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Abstract

This paper gives a theorem for the continuous time super-replication cost of European options in an unbounded multinomial market. An approximation multinomial scheme is put forward on a finite time interval [0,1] corresponding to a pure jump Lévy model with unbounded jumps. Under the assumption that the expected underlying stock price at time 1 is bounded, the limit of the sequence of the super-replication cost in a multinomial model is proved to be greater than or equal to an optimal control problem. Furthermore, it is discussed that the existence conditions of a super-replication cost and a liquidity premium for the multinomial model. This paper concentrates on a multinomial tree with unbounded jumps, which can be seen as an extension of the work of(Xing, 2015). The super-replication cost and the liquidity premium under the variance gamma model and the normal inverse Gaussian model are calculated and illustrated.

Keywords: multinomial model, super-replication cost, Lévy process

1. Introduction

This paper examines the super-replication cost and the liquidity premium of a European claim in models where the stock price follows an exponential multinomial tree scheme. Super-replication cost or related problems were studied in (Cetin, Soner & Touzi, 2010), (Gokay & Soner, 2012), (Dolinsky & Soner, 2012) and (Xing, 2015). In (Cetin, Soner & Touzi, 2010), the existence of the continuous time super-replication cost from a binomial model was shown. (Gokay & Soner, 2012) showed that a stochastic optimal control problem could be seen as a dual of the super-replication problem. A duality result implicitly through a partial differential equation was obtained by (Gokay & Soner, 2012)though their result was not given explicitly. And, the proof given in it was limited to Markovian claims. The same one-dimensional binomial model introduced by (Cetin, Jarrow & Protter, 2004), (Cetin & Roger, 2007) and (Gokay & Soner, 2012) was also used in (Dolinsky & Soner, 2012). That paper was based on nonmarkovian claims and more general liquidity functions. In (Dolinsky & Soner, 2012), the dual of the discrete model was derived as an optimal control problem and the construction given in (Kusuoka, 1995) was applied to prove that a liquidity premium exists. All the continuous time or discrete time super-replication works above were based on a binomial model. However, the restriction to a binomial model does not actually frequently used for building stochastic models in finance, economics and many other fields. (Xing, 2015) extended the super-replication problem to a one-dimensional multinomial model. The existence of the liquidity premium is proved. However, The multinomial tree in that paper is bounded and the super-replication cost was not calculated in that paper.

In this present paper, the super-replication cost and the liquidity premium problem is extended to an unbounded multinomial model which should have the practical importance in the real world. The conditions under which the existence of the liquidity premium is discussed. Moreover, the super-replication cost and the liquidity premium under the multinomial model with respect to the variance gamma (VG) model and the normal inverse Gaussian (NIG) model are calculated . The multinomial approximation used in this present paper is similar to that proposed by (Maller, Solomon & Szimayer, 2006). In a finite time interval [0, 1], this multinomial scheme is based on a discrete grid with finite number of states. The range of the state value will goes to infinity as $n \to \infty$. It is proved that the discrete multinomial scheme converges to a continuous time unbounded Lévy model in probability under the Skorokhod J_1 topology. To show this, the boundedness of the expected stock price at time 1 is assumed. The prove method used in this present paper overcomes the difficulties from the unbounded jump size or unbounded state value in the multinomial tree.

The remainder of this paper is organized as follows. The stochastic setup is outlined in section 2. In section 3, the main results of this paper are presented, proved and discussed. Section 4 calculate, compare and illustrate the super-replication cost and the liquidity premiums for an European put option under the variance gamma (VG) model and the normal inverse Gaussian (NIG) model. The conclusions and discussions are given in section 5.

2. Overview, and Stochastic Setup

2.1 Setup

Let $\mathbb{Z}^{\mathbb{N}}$ be the space of infinite sequences $\omega = (\omega_1,\ \omega_2,\ \dots)$ with the product probability, in which $\omega_i \in \mathbb{Z}$. Define the canonical sequence of random variables $X_1(n),\ X_2(n),\ \cdots$ by

$$X_i(\omega(n)) = \omega_i(n),\ i \in \mathbb{N},$$

and consider the natural filtration $\mathcal{F}_k(n) = \sigma\{X_1(n),\ X_2(n),\ \cdots,\ X_k(n)\},\ k \in \mathbb{N}$, and let $\mathcal{F}_0(n)$ be trivial.

