Step 5: In the “thousands” column subtract the subtrahend from the minuend.

Remember When the units in the minuend are smaller than the units in the subtrahend, “borrow” 1 from the “ten thousands” digit.

$$\begin{array}{r} 2,586,714 \quad \text{Minuend} \\ - \underline{2,497,318} \quad \text{Subtrahend} \\ \hline 9,396 \quad \text{Difference} \end{array}$$

Table 22: Step 5 Subtraction

Step 6: In the “ten thousands” column subtract the subtrahend from the minuend.

Remember When the units in the minuend are smaller than the units in the subtrahend, “borrow” 1 from the “hundred thousands” digit.

$$\begin{array}{r} 2,586,714 \quad \text{Minuend} \\ - \underline{2,497,318} \quad \text{Subtrahend} \\ \hline 89,396 \quad \text{Difference} \end{array}$$

Table 23: Step 6 Subtraction

We have now completed the subtraction. In 2018, sales in the U.S. for the Ford Motor Company fell 89,396 vehicles from 2017 levels.

Checking your answer: Add the difference to the subtrahend. The sum must equal the minuend.

$$\begin{array}{r} 89,396 \quad \text{Difference} \\ + \underline{2,497,318} \quad \text{Subtrahend} \\ \hline 2,586,714 \quad \text{Minuend} \end{array}$$

Table 24: Subtraction Check - Add the subtrahend to the difference

Note that unlike the operation of addition, **subtraction is not commutative**. Flipping the order of the *minuend* and *subtrahend* will result in a different answer. And, unlike addition, subtraction is not associative. This means that the order in which the numbers are arranged will affect the answer.

$$\begin{array}{r}
 2,586,714 \\
 - 2,497,318 \\
 \hline
 89,396
 \end{array}
 \begin{array}{l}
 \textit{Minuend} \\
 \textit{Subtrahend} \\
 \textit{Difference} \neq
 \end{array}
 \begin{array}{r}
 2,497,318 \\
 - 2,586,714 \\
 \hline
 -89,396
 \end{array}
 \begin{array}{l}
 \textit{Minuend} \\
 \textit{Subtrahend} \\
 \textit{Difference}
 \end{array}$$

Table 25: Subtraction is not commutative

Subtraction Practice Problems: Complete these subtraction problems using a pencil and paper. Check your answers using a hand-held calculator.

1. According to CNBC, Sears Holdings lost \$558 million during third quarter 2017 compared to \$748 million during third quarter 2016. How much lower were Sears' losses during 3rd Quarter 2017 compared to 3rd Quarter 2016?
2. According to the FBI Uniform Crimes Reports for 2017, the rate of violent crimes per 100,000 people in American counties along the Mexican border was 347.8 per 100,000 compared to 377.8 per 100,000 for the entire USA.^v How many fewer crimes are there in border counties than in the United States as a whole?
3. On February 6, 2016, the Cato Institute, an American libertarian think tank located in Washington DC, published a study on criminal immigrant convictions in Texas.^{vi} The conviction rate for homicides among all immigrants (legal and illegal) was 1.6 per 100,000 residents of that subpopulation. The conviction rate for homicides among native Texans was 3.1 per 100,000 of that subpopulation. What is the difference in the conviction rates for homicides per 100,000 residents?
4. On September 12, 2017, CNBC reported that the median weekly income for women 35 to 44 years old was \$877 compared to \$1,058 for men the same age.^{vii} How much lower were the earnings of female workers?
5. According to [Student Load Hero](#), during the 2015-2016 school year that average tuition for in-state students at a public university was \$9,410 a year compared to \$32,405 for a private university.^{viii} How much cheaper is tuition per year for an in-state student at a public university?

Answers to the subtraction problems:

1. \$190 million
2. 30 per 100,000
3. Immigrants had 1.5 per 100,000 residents fewer convictions for murder than native Texans
4. \$181 a week
5. \$22,995

C) Multiplication

The arithmetic operation of multiplication is denoted by the following symbols: \times , $*$, or \bullet .

Multiplication is often considered repetitive addition. Consider the following problem. A landlord owns a small building with five apartments. Each apartment rents for \$1,500 a month. What is this landlord’s monthly revenue from the five apartments? You can solve this problem using addition; $\$1,500 + \$1,500 + \$1,500 + \$1,500 + \$1,500 = \$7,500$. Or, you can use multiplication: $5 \times \$1,500 = \$7,500$, or five times \$1,500. In this case, “\$1,500” is the *multiplicand*, the number of things we want to multiply. The number 5 is called the *multiplier*, the number by which we multiply the multiplicand. Our answer is called the ***product***.

$$\begin{array}{r}
 \$1,500 \quad \text{Multiplicand} \\
 \times \quad 5 \quad \times \quad \text{Multiplier} \\
 \hline
 = \$7,500 \quad = \text{Product}
 \end{array}$$

Table 22: Placement of Multiplication Terms

The *multiplicand* and the *multiplier* are often called *factors*. Like addition, the operation of multiplication is commutative; which is to say, we can switch the positions of the *multiplicand* and *multiplier*: 3 times 800 = 800 times 3.

One common way to multiply numbers is to use a multiplication or “times” table. Here is a multiplication table for 1 through 12. Consider the numbers on the top row and left column as *factors*, *multiplicands*, or *multipliers*. **Remember:** Because of the commutative property of multiplication, *factors* are interchangeable.

Multiplication Table:

| x | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|----|----|----|----|----|----|----|----|----|-----|-----|-----|-----|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 2 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 |
| 3 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 | 33 | 36 |
| 4 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 | 44 | 48 |
| 5 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 |
| 6 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 | 66 | 72 |
| 7 | 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 | 77 | 84 |
| 8 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 | 88 | 96 |
| 9 | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 | 99 | 108 |
| 10 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
| 11 | 11 | 22 | 33 | 44 | 55 | 66 | 77 | 88 | 99 | 110 | 121 | 132 |
| 12 | 12 | 24 | 36 | 48 | 60 | 72 | 84 | 96 | 108 | 120 | 132 | 144 |

Table 26: Completed Multiplication Table for 1 through 12

Here is a basic approach to multiplying numbers.

1. Align the *multiplicand* and the *multiplier* in a column according to their place-value.
2. Place the *multiplicand* on the first row and the multiplier on the second row.
3. Start by multiplying the digit in the “units” column.
4. Keep moving to the left through the multiplicand.
5. Move left through the multiplier and continue multiplying the multiplicand.
6. Continue until you complete the process

A Word Problem

Multiplication Example: Bayside State College has 14,555 full-time students registered this semester. The average student spends \$229 on books per semester. What is the total amount full-time students spent on books this semester?

Step 1:

14,555 *Multiplicand*

$$\begin{array}{r} \underline{\times 229} \quad \times \quad \underline{\text{Multiplier}} \\ = ? \quad = \quad \text{Product} \end{array}$$

Table 27: Step 1 Multiplication

Step 2:

$$\begin{array}{r} 14,555 \quad \text{Multiplicand} \\ \underline{\times 229} \quad \times \quad \underline{\text{Multiplier}} \\ 130995 \quad = \quad \text{Partial Product} \end{array}$$

Table 28: Step 2 Multiplication

Step 3:

$$\begin{array}{r} 14,555 \quad \text{Multiplicand} \\ \underline{\times 229} \quad \times \quad \underline{\text{Multiplier}} \\ 130995 \quad = \quad \text{Partial Product} \\ 29110 \quad = \quad \text{Partial Product} \end{array}$$

Note: The second partial product is not lined up with the previous line. It moves one digit to the left,
Table 29: Step 3 Multiplication

Step 4:

$$\begin{array}{r} 14,555 \quad \text{Multiplicand} \\ \underline{\times 229} \quad \times \quad \underline{\text{Multiplier}} \\ 130995 \quad = \quad \text{Partial Product} \\ 29110 \quad = \quad \text{Partial Product} \\ 29110 \quad = \quad \text{Partial Product} \end{array}$$

Table 30 Step 4 Multiplication

Step 5:

$$\begin{array}{r} 14,555 \quad \text{Multiplicand} \\ \underline{\times 229} \quad \times \quad \underline{\text{Multiplier}} \\ 130995 \quad + \quad \text{Partial Product} \\ 29110 \quad + \quad \text{Partial Product} \\ \underline{29110} \quad + \quad \text{Partial Product} \\ = \quad = \quad \text{Product} \\ \$3,333,095 \end{array}$$

Table 31: Step 5 Multiplication

Multiplication Practice Problems: Complete these multiplication problems using a pencil and paper. Check your answers using a hand-held calculator.

1. Bayside State University is planning on buying new computers for its computer labs. Each computer costs \$1,155. Each computer lab has 36

computers. How much will it cost the university to buy new computers for one computer lab?

2. Bayside State University has 45 computer labs on campus. Using your answer from the previous question, how much would it cost to buy new computers for all 45 computer labs?
3. Bayside Energy is planning on increasing its fee for electricity for residential customers by \$3.35 a month. Bayside Energy has 25,849 customers. How much money will the increased fees raise in a month?
4. Bayside Used Cars received 18 used cars this week. As part of routine maintenance on these vehicles, they will replace the battery on each car. Car batteries cost Bayside Used Cars \$62.99 each. How much will it cost to buy new batteries for the 18 cars?
5. Larry Luftmensch is working as a travelling salesman. This month he drove 1,525 miles for work. His employer will compensate him at a rate of 33¢ a mile. How much will Mr. Luftmensch's employer compensate him for the use of his car this month?

Answers to the multiplication problems:

1. \$41,580
2. \$1,871,100
3. \$86,594.15
4. \$1,133.82
5. \$503.25

D) Division

The arithmetic operation of division is denoted by the following symbols: \div , $/$, or $\overline{)$.

Division is the inverse, or reverse of multiplication. We use division to calculate how many times one number, called a *divisor*, is contained in another number called a *dividend*. The answer is called a *quotient*.

$$\begin{array}{l} \text{Quotient} \\ \text{Divisor} \overline{) \text{Dividend}} \end{array} \Rightarrow \begin{array}{l} 9 \\ 5 \overline{) 45} \end{array} \text{ or } 45/5 = 9 \text{ or } 45 \div 5 = 9$$

Figure 2: This figure shows the positions of the divisor, dividend, and quotient

For the problem shown above, we express the term as “45 divided by 5.” The standard way to double-check the accuracy of your calculation is to multiply the *divisor* (5, in this case) by the *quotient* (9). The result must equal the *dividend*: $5 \times 9 = 45$.

Short Division vs. Long Division: We use short division when the *divisor* has only one digit: $45 \div 5$ is an illustration of short division. Long division is used when the *divisor* has more than two digits.

Here is an example of long division: The bonus pool for junior managers at Dewey, Cheatem, and Howe is \$320,000. There are 25 people at the firm eligible for a bonus. Management has decided that every junior manager will get an equal share of the bonus pool. How much will each junior manager receive? $\$320,000 \div 25$.

| Step | Example |
|--|--|
| Step 1: Set up the problem: 25 is the <i>divisor</i> , 320,000 the is the <i>dividend</i> . | $25 \overline{) 320,000}$ |
| Step 2: Divide the first digit of the <i>dividend</i> (1) by the divisor. At this point any remainder is ignored. Place the answer, 0, in the <i>quotient</i> . | $25 \overline{) 320,000} \\ 0$ |
| Step 3: The answer from the first operation is multiplied by the <i>divisor</i> . The result is placed under the number being <i>divided</i> . | $25 \overline{) 320,000} \\ 0$ |
| Step 4: Subtract the bottom number from the top number. | $25 \overline{) 320,000} \\ 0 \\ \underline{3}$ |
| Step 5: Bring down the next digit of the <i>dividend</i> . | $25 \overline{) 320,000} \\ 0 \\ \underline{32}$ |

| | |
|--|--|
| <p>Step 6: Divide 32 by the <i>divisor</i>. Place the number, in this case 1, in the <i>quotient</i>. Any remainder is ignored at this point.</p> | $\begin{array}{r} 01 \\ 25 \overline{) 320,000} \\ 0 \\ \underline{32} \end{array}$ |
| <p>Step 7: The answer from the second operation is multiplied by the divisor. The result is placed under the number being divided.</p> | $\begin{array}{r} 01 \\ 25 \overline{) 320,000} \\ 0 \\ 32 \\ \underline{25} \end{array}$ |
| <p>Step 8: Subtract the bottom number from the top number.</p> | $\begin{array}{r} 01 \\ 25 \overline{) 320,000} \\ 0 \\ 32 \\ \underline{25} \\ 7 \end{array}$ |
| <p>Step 9: Bring down the next digit of the <i>dividend</i>.</p> | $\begin{array}{r} 01 \\ 25 \overline{) 320,000} \\ 0 \\ 32 \\ \underline{25} \\ 70 \end{array}$ |
| <p>Step 10: Divide 70 by the divisor. Place the number, in this case 2, in the <i>quotient</i>. Any remainder is ignored at this point.</p> | $\begin{array}{r} 012 \\ 25 \overline{) 320,000} \\ 0 \\ 32 \\ \underline{25} \\ 70 \\ \underline{50} \end{array}$ |

| | |
|--|--|
| <p>Step 11: Subtract the bottom number from the top number.</p> | $\begin{array}{r} 012 \\ 25 \overline{) 320,000} \\ 0 \\ 32 \\ 25 \\ \hline 70 \\ 50 \\ \hline 20 \end{array}$ |
| <p>Step 12: Bring down the next digit of the <i>dividend</i>.</p> | $\begin{array}{r} 012 \\ 25 \overline{) 320,000} \\ 0 \\ 32 \\ 25 \\ \hline 70 \\ 50 \\ \hline 200 \end{array}$ |
| <p>Step 13: Divide this number, 200, by the <i>divisor</i> and multiply that number by the <i>divisor</i>.</p> | $\begin{array}{r} 012,8 \\ 25 \overline{) 320,000} \\ 0 \\ 32 \\ 25 \\ \hline 70 \\ 50 \\ \hline 200 \end{array}$ |
| <p>Step 14: Subtract the bottom number from the top number., $200 - 200 = 0$. Given that the remaining two digits in the <i>dividend</i> are zeros, we add two zeros to the <i>quotient</i>.</p> | $\begin{array}{r} 012,800 \\ 25 \overline{) 320,000} \\ 0 \\ 32 \\ 25 \\ \hline 70 \\ 50 \\ \hline 200 \\ 200 \\ \hline 000 \end{array}$ |
| <p>Each junior manager gets a bonus of \$12,800.</p> | <p>\$12,800</p> |
| <p>Check your answer: <i>Divisor x Dividend = Quotient</i></p> | <p><u>$25 \times 12,800 = 320,000$</u></p> |