Let $\mathbb{D}[0,\ 1]$ be the space of cádlág real-valued functions on $[0,\ 1]$ and $\rho(\cdot,\ \cdot)$ be the Skorokhod J_1 topology defined on $\mathbb{D}[0,\ 1]$. For the Skorokhod J_1 topology, the readers are referred to (Karatzas & Shreve, 1991) and (Jacod & Shiryaev, 2003). Let $F : \mathbb{D}[0,\ 1] \to \mathbb{R}_+$ be a continuous map. Assume that there exist constants $L,\ p > 0$ for which

$$F(y) \le L\left(1+ \parallel y \parallel_{Sk}^{p}\right),\ \ \forall y \in \mathbb{D}[0,\ 1], \tag{1}$$

where $\parallel \cdot \parallel_{Sk}$ is the Skorokhod norm.

Assume that $N(n) \uparrow \infty$ and that $\Delta(n) \downarrow 0$. For any n, suppose the n-step multinomial model of a financial market which is active at times $0,\ \frac{1}{N(n)},\ \frac{2}{N(n)},\ \cdots,\ 1$. Assume that the discrete stock price at time $\frac{k}{N(n)}$ is given by

$$S_n(k) = S_0 \exp\left(\sum_{j=1}^{k} X_j(n)\Delta(n)\right),\ \ k = 1,\ \cdots,\ N(n). \tag{2}$$

This multinomial scheme is similar to that proposed by (Maller et al., 2006) and (Xing, 2015).

Let the payoff function of a European claim with maturity $T = 1$ be

$$F_n := F(S_n). \tag{3}$$

Let $g(t,\ S,\ \lambda)$ be the trading cost function at time t, where S is the stock price and λ is the trading volume at time t. Assume that $g :\ [0,\ 1] \times \mathbb{D}[0,\ 1] \times \mathbb{R} \to [0,\ \infty)$ is nonnegative, adapted, convex for every $(t,\ S) \in [0,\ 1] \times \mathbb{D}[0,\ 1]$, and $g(t,\ S,\ 0) = 0$. The convex conjugate of g is denoted by G. For the definition of convex conjugate, the readers are referred to (Dolinsky & Soner, 2012).

The Theorem 3.1 in (Dolinsky & Soner, 2012) showed that the super-replication cost of a European claim with payoff F_n, denoted by V_n, is an optimal control problem in which the controller is allowed to choose any probability measure on $(\mathbb{Z}^{\mathbb{N}},\ \mathcal{F}(n))$ in their binomial model. (Xing, 2015) pointed out this conclusion is also true for the multinomial model. That is, the super-replication cost

$$V_n = \sup_{\mathbb{P} \in Q_n} \mathbb{E}^{\mathbb{P}}\left(F_n - \sum_{k=0}^{N(n)-1} G\left(\frac{k}{N(n)},\ S_n,\ \mathbb{E}^{\mathbb{P}}(S_n(N(n))|\mathcal{F}_k(n)) - S_n(k)\right)\right),$$

where Q_n is the set of all probability measures on $(\mathbb{Z}^{\mathbb{N}},\ \mathcal{F}(n))$ and $\mathbb{E}^{\mathbb{P}}$ denotes the expectation with respect to a probability measure $\mathbb{P} \in Q_n$.