Table 32: Long Division Example

Division Practice Problems: Complete these division problems using a pencil and paper.

Check your answers using a hand-held calculator. Round off your answers to the hundredths column (0.00).

1. Bayside Gardener has a truck. This month the truck was driven 1,751 miles. The truck consumed 105 gallons of diesel fuel. How many miles per gallon does this truck get? Round your answer to the nearest hundredths (0.00).
2. The top 30 home run hitters in baseball hit a total of 17,251 home runs. What is the average number of home runs hit by a baseball player included in the top 30 home run hitters? Round your answer to the nearest hundredths (0.00).
3. The top 30 home run hitters in baseball struck out a total of 47,513 times. What is the average number of strikeouts for a player on the top 30 home run hitters list? Round your answer to the nearest hundredths (0.00).
4. According to the Nielsen company which monitors sales in food stores, vegan foods, or plant-based foods, had \$3,327 million in sales in 2018. The milk alternatives segment of this market had \$1,600 million in sales. What proportion of the total vegan foods market does the milk alternatives segment represent?
5. John has to buy a carpet for a 130 square foot room. He has a budget of \$2,500. What is the most John can spend per square foot of carpeting? Round your answer to the nearest hundredths (\$0.00).

Answers to the division problems:

1. $1,751 \div 105 = 16.67$ miles per gallon
2. $17,251 \div 30 = 575.03$ home runs
3. $47,513 \div 30 = 1,583.77$ strikeouts
4. $\$3,327 \div \$1,600 = 0.48$ or 48%
5. $\$2,500 \div 130 = \19.23 per square foot

VI. Fractions, Decimals, Percentages, and Ratios

A. Fractions

A [fraction](#) represents a part of a whole unit or 1.

In its simplest form, a fraction consists of two numbers, one above a line or before a slash, and the other below the line or after the slash.

$$\frac{3}{8} \text{ or } \frac{3}{8}$$

Figure 4: Fraction



Figure 3: Whole Pizza

The **denominator** is the number displayed below the line or after the slash. The

$$\frac{3}{8} \text{ or } \frac{3}{8}$$

denominator indicates how many equal parts the whole has been divided into.

Figure 5: Fraction Denominator – Bottom Number

The **numerator** is the number displayed above the line or before the slash. The numerator indicates just how many of these parts we have.

$$\frac{3}{8} \text{ or } \frac{3}{8}$$

Figure 6: Fraction Numerator – Top Number

In the fraction above, $\frac{3}{8}$, the whole item has been divided into eight equal parts based on the denominator, and there are three of these parts based on the numerator.

If you can cut a thing into two equal parts, each part represents half of the whole. We can express this mathematically as:

$$\frac{1}{2} + \frac{1}{2} = 1$$

Figure 7: Addition with Fractions, Example 1

If we cut the whole into five equal parts, each part represents a fifth of the whole. We can express this mathematically as:

$$1/5 + 1/5 + 1/5 + 1/5 + 1/5 = 1$$

Figure 8: Addition with Fractions, Example 2

Fractions are widely used in statistics; therefore, students of statistics must understand how to work with fractions.

B. Types of Fractions:

1. Common Fractions (Also known as proper fractions or vulgar fractions)

This type of fraction is smaller than one, but greater than zero. Notice that the numerator, the top number in the fraction, is smaller than the denominator, the bottom number. Here are examples of common five fractions:

| | | | | |
|-------|------|-----|-----|-----|
| 40/50 | 3/40 | 2/3 | 1/2 | 1/4 |
|-------|------|-----|-----|-----|

Figure 9: Common Fractions

2. Improper Fractions

These are fractions where the numerator (top number) is larger than the denominator (bottom number). This, of course, means that these numbers are greater than one. Here are five examples of improper fractions.

| | | | | |
|-----|-------|-----|-----|-----|
| 6/5 | 80/40 | 4/3 | 3/2 | 5/4 |
|-----|-------|-----|-----|-----|

Figure 10: Improper Fractions

3. Mixed Fractions

In mixed fractions, the fraction is written after the whole number. Let's convert the improper fraction $6/5$ into a mixed fraction. To convert an improper fraction to a mixed fraction, divide numerator by denominator. The quotient is 1 and the remainder is 1, hence the mixed fraction is $1\frac{1}{5}$; $1/5$ is the remainder over the divisor.

5. Equivalent Values

Equivalent Values are fractions that have the same value. Here are three equivalent fractions: $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{3}{6}$. Each represents one half of the whole.

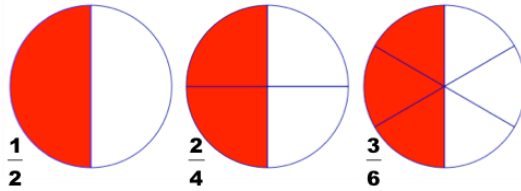


Figure 11: Fractions - Equivalent Values

We know that $\frac{2}{4}$ and $\frac{3}{6}$ are equal to $\frac{1}{2}$ when we reduce these numbers by the lowest terms. In our example, “2” is the lowest term for the three denominators: 2, 4, and 6.

C. Arithmetic With Fractions

We can add, subtract, multiply, and divide fractions. To simplify the arithmetic, we will have to find the [least common denominator \(LCD\)](#).

In our example below, we have two denominators, 2 and 5. The LCD is 10.

$$\frac{1}{2} + \frac{3}{5} = \frac{5}{10} + \frac{6}{10} = \frac{11}{10} = 1\frac{1}{10}$$

Ten is the least common multiple of 2 & 5

Figure 12: Least Common Denominator

To illustrate arithmetic with fractions, let’s start with something that is “whole,” like one pizza fresh from the oven.



Figure 13, Whole Pizza = 1 Pizza

Now let's cut the pizza into eight slices.



Figure 14: Pizza divided into 8 slices = 8/8 of a Pizza. Click on pizza to play animated GIF.

Three students living in a dormitory. The students are Barry, Garry, and Harry. Here are how many slices each student ate.

| | | |
|---------------------|---|---------------|
| Barry eats 1 slice |  | $\frac{1}{8}$ |
| Garry eats 3 slices |  | $\frac{3}{8}$ |


| | | |
|---------------------|---|---------------|
| | | |
| Harry eats 2 slices |  | $\frac{2}{8}$ |

Table 33: Pizza Fractions

Addition:

How much of the pizza did Barry, Garry, and Harry eat?

$$\frac{1}{8} + \frac{3}{8} + \frac{2}{8} = \frac{6}{8} \text{ or } \frac{3}{4}$$

Table 34: Addition with Fractions

Subtraction:

How much of the pizza is left?

$$\frac{8}{8} - \frac{6}{8} = \frac{2}{8} \text{ or } \frac{1}{4}$$

Table 35: Subtraction with Fractions

Multiplication:

Let's multiply: $\frac{1}{2} \times \frac{4}{6}$ or $\frac{1}{2} \times \frac{4}{6} = ?$

To multiply with fractions, follow these three steps:

Step 1: Multiply the numerators

Step 2: Multiply the Denominators

Step 3: Reduce answer to the lowest terms

Table 36: Multiplication with Fractions

$$\begin{aligned} \frac{1 \cdot 4}{2 \cdot 6} &= \frac{4}{12} \\ \frac{4}{12} &= \frac{2}{6} = \frac{1}{3} \end{aligned}$$

Division: $\frac{1}{2} \div \frac{2}{6}$

Let's divide: $\frac{1}{2} \div \frac{4}{6}$ or $\frac{1}{2} \div \frac{4}{6} = ?$

To divide with fractions, follow these three steps:

Step 1: Change the second fraction into the reciprocal (flip the numerator and the denominator, and

$$\frac{1}{2} * \frac{6}{2}$$

change the division symbol to a multiplication symbol)

Step 2: Multiply the first fraction by the reciprocal of the second fraction

$$1 \cdot 6 / 2 \cdot 2 = 6/4$$

Step 3: Reduce answer to the lowest terms

$$6/4 = 1 \frac{2}{4} = 1 \frac{1}{2}$$

Table 37: Division with Fractions

Most people find performing arithmetic with fractions awkward. This is especially true when working with a hand-held calculator. Using decimals instead of fractions is much easier. Many calculations in statistics are technically fractions because there is a numerator and a denominator. The calculations, however, are performed using decimal numbers.

B. Decimals

Decimals, like fractions, represent a part of a whole. Decimals are, in fact, a special form of fractions. Decimals are structured using the base-10 number system. The denominators are powers of 10 (10, 100, 1,000, etc.).

| |
|-------------------|
| Millions |
| Comma |
| Hundred Thousands |
| Ten Thousands |
| Thousands |
| Comma |
| Hundreds |
| Tens |
| Units |
| Decimal Point |
| Tenths |
| Hundredths |
| Thousandths |

Figure 15: Decimals and Place-Values

Example 1:

Instead of writing $\frac{1}{2}$, we write 0.5, where “0” is the units place, and after the decimal point (.), the “5” is in the tenths place: $0.5 = \frac{5}{10}$, or $\frac{1}{2}$.

| | |
|-------------------|---|
| Millions | |
| Comma | , |
| Hundred Thousands | |
| Ten Thousands | |
| Thousands | |
| Comma | , |
| Hundreds | |
| Tens | |
| Units | 0 |
| Decimal Point | . |
| Tenths | 5 |
| Hundredths | |
| Thousandths | |

Figure 16: Decimals and Place-Values, Example 1
Example 2:

Instead of writing $\frac{1}{4}$, we write 0.25, where “0” is in the units place, and after the decimal point (.), the “2” is in the tenths place and a “5” in the hundredths place; 0.25 means $\frac{25}{100}$ or $\frac{1}{4}$.

| | |
|-------------------|---|
| Millions | |
| Comma | , |
| Hundred Thousands | |
| Ten Thousands | |
| Thousands | |
| Comma | , |
| Hundreds | |
| Tens | |
| Units | 0 |
| Decimal Point | . |
| Tenths | 2 |
| Hundredths | 5 |
| Thousandths | |

Figure 17: Decimals and Place-Values, Example 2

Converting Fractions into Decimals:

Example 1: Convert $\frac{2}{5}$ to a decimal

Divide 2 by 5 for the answer: $2 \div 5 = 0.4$

Example 2: Convert $\frac{6}{8}$ to a decimal

Divide 6 by 8 for the answer: $6 \div 8 = 0.75$

Converting Decimals into Fractions:

Convert 0.75 to a fraction. Follow these three steps:

| | |
|---|---------------------------|
| Step 1: Place the decimal in the numerator and 1 in the denominator: | $0.75/1$ |
| Step 2: Multiply the numerator and denominator by 100 | $0.75*100/1*100 = 75/100$ |
| Step 3: Reduce answer to the LCD | $75/100 = \frac{3}{4}$ |

Table 38: Converting Decimals into Fractions

C. Percentages

A percentage is a rate, number, or amount per one hundred. Percentages are used all the time: The sales tax rate, the rate of interest, income tax rates, or the proportion of your high school graduating class that went to college. Percent literally “means per 100.” That is why there are 100 pennies (cents) in a dollar.

Percentages, or percents for short, are a special kind of fractions just like decimals. What makes percentages special is that the denominator is always 100. Because this denominator is always 100, we do not write it. Instead we designate the number as a percentage by placing the percentage sign (%) after the number.

| |
|--|
| $0\% = \frac{0}{100} = 0$ |
| $2\% = \frac{2}{100} = 0.02$ |
| $5\% = \frac{5}{100} = 0.05$ |
| $25\% = \frac{25}{100} = 0.25$ |
| $50\% = \frac{50}{100} = 0.50$ |
| $85\% = \frac{85}{100} = 0.85$ |
| $100\% = \frac{100}{100} = 1$ |
| $135\% = \frac{135}{100} = 1\frac{35}{100} = 1.35$ |

Figure 18: Percentages to Fractions to Decimals

Converting Decimals to Percentages

$$0.25 = 25\%$$

I decided to give Larry a quick lesson on percentages and rate changes. I gave him the following formula: **Base Rate x Change in Base Rate equals the New Rate**. Larry immediately asked for help with the calculation. Larry's Base Rate was \$1,000. His rate cut was 0.25 (a cut of 25%). The **Change in Rate is 75 percent of his base, found by $1.00 - 0.25 = 0.75$** . This means that Larry's new rate is 75% of his old rate. Larry's new weekly fee is \$750, found by $\$1,000 \times .75 = \750 . Larry has lost \$250 in weekly fees. More tears flowed. I paid for lunch.

A few months later, Larry and I met for lunch again. He had good news. A new executive had replaced the one who cut his fee. The new executive likes Larry and decided to raise his weekly fee by 25 percent. Larry was ecstatic. He told me, "This 25 percent raise will offset that 25 percent fee reduction. I'm back to my old rate of \$1,000. Right?"