2.2 Continuous Time Lévy Model

Let $(\Omega,\ \mathcal{F},\ \mathbb{P})$ be a completed probability space on which a Lévy process, $L = (L_t,\ 0 \le t \le 1)$, with càdlàg paths is defined. Let $\mathbb{F}^L = (\mathcal{F}_t^{\ L})_{0 \le t \le 1}$ be the right continuous filtration generated by L. This process is characterized by its Lévy triplet $(\gamma,\ \sigma,\ \nu)$, where γ is the drifting term, σ is the volatility of the Brownian motion part, and the Lévy measure ν is the intensity of the jump process of the Lévy process. Assume that $(\gamma,\ \sigma,\ \nu) = (0,\ 0,\ \nu)$. That means this Lévy process is determined by the Lévy measure, ν. By the Lévy-Ito Decomposition, L_t can be written as:

$$L_t = \int_0^t \int_{|x|<1} x\tilde{N}(dt, dx) + \int_0^t \int_{|x|\ge 1} xN(dt, dx), \tag{4}$$

where $N(\cdot,\ \cdot)$ is the associated independent Poisson random measure process on $\mathbb{R}^+ \times (\mathbb{R}\backslash\{0\})$ with intensity ν. And, for any $t \in [0,\ 1]$ and $A \in \mathcal{B}(\mathbb{R}\backslash\{0\})$, $\tilde{N}(t, A) = N(t, A) - t\nu(A)$ is the compensate Poisson process. For background of Lévy processes, the readers are referred to (Applebaum, 2004), (Bertoin, 1996) and (Sato, 1999). Special emphasis is placed on Lévy processes that have infinitely many jumps, almost surely, in any finite time interval.

Let μ be the Lebesgue measure and let $\mathcal{H}_2(1)$ be the linear space of all equivalent classes of predictable mappings f : $[0,\ 1]\times\Omega \to \mathbb{R}$ which coincide almost everywhere with respect to $\mu\times\mathbb{P}$ and $\|f\|^2 \triangleq \int_0^1 \mathbb{E}f(t)^2dt < \infty$. Let $\mathbb{F}^f = (\mathcal{F}_t^f)_{t\in[0,1]}$ be the natural filtration generated by $(f(t),\ t \in [0, 1])$. Define that $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$, where for any $t \in [0, 1]$, $\mathcal{F}_t$ is the smallest $\sigma-$algebra containing $\mathcal{F}_t^f$ and $\mathcal{F}_t^L$. Suppose that $\mathcal{F}_0$ contains all $\mathbb{P}-$null sets and that $\mathcal{F}_1 = \mathcal{F}$.

Definition 1 *For any $f(t) \in \mathcal{H}_2(1)$, let $S^f(t) \triangleq s_0 \exp(\varepsilon_t)$, where*

$$\varepsilon_t = \int_0^t \int_{\mathbb{R}} f(s)\cdot x\, N(ds,\ dx) - \int_0^t \int_{\mathbb{R}} \left[e^{f(s)\cdot x} - 1\right] \nu(dx)ds.$$

By Corollary 5.2.2 of (Applebaum, 2004), $S_H(t)$ is a local martingale.

Remark 1 (Xing, 2015) assume that $S_H = s_0exp(\epsilon_t)$, where $\epsilon_t = \int_0^t \int_{\mathbb{R}} H(s,\ x)\, N(ds,\ dx) - \int_0^t \int_{\mathbb{R}} \left[e^{H(s,\ x)} - 1\right] \nu(dx)ds$. In that paper, the boundedness of the integrand $H(t,\ x)$ is required. The boundedness restriction in fact limits the multinomial scheme to that with finite size of states only and in turn limits the corresponding Lévy process to be that with finite size of jumps. In this present paper, the integrand in the exponent of S^f is set to be $f(x)\cdot x$ as in above definition, where $x \in \mathbb{R}$ is unbounded, which is actually frequently used for building stochastic models in finance, economics and many other fields.

3. Main Results

3.1 Main Results

Assumption 1 We assume that the Lévy process, L, and the dual function G satisfy the following conditions:

(a) The moment generating function of the Lévy process, L, is finite, such as Variance gamma process (VG) and normal inverse Gaussian process (NIG). That is, $e^{L_1} < \infty$.

(b) There exists $m(n) \downarrow 0$ and a continuous function

$$\hat{G}:\ [0,\ 1] \times \mathbb{D}[0,\ 1] \times \mathbb{R} \to [0,\ \infty]$$

such that $\lim_{n\to\infty} \nu(I_n)\Delta(n) = 0$, $\lim_{n\to\infty} \frac{\bar{\nu}(m(n))^2}{N(n)} = 0$, where $\bar{\nu}(m(n)) = \nu((m(n),\ \infty) \cup (-\infty,\ -m(n)))$ and, for any bounded sequence $\{x_n\}$ and convergent sequences $t_n \to t,\ S_n \to S$ in the Skorohod norm,

$$\lim_{n\to\infty} \left|N(n)G\left(t_n,\ S_n,\ x_n\bar{\nu}(m(n))\Delta(n)O(1)S_n(t)\right) - \hat{G}(t,\ S,\ x_nO(1)S(t))\right| = 0.$$

Theorem 1 *Let G be a dual function of the trading cost function $g(t,\ S,\ \lambda)$ that satisfies Assumption 1. Then,*

$$\lim_{n\to\infty} V_n \geq \sup_{f\in\mathcal{A}_\alpha} J(S^f),$$

where

$$J(S^f) := \mathbb{E}\left\{F(S^f) - \int_0^1 \widehat{G}(t,\ S^f,\ (t-1)S^f(t))dt\right\}, \tag{5}$$

$\mathcal{A}_\alpha := \{f(t) \in \mathcal{H}_2(1):\ |f| < \alpha$ *and* $\mathbb{E}(S^f(t) = 1)$ *for any* $t \in [0,\ 1]\}$.

Remark 2 Corollary 5.2.2 of (Applebaum, 2004) showed that $S^f(t)$ is a local martingale if $f(t) \in \mathcal{H}_2(1)$. Moreover, by the Theorem 5.2.4 in (Applebaum, 2004), $S^f(t)$ is a martingale if $\mathbb{E}(S_H(t)) = 1$ for any $t \in [0,\ 1]$.

Remark 3 The goal of this paper is to discuss the conditions under which the right hand side of (5) is non-negative and, moreover, greater than $\mathbb{E}(F(S^{f_0}))$, where $f_0 = 1$ almost surely.

3.2 Multinomial Approximation Schemes and Proofs

To show Theorem 1, we need the following multinomial schemes and lemmas.

Fix $f(t) \in \mathcal{A}_\alpha$. By the proof of Lemma 4.1.4 of (Applebaum, 2004), a sequence of simple processes, $f_n(t)$, can be constructed such that, as $n \to \infty$,

$$\int_0^1 \mathbb{E}|f_n - f|^2dt \to 0. \tag{6}$$

Actually,

$$f_n(t) = f_n(t_{j-1}(n)) \tag{7}$$

if $t \in [t_{j-1}(n),\ t_j(n))$, $j = 1, \cdots, N(n)$, where $t_j(n)$'s are the grid points of an equal partition of the time interval $[0,\ 1]$. Let $\hat{S}_n(t) := s_0 \exp(\hat{\varepsilon}_n(t))$, where

$$\hat{\varepsilon}_n(t) := \int_0^t \int_{m(n)<|x|<M(n)} f_n(s) \cdot x\, N(ds,\ dx) - \int_0^t \int_{m(n)<|x|<M(n)} \left[e^{f_n(s)\cdot x} - 1\right] \nu(dx)ds, \tag{8}$$

with $m(n) \downarrow 0$ and $M(n) \uparrow \infty$ as $n \to \infty$.

Lemma 2 *Assume that Assumption 1(a) holds.*

$$\mathbb{E}\left(\sup_{t\in[0,\ 1]} |\hat{\varepsilon}_n(t) - \varepsilon(t)|\right) \longrightarrow 0.$$

That is, $\hat{\varepsilon}_n \overset{L^1}{\to} \varepsilon$ *as* $n \to \infty$. *Here and later,* "$\overset{L^1}{\to}$" *denotes convergence in mean.*