Should Larry be happy? Does his additional 25 percent really cancel his earlier 25 percent cut? I told Larry that it was time for another lesson. I reminded him about the formula I had given him: **Base Rate x Change in Base Rate equals New Rate**. His Base Rate is now \$750. Given the fact that he was getting a 25 percent increase, the Change in Base Rate is 1.25 (that is to say, a 25 percent increase over his Base Rate). His new weekly consulting fee is \$937.50, found by $\$750 \times 1.25 = \937.50 . Larry was [crestfallen](#) when he realized his new fee was \$62.50 below his original fee. As I paid for lunch, I told Larry that a 25 percent cut of his old base is not restored by a 25 percent increase on his reduced base.

Here is another example: Let's say you are at a shopping center and you see a sweater in a store window. The price of the sweater is \$100. You decide that you cannot afford to buy it. You know that if you wait until the end of the season, you might be able to buy the sweater at a much lower price. A month goes by and a friend calls to tell you that

the sweater is now on sale for 50 percent off: $\$100 \times 0.50 = \50 . Half price is a good deal, but you decide to wait. You think that the price of the sweater may be reduced again by 50 percent. If there is a second 50 percent reduction, would the sweater be free? $\$100 - \$50 = \$50 = \0 ? The answer, of course, is no. The new reduced price is 50 percent of \$50: $\$50 \times 0.50 = \25 . To repeat, the second 50 percent reduction would bring the price of the sweater down to \$25, a quarter of the original price.

When comparing percentage changes, we need to know both the rate of change and the base. Let's compare rates of change for sales of two lemonade makers: Minute Maid and your two cousins, eight-year-old Coco and her ten-year-old sister Cola. Suppose sales for both enterprises increase 10 percent from 2018 to 2019. This information tells us very little if we do not know the base from which both suppliers' sales grew. This context is very important. According to IRI, Minute Maid's total retail sales for the 52-week period ending on June 17, 2018 was \$81.53 million, or \$81,530,000.^{ix} A ten percent increase in annual sales for Minute Maid lemonade would represent total annual sales of \$89.68 million ($\81.53×1.1) or an increase of \$8.15 million over the base year. This is a huge increase. Executives will get big promotions and year-end bonuses. Now your two cousins had only \$8.15 in total sales when they set up their lemonade stand in front of your aunt's house in June 2018. Their sales also rose 10 percent when they set up their lemonade stand in June 2019. Your cousin's sales are now \$8.97 ($\8.15×1.1). Sadly, there is not much they can buy with their additional revenue of 82¢. The big lesson here is that whenever percentage increases or decreases are reported, we should know the base. The second lesson is when the base is small, large percentage increases or decreases are relatively common. Going

from sales of \$8.15 to \$16.30 should not be too difficult. But, doubling sales of \$81.53 million, going from \$81.53 million to \$163.06 million, is very unlikely.

People mishandle percentages all the time. In 1993, A. K. Dewdney published *200% of Nothing*, a wonderful book about this phenomenon and other forms of innumeracy.

Dewdney begins his first chapter by reporting on a magazine advertisement for Northeast Utilities, a New England utility company. The advertisement is about savings its customers could reap by switching light bulbs and fixtures. The ad promise customers “amazing savings of ‘200 percent on energy.’”^x

Is it possible to save 200 percent on anything? What is the highest rate reduction possible? To find this value we need a new formula: Price Reduction in Dollars over the Current Price equals the Percent Decrease. Suppose a customer of Northeast Utility was paying an average monthly fee of \$100. If this customer’s fee were lowered by \$75, the fee would be reduced by 75%, found by $\$75/\$100 = 0.75$ or 75%. Now let’s say that Northeast Utilities decides to offer its services for free. What is the rate reduction? The answer is 100%, found by $\$100/\$100 = 1.00$ or 100%. 100 percent is the largest possible price reduction. A 200 percent price reduction is absurd. Of course, Northeast Utilities could raise this customer’s monthly fee by 200 percent. That would be a \$200 increase on top of the base rate of \$100. The new monthly fee would be \$300.

The foolishness about price reductions can be found everywhere. A rather disturbing example can be found on a book sold on [Amazon](#). While one normally should not judge a book by its cover, we can make an exception in this case. To prove my point, let me

show you just part of this book's lengthy title: *Get What You Pay For: Save 200% on stocks, mutual funds, 40% on vehicle insurance, 300% on life insurance, 100% on long-term care...*

How can I save 200% on my investments? 300% on life insurance? If I get 100 percent off my long-term care or disability insurance, I my long-term care is free, how can I actually get a larger price reduction? I fear that I will

get what I paid for: nothing. I do not think that I want to take investment advice from the author, Dan Keppel, who seems confused about how to calculate price decreases.

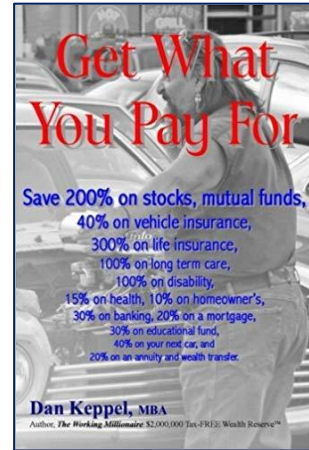


Figure 21: *Get What You Pay For*

E. Ratios

Ratios are expressions that compare qualities. They measure how much of one thing there is compared to another thing.

Here is a simple example that illustrates what ratios look like:

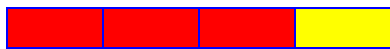


Figure 20: 3:1 Ratio

The figure has four cells. Three are red, and one is yellow. The ratio of red to yellow cells is expressed as 3:1, or 3 to 1. The first term, 3, is called the “antecedent.” The second term, 1, is called the “consequent.”

With the 9-cell table below, what is the ratio of red to yellow cells?



Figure 21: 6:3 Ratio

The ratio of red to yellow cells is 6:3. Typically we reduce the ratio to the least common denominator, which reduces the ratio to 2:1.

Ratios and the Pizza Example:

Barry
eats 3 slices



$$\frac{3}{12} = \frac{1}{4}$$

Garry
eats 2 slices



$$\frac{2}{12} = \frac{1}{6}$$

Harry
eats 1 slice



$$\frac{1}{12}$$

Figure 22: 3:1 Ratio

For each slice of pizza Harry eats, Garry eats two and Barry eats three. We can express these ratios mathematically as:

$$1 : 2 : 3$$

Figure 23: Fractions to Ratios

VII. Dealing with Negative Numbers

With the exception of zero, numbers are either positive or negative. Negative numbers are reported with a minus sign (-) sign. A negative number is defined as a real number that is less than zero while a positive number is greater than zero. A real number is any number that can be found on a number line. Numbers on the numbers line include [fractions](#), [whole numbers](#), [rational numbers](#), and [irrational numbers](#).