Proof. Let

$$\varepsilon_1^{(n)}(t) = \int_0^t \int_{|x|<m(n)} f(s) \cdot x\, N(ds,\ dx) - \int_0^t \int_{|x|<m(n)} \left[e^{f(s)\cdot x} - 1\right] \nu(dx)ds,$$

and

$$\varepsilon_2^{(n)}(t) = \int_0^t \int_{|x|>M(n)} f(s) \cdot x\, N(ds,\ dx) - \int_0^t \int_{|x|>M(n)} \left[e^{f(s)\cdot x} - 1\right] \nu(dx)ds.$$

Since $e^u - 1 - u \le u^2 e^{|u|}$,

$$\sup_{t\in[0,\ 1]} |\varepsilon_1^{(n)}(t)| \le \sup_{t\in[0,\ 1]} \left|\int_0^t \int_{|x|<m(n)} f(s) \cdot x\, \tilde{N}(ds,\ dx)\right| + \sup_{t\in[0,\ 1]} \left|\int_0^t \int_{|x|<m(n)} f(s)^2 \cdot x^2 e^{|f(s)\cdot x|} \nu(dx)ds\right|. \tag{9}$$

The second term on right of (9) above is bounded by $\int_0^1 \int_{|x|<m(n)} \alpha^2 \cdot x^2 e^{|\alpha m(n)|} \nu(dx)ds$ since $|f(s)| \le \alpha$ for all $s \in [0,\ 1]$. It follows from the definition of Lévy measure, $\int_0^1 \int_{x\neq 0} x^2 \nu(dx)ds < \infty$. Thus, $\int_0^1 \int_{|x|<m(n)} x^2 \nu(dx)ds \to 0$ as $n \to \infty$. And so does the second term on right of (9). For the first term on right of (9) is a martingale,

$$\begin{aligned}
\mathbb{E}^2\left(\sup_{t\in[0,\ 1]} \left|\int_0^t \int_{|x|<m(n)} f(s) \cdot x\, \tilde{N}(ds,\ dx)\right|\right) &\le \mathbb{E}\left(\sup_{t\in[0,\ 1]} \left|\int_0^t \int_{|x|<m(n)} f(s) \cdot x\, \tilde{N}(ds,\ dx)\right|^2\right) \\
&\le \mathbb{E}\left(\int_0^t \int_{|x|<m(n)} f(s) \cdot x\, \tilde{N}(ds,\ dx)\right)^2 \\
&= \int_0^1 \int_{|x|<m(n)} f(s)^2 x^2\, \nu(dx)ds \to 0,
\end{aligned}$$

as $n \to \infty$, where the second inequality is from the Doob's martingale inequality. Thus, $\mathbb{E}\left(\sup_{t\in[0,\ 1]} |\varepsilon_1^{(n)}(t)|\right) \to 0$ as $n \to \infty$. Consider that

$$\sup_{t\in[0,\ 1]} |\varepsilon_2^{(n)}(t)| \le \int_0^1 \int_{|x|>M(n)} |f(s) \cdot x|\, N(ds,\ dx) + \int_0^1 \int_{|x|>M(n)} e^{f(s)\cdot x} \nu(dx)ds + \bar{\nu}(M(n)). \tag{10}$$

Since $M(n) \to \infty$ as $n \to \infty$, $\bar{\nu}(M(n)) \to 0$. Moreover, by the assumption 1(a), $\int_0^1 \int_{|x|>M(n)} e^{f(s)\cdot x} \nu(dx)ds \to 0$ almost surely as $n \to \infty$. Also, $\mathbb{E}\left(\int_0^1 \int_{|x|>M(n)} |f(s) \cdot x|\, N(ds,\ dx)\right) \le \int_0^1 \int_{|x|>M(n)} \alpha|x|\, \nu(dx)ds \le \alpha \int_{|x|>M(n)} x^2\, \nu(dx) \to 0$ as $n \to \infty$. Thus, $\mathbb{E}\left(\sup_{t\in[0,\ 1]} |\varepsilon_2^{(n)}(t)|\right) \to 0$ as $n \to \infty$.