The number line is a straight line with zero in its center. On the right of zero, numbers increase on size from 1, 2, 3 to infinity. On the left of zero, numbers decrease in size from -1, -2, -3 to negative infinity. At the center is the number zero.

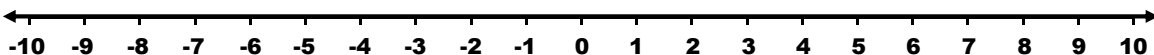


Figure 24: The Number Line

Rational numbers are numbers that can be written as a ratio between two integers ($5.0 = 10/2$, $2.5 = 5/4$, or $1/3 = 0.3333$). Irrational numbers are numbers that cannot be expressed as a ratio between two numbers or written as a simple fraction because there is not a finite number of decimal places when written as a decimal. The square root of 2 ($1.4142135623731\dots$) and π ($3.1415926535898\dots$) are examples of irrational numbers.

Negative numbers signify a loss or a deficiency. The number -5 is pronounced as “negative five” or “minus five.” Positive numbers carry plus, “+” sign or no sign at all; $+5 = 5$. A negative number is on the opposite side of its corresponding positive number. In the case of -5, the opposite number is +5. For example, $-(-5) = 5$, and $-5 + 5 = 0$. The negative example of a positive number is referred to as its negation or opposite number.

We use negative numbers in statistics. We see negative numbers in the media all the time. Here is one example: During the first quarter of 2018, Sears Holdings Corporation, the parent company of Sears and Kmart, reported a net loss of \$424 million.^{xi} The Sears Holding Corporation’s loss for first quarter 2018 was -\$424 million.

Dealing with Negative Numbers with Arithmetic Calculations

1) Addition

a. When adding positive numbers, the sum will always be positive

$$\begin{array}{r} \text{Addend} + \text{Addend} + \text{Addend} = \text{Sum} \\ 96 + 136 + 1,530 = 1,762 \end{array}$$

Table 39: Adding Positive Numbers

b. When adding negative numbers, add the addends and keep the negative sign

$$\begin{array}{r} \text{Addend} + \text{Addend} + \text{Addend} = \text{Sum} \\ -96 + -136 + -1,530 = -1,762 \end{array}$$

Table 40: Adding Negative Numbers

c. When adding two numbers of opposite signs, the sum is obtained by finding the difference between the two addends and keeping the sign of the more extreme number.

$$\begin{array}{r r r r r} \text{Addend} & + & \text{Addend} & = & \text{Sum} \\ -36 & + & 49 & = & 13 \end{array}$$

$$\begin{array}{r r r r r} \text{Addend} & + & \text{Addend} & = & \text{Sum} \\ 36 & + & -49 & = & -13 \end{array}$$

Table 41: Adding Numbers with Opposite Signs

d. When adding more than two signed numbers, add the positive addends (the sign will be positive) add all the negative addends (the sign will be negative), then add the two sums.

$$\begin{array}{r r r r r r r} \text{Addend} & + & \text{Addend} & + & \text{Addend} & + & \text{Addend} & = & \text{Sum} \\ 96 & + & -136 & + & -1,530 & + & 359 & = & ? \end{array}$$

$$\begin{array}{r r r r r} \text{Addend} & + & \text{Addend} & = & \text{Sum} \\ 455 & + & -1,666 & = & -1,211 \end{array}$$

Table 42: Adding More than Two Signed Numbers

2) Subtraction

a. When subtracting positive numbers, if the minuend is larger than the subtrahend then the difference is positive. If the minuend is equal to the subtrahend, the difference is zero. If the minuend is smaller than the subtrahend, the difference is negative.

$$\begin{array}{r r r r r} \text{Minuend} & - & \text{Subtrahend} & = & \text{Difference} \\ & & d & & \\ 136 & - & 96 & = & 40 \end{array}$$

Table 43: Subtracting Positive Numbers

b. When subtracting negative numbers, the more extreme number is the minuend, the less extreme number is the subtrahend. Add the two numbers and keep the negative sign.

$$\begin{array}{r r r r r} \text{Minuend} & - & \text{Subtrahend} & = & \text{Difference} \\ & & d & & \\ -136 & - & -96 & = & ? \end{array}$$

Table 44: Subtracting Negative Numbers

Step 1: Rewrite the problem as an addition problem: $-136 - -96 = -136 + 96$.

Step 2: Complete the problem as if it were an addition problem: $-136 + 96 = -40$.

Subtraction of a positive number yields the same result as adding a negative number:

$$\$150 - \$75 = \$75 \text{ equals } \$150 + -\$75 = \$75.$$

Subtracting a negative number returns the same result of adding a positive number of equal magnitude: $\$75 - \$150 = -\$75$ equals $\$75 + -\$150 = -\$75$.

2) Multiplication

When multiplying two negative numbers, the product is always positive: $-2 \times -5 = 10$. When multiplying one negative and one positive number, the product is always negative: $-2 \times 5 = -10$ or $2 \times -5 = -10$.

4) Division

With division, negative numbers follow the same rules as negative numbers in multiplication. When both the divisor and dividend are of the same sign, both are positive or both negative, the result is positive: $-10 \div -2 = 5$, $-2 \div -10 = 0.2$, $10 \div 2 = 5$, $2 \div 10 = 0.2$

If only the divisor or the dividend is negative, the result will be negative:

- $10 \div -2 = -5$
- $-10 \div 2 = -5$.

Absolute Values

An absolute value represents the distance from that number and zero regardless of whether the number is positive or negative. Absolute values are symbolized by surrounding the number with two vertical lines, $|X|$. For example, $|-7|$ is seven units away from zero regardless whether the original number is -7 or $+7$. You can think of absolute values as removing the negative sign in front of a number and thereby turning that number into a positive number.

VIII. Exponents and Square Roots

A) Exponents

An exponent refers to the number of times a number is multiplied by itself. The exponent of a number is shown in a superscript. We speak of exponents as the number of times the base number is raised, or the power of a number: $\text{Base}^{\text{exponent}}$. Exponents are used all the time in statistics. Let's look at the "power" of 10 and the "power" of 2.

| n^x | Value | How it is pronounced |
|--------|--|---------------------------------|
| 10^0 | 1 | 10 to the zero power |
| 10^1 | $10 \times 1 = 10$ | 10 to the 1 st power |
| 10^2 | $10 \times 10 = 100$ | 10-squared |
| 10^3 | $10 \times 10 \times 10 = 1,000$ | 10-cubed |
| 10^4 | $10 \times 10 \times 10 \times 10 = 10,000$ | 10 to the fourth power |
| 10^5 | $10 \times 10 \times 10 \times 10 \times 10 = 100,000$ | 10 to the fifth power |
| 10^6 | $10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1,000,000$ | 10 to the sixth power |

Table 45: The Power of 10

| n^x | Value | How it is pronounced |
|-------|---|--------------------------------|
| 2^0 | 1 | 2 to the zero power |
| 2^1 | $2 \times 1 = 2$ | 2 to the 1 st power |
| 2^2 | $2 \times 2 = 4$ | 2-squared |
| 2^3 | $2 \times 2 \times 2 = 8$ | 2-cubed |
| 2^4 | $2 \times 2 \times 2 \times 2 = 16$ | 2 to the fourth power |
| 2^5 | $2 \times 2 \times 2 \times 2 \times 2 = 32$ | 2 to the fifth power |
| 2^6 | $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$ | 2 to the sixth power |

Table 46: The Power of 2

Please note: Zero to the zero power, 0^0 , is not zero. It is considered "undefined."^{xii} The good news is that you are unlikely to see 0^0 in your statistics class.

Mistakes to Avoid With Exponents

A. Never multiply the base by the exponent

Mistake 1: $10^2 = 20$, found by 10×2 (This is the wrong answer). Note: $10^2 \neq 20$.

Correct Answer: $10^2 = 10 \times 10 = 100$.

B. Do not add the exponents

Mistake 2: $10^2 + 10^3 = 10^5 = 10,000$ (This is the wrong answer).

Correct Answer: $10^2 + 10^3$ equals $100 + 1,100 = 1,100$.

C. Never multiply the bases

Mistake 3: $10^2 \times 10^3 = 100^5 = 10,000,000,000$ (This is the wrong answer).

Correct Answer: $10^2 \times 10^3 = 100 \times 1,000 = 1,000,000$.

2) Square Roots

Mathematically the square root of a number is the number when multiplied by itself results in the original number. For example, the square root of 100 is 10 and $10 \times 10 = 100$. Square roots are symbolized by the *radical* symbol, $\sqrt{\quad}$. The number under the *radical* symbol, $\sqrt{100}$, is called the radicand. The $\sqrt{x^2} = |x|$, the square root of 100 ($\sqrt{100}$) is 10.

Here is a table of square roots that are whole numbers.

| \sqrt{n} | Value |
|--------------|-------|
| $\sqrt{100}$ | 10 |
| $\sqrt{81}$ | 9 |
| $\sqrt{64}$ | 8 |
| $\sqrt{49}$ | 7 |
| $\sqrt{36}$ | 6 |
| $\sqrt{25}$ | 5 |
| $\sqrt{16}$ | 4 |
| $\sqrt{9}$ | 3 |
| $\sqrt{4}$ | 2 |

Table 47: Square Roots that are Whole Numbers

Calculating the square root of any number using a hand-held calculator is very easy.

All you do is enter the number, then hit the $\sqrt{\quad}$ key on your calculator, although some calculators may require you to press the square root key and then enter the number.

Calculating the square root of a number by hand is not easy. It is a matter of trial and error and it takes a bit of time. This is why most people who learned how to calculate a square root by hand in high school and have become accustomed to using a hand-held

calculator forgot how to do this operation. Let's try to calculate the $\sqrt{12}$. We know the $\sqrt{9} = 3$ and the $\sqrt{16} = 4$, so the $\sqrt{12}$ must be between 3 and 4. We can estimate $\sqrt{12}$ through a process of trial and error. Our first guess is 3.5.

| Attempt | Estimate | Square of Estimate | Comment |
|---------|----------|--------------------|----------|
| #1 | 3.5 | 12.25 | too high |
| #2 | 3.4 | 11.56 | too low |
| #3 | 3.45 | 11.9025 | too low |
| #4 | 3.475 | 12.075625 | too high |
| #5 | 3.47 | 12.0409 | too high |
| #6 | 3.46 | 11.9716 | too low |
| #7 | 3.465 | 12.006225 | too high |
| #8 | 3.463 | 11.992369 | too high |
| #9 | 3.464 | 11.999296 | too low |
| #10 | 3.46411 | 12.0000581 | too high |
| #11 | 3.464101 | 11.9999957 | too low |
| #12 | 3.464102 | 12.0000027 | too high |

Table 48: $\sqrt{12}$

Based on this table of our trial and error estimates, the square root of 12 is between 3.464101 and 3.464102. The exact answer, calculated using Microsoft Excel, is 3.46410161513775.

There are several ways of calculating square roots, cube roots, and other roots using Microsoft Excel. We will review how to do this throughout *Clear-Sighted Statistics*.

IX. The Mathematical Order of Operations (PEMDAS)

The accuracy of the calculations we perform doing statistical analysis requires an understanding of the mathematical order of operations. The mathematical order of operations is associated with the [acronym](#), PEMDAS, which stands for: Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction. The handy [mnemonic](#) for remembering PEMDAS, is “Please Excuse My Dear Aunt Sally.” My favorite version of this mnemonic is “Poor Emperor Maximilian Drowned At Sea.” It is a good mnemonic, but bad

history. Austrian-born Emperor Maximilian I of Mexico was executed by a firing squad on June 19, 1867 in Santiago de Querétaro, Mexico. The Holy Roman Emperor, Maximilian I, who is said to have travelled with his coffin, died on land in 1514. The Holy Roman Emperor, Maximilian II, died in 1576 in Regensburg, Germany in 1576 while preparing to invade Poland. Maximilian II's son, Maximilian III, was only an Archduke and he died in Vienna.

PEMDAS provides the operational order for the performance of arithmetical procedures to solve mathematical expressions. First, we solve the expression within the parentheses. Then we calculate exponents, which includes square roots. We then complete the operations of multiplication, division, addition, and subtraction. The mathematical order of operations applies whether done with a pencil and paper, a hand-held calculator, or with a computer program like Microsoft Excel.

Here are the details:

Step 1, P: The **P** stands for parentheses and any other grouping symbol like brackets [], or braces {}. The first step is to find the value of the expression within the parentheses or other grouping symbol.

Step 2, E: The **E** stands for exponents (the power to which a number is to be raised) or roots (square roots, cube roots, etc.). The second step is to find the value of the roots, which in introductory statistics are almost always square roots.

Step 3, MD: The **MD** stands for multiplication and division. Multiplication and division have equal rank in the order of operations. The calculations must be performed moving left to right.

Step 3, AS: The **AS** stands for addition and subtraction. Addition and subtraction have equal rank in the order of operations. Just like multiplication and division, the calculations must be performed moving left to right.