Now, to show this lemma, it suffices to show that

$$\mathbb{E}\left(\sup_{t\in[0,\ 1]} \left|\int_0^t \int_{m(n)<|x|\le M(n)} (f_n(s) - f(s)) \cdot x\, N(ds,\ dx) - \int_0^t \int_{m(n)<|x|\le M(n)} \left[e^{f_n(s)\cdot x} - e^{f(s)\cdot x}\right] \nu(dx)ds\right|\right) \to 0.$$

When $|x|$ is bounded, say $|x| \le 1$, the convergence in mean was proved in (Xing, 2015). Here, the convergence will be proved for $1 < |x| \le M(n)$. Indeed,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,\ 1]} \left|\int_0^t \int_{1<|x|\le M(n)} (f_n(s) - f(s)) \cdot x\, N(ds,\ dx) - \int_0^t \int_{1<|x|\le M(n)} \left[e^{f_n(s)\cdot x} - e^{f(s)\cdot x}\right] \nu(dx)ds\right|\right) \\
\le\ &\mathbb{E}\left(\int_0^1 \int_{1<|x|\le M(n)} |f_n(s) - f(s)| \cdot |x|\, N(ds,\ dx)\right) + \int_0^1 \int_{1<|x|\le M(n)} \mathbb{E}\left|e^{f_n(s)\cdot x} - e^{f(s)\cdot x}\right| \nu(dx)ds. \qquad (11)
\end{aligned}$$

Consider firstly the first term on right of (11). By Hölder inequality,

$$\begin{aligned}\mathbb{E}\left(\int_0^1\int_{1<|x|\le M(n)}|f_n(s)-f(s)|\cdot|x|\,N(ds,\ dx)\right) &= \int_0^1\int_{1<|x|\le M(n)}\mathbb{E}|f_n(s)-f(s)|\cdot|x|\,\nu(dx)ds\\ &\le \left[\int_{1<|x|\le M(n)}x^2\,\nu(dx)\right]^{1/2}\cdot\left[\int_0^1\mathbb{E}|f_n(s)-f(s)|^2ds\right]^{1/2}\\ &\to 0\end{aligned}$$

as $n\to\infty$ due to (6) and $\int_{1<|x|\le M(n)}x^2\,\nu(dx)<\infty$.

Next, for the second term on right of (11),

$$\begin{aligned}&\int_0^1\int_{1<|x|\le M(n)}\mathbb{E}\left|e^{f_n(s)\cdot x}-e^{f(s)\cdot x}\right|\nu(dx)ds\\ =&\int_0^1\int_{1<|x|\le M(n)}\mathbb{E}\left(e^{f(s)\cdot x}\left|e^{f_n(s)\cdot x-f(s)\cdot x}-1\right|\right)\nu(dx)ds\\ \le&\int_0^1\int_{1<|x|\le M(n)}\mathbb{E}\left(|(f_n(s)-f(s))x|\cdot e^{f(s)x+|f_n(s)-f(s)|x}\right)\nu(dx)ds\\ \le&\int_0^1\int_{1<|x|\le M(n)}\mathbb{E}\left(|(f_n(s)-f(s))x|\cdot e^{3\alpha|x|}\right)\nu(dx)ds\\ \le&\left[\int_0^1\mathbb{E}|f_n(s)-f(s)|^2ds\right]^{1/2}\cdot\left[\int_{1<|x|\le M(n)}x^2e^{6\alpha|x|}\,\nu(dx)\right]^{1/2}\\ \le&\left[\int_0^1\mathbb{E}|f_n(s)-f(s)|^2ds\right]^{1/2}\cdot\left[\int_{1<|x|\le M(n)}e^{(6\alpha+1)|x|}\,\nu(dx)\right]^{1/2},\end{aligned}$$

where the second inequality is from the assumption of $|f|,\ |f_n|<\alpha$ and the third inequality from the Hölder inequality. Then, by (6) and Assumption 1(a), $\int_0^1\int_{1<|x|\le M(n)}\mathbb{E}\left|e^{f_n(s)\cdot x}-e^{f(s)\cdot x}\right|\nu(dx)ds\to 0$ as $n\to\infty$.

Above all, the lemma is obtained.