PEMDAS Problem 1: Solve this expression using PEMDAS: $3 \times (4 + 9) - 3^2 / 4 + 3$

Step 1: Perform the Operations Within the Parentheses

$$\begin{aligned} 3 \times (4 + 9) - 3^2 / 4 + 3 &= \\ 3 \times (13) - 3^2 / 4 + 3 & \end{aligned}$$

Step 2: Calculate the Exponents

$$\begin{aligned} 3 \times (13) - 3^2 / 4 + 3 &= \\ 3 \times (13) - 9 / 4 + 3 & \end{aligned}$$

Step 3: Perform the Multiplication and then the Division

$$\begin{aligned} 3 \times (13) - 9 / 4 + 3 &= \\ 39 - 2.25 + 3 & \end{aligned}$$

Step 4: Perform the Addition and then the Subtraction

$$\begin{aligned} 39 - 2.25 + 3 &= \\ \text{Answer: } 39 - 5.25 &= 33.75 \end{aligned}$$

PEMDAS Problem 2: $25 - 4 \times 6$

Step 1: Perform the Operations Within the Parentheses

$$\begin{aligned} 25 - 4 \times 6 &= \\ \text{There are no parentheses, go to step 2.} & \end{aligned}$$

Step 2: Calculate the Exponents

$$\begin{aligned} 25 - 4 \times 6 &= \\ \text{There are no exponents, go to step 3.} & \end{aligned}$$

Step 3: Perform the Multiplication and then the Division

$$\begin{aligned} 25 - 4 \times 6 &= \\ 25 - 24 & \end{aligned}$$

Step 4: Perform the Addition and then the Subtraction

$$\text{Answer: } 25 - 24 = 1$$

PEMDAS Problem 3: $100 - 5 \times 3^2 + (22 + 2)$

Step 1: Perform the Operations Within the Parentheses

$$\begin{aligned} 100 - 5 \times 3^2 + (22 + 2) &= \\ 100 - 5 \times 3^2 + 24 & \end{aligned}$$

Step 2: Calculate the Exponents

$$\begin{aligned} 100 - 5 \times 3^2 + 24 &= \\ 100 - 5 \times 9 + 24 & \end{aligned}$$

Step 3: Perform the Multiplication and then the Division

$$\begin{aligned} 100 - 5 \times 9 + 24 & \\ 100 - 45 + 24 & \end{aligned}$$

Step 4: Perform the Addition and then the Subtraction

$$\begin{aligned} 100 - 45 + 24 &= \\ 55 + 24 &= \\ \text{Answer: } 55 + 24 &= 79 \end{aligned}$$

PEMDAS Problem 4: $8 \times (4 + 4^2) \div 5$

Step 1: Perform the Operations Within the Parentheses

$$\begin{aligned} 8 \times (4 + 4^2) \div 5 & \\ 8 \times (4 + 16) \div 5 & \\ 8 \times (20) \div 5 &= \end{aligned}$$

Step 2: Calculate the Exponents

$$\begin{aligned} 8 \times (20) \div 5 &= \\ \text{There are no exponents, go to step 3.} & \end{aligned}$$

Step 3: Perform the Multiplication and then the Division

$$\begin{aligned} 8 \times (20) \div 5 &= \\ 160 \div 5 &= 32 \end{aligned}$$

Step 4: Perform the Addition and then the Subtraction

There are no addition or subtraction steps.
Answer = 32.

PEMDAS Problem 5: $(3 - 6)^2 + (16/2)^2$

Step 1: Perform the Operations Within the Parentheses

$$(3 - 6)^2 + (16/2)^2 =$$
$$(-3)^2 + (8)^2$$

Step 2: Calculate the Exponents

$$(-3)^2 + (8)^2 =$$
$$9 + 64$$

Step 3: Perform the Multiplication and then the Division

$$9 + 64$$

There are no multiplication or division steps, go to step 4.

Step 4: Perform the Addition and then the Subtraction

$$\text{Answer} = 9 + 64 = 73$$

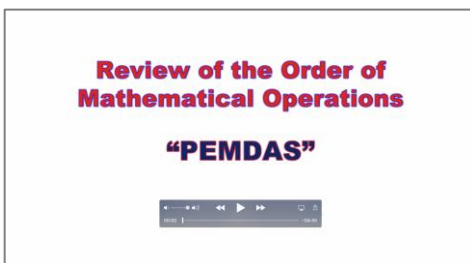


Figure 25: Link to PEMDAS Video

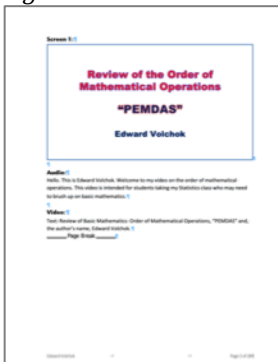
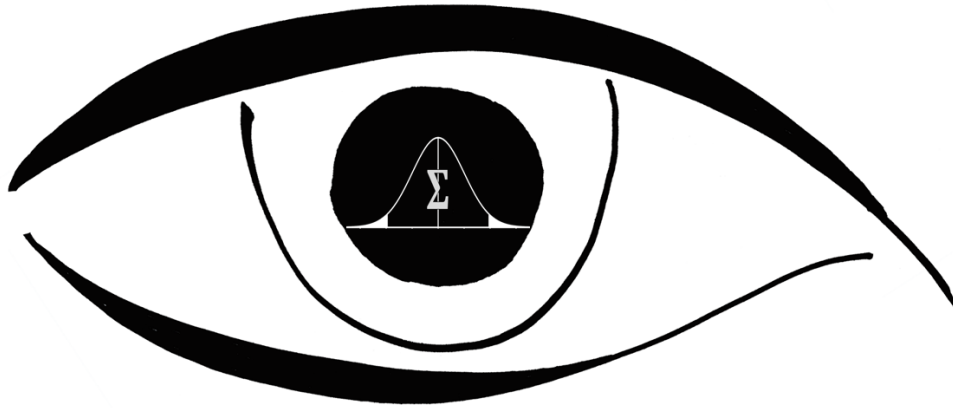


Figure 26: Link to PEMDAS Video Transcript

CLEAR-SIGHTED STATISTICS



EDWARD VOLCHOK



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* * *

ⁱ L. V. Jones (ed.) *The Collected Works of John W. Tukey: Philosophy and Principles of Data Analysis 1949-1964, Volume III.* (New York: Chapman & Hall, 1986), p. 66.

ⁱⁱ Liaqat Aki Khan, "What is Mathematics – An Overview," *International Journal of Mathematics and Computational Science*. Vol. 1, No. 3, 2015. p. 98. [Link to "What is Mathematics – An Overview"](#).

ⁱⁱⁱ W. H. Cockcroft. *Mathematics Counts: Report of the Committee of Inquiry into the Teaching of Mathematics in Schools Under the Chairmanship of Dr. WH Cockcroft.* (London: Her Majesty's Stationery Office, 1982), p. 11. [Link to Mathematics Counts](#).

^{iv} U.S. Census Bureau, <https://www.census.gov/quickfacts/fact/table/US/PST045218>

^v [Source: FBI Uniform Crime Reports 2017](#)

^{vi} [Source: Cato Institute, Immigration Research and Policy Brief No. 4](#)

^{vii} [Source: CNBC, 9/12/17](#)

^{viii} [Source: Student Loan Hero, 10/14/16](#)

^{ix} <https://www-statista-com/772875/us-dollar-sales-of-the-leading-refrigerated-lemonade-brands/>

^x A. K. Dewdney (1993). *200% of Nothing*. (New York: J. Wiley & Sons), p. 4.

^{xi} "Sears Holdings Reports First Quarter 2018 Results," May 31, 2018. Sears Holdings, [1st Q 2018 Results](#).

^{xii} <http://www.math.utah.edu/~pa/math/Oto0.html>