Definition 4 *Let $\vartheta_j(n)$ be the number of jumps of $\varepsilon(t)$ in the subinterval, $(t_{j-1}(n),\ t_j(n)]$, with magnitude in $(m(n),\ M(n)]$, and $Y_j^k(n)$ be the size of the kth such jump that occurs at time $t_j^k(n)$, $k=1,2,3,\cdots,\vartheta_j(n)$.*

$$\begin{aligned}&J_n^{(2)}(t)\\ :=&\sum_{j=1}^{N(n)}1_{\{\vartheta_j(n)\ge 2\}}\sum_{k=2}^{\vartheta_j(n)}1_{\{t_j^k(n)\le t\}}\left[f_n(t_j^k(n))\cdot Y_j^k(n)-ln\left(1+\frac{\int_{\mathbb{R}}(e^{f_n(t_{j-1}(n))x}-1)\nu(dx)}{\bar{\nu}(m(n))}\right)\right],\end{aligned}$$

That means, $J_n^{(2)}(t)$ collects all of the jumps in each subinterval with magnitude in $(m(n),\ M(n)]$ except for the first one.

Lemma 3 *Assume that Assumption 1 holds. Let*

$$\check{\varepsilon}_n(t)=\sum_{j=1}^{N(n)}1_{\{t_j^1(n)\le t\}}\left\{(f_n(t_j^1(n))Y_j^1(n)-\ln\left[1+\frac{B_n^j}{\nu(I_n)}\right]\right\},$$

where $I_n=[-M(n),\ -m(n))\cup(m(n),\ M(n)]$ and $B_n^j=\int_{I_n}\left(e^{f_n(t_j^1(n))x}-1\right)\nu(dx)$. Then, $\mathbb{P}\left(\sup_{t\in[0,\ 1]}|\check{\varepsilon}_n(t)-\varepsilon(t)|\right)\longrightarrow 0$. That is, $\check{\varepsilon}_n\overset{P}{\to}\varepsilon$ as $n\to\infty$. Here and later, "$\overset{P}{\to}$" denotes convergence in probability.

Proof. Notice that

$$\hat{\varepsilon}_n(t)-\check{\varepsilon}_n(t)=J_n^{(2)}(t)+\sum_{j=1}^{N(n)}1_{\{\vartheta_j(n)\ge 1\}}\sum_{k=1}^{\vartheta_j(n)}1_{\{t_j^k(n)\le t\}}\ln\left[1+\frac{B_n^j}{\nu(I_n)}\right]-\int_0^t\int_{I_n}\left(e^{f_n(s)x}-1\right)\nu(dx)ds. \tag{12}$$

By Definition 4 above, $\vartheta_j(n)$, $j = 1,\ 2,\ \cdots,\ N(n)$, are independent Poisson rvs with expectation $\nu(I_n)/N(n)$. Now,

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,\ 1]}\left|J_n^{(2)}(t)\right|>0\right) &\le \mathbb{P}\left(\sum_{j=1}^{N(n)} 1_{\{\vartheta_j(n)\ge 2\}}\sum_{k=2}^{\vartheta_j(n)}\left|f_n(t_j^k(n))\cdot Y_{j,k}(n)\right|>0\right)\\ &\le \sum_{j=1}^{N(n)}\mathbb{P}(\vartheta_j(n)\ge 2)\\ &= \sum_{j=1}^{N(n)}\left[1-e^{-\frac{\nu(I_n)}{N(n)}}\left(1+\frac{\nu(I_n)}{N(n)}\right)\right] = O\left(\frac{\nu(I_n)^2}{N(n)}\right)\to 0,\end{aligned}$$

as $n\to\infty$ by Assumption 1(c).

Next, let the last two terms on left of (12) be $H_n(t)$. Actually, $\mathbb{E}\left(e^{H_n(t)}\right)\equiv 1$, $\forall t\in[0,\ 1]$. Indeed, assume that $t\in[t_l(n),\ t_{l+1}(n))$ and in $[t_l(n),\ t)$, there are $\bar{\vartheta}_{l+1}(n)$ jumps with size in I_n, then

$$\mathbb{E}\left(e^{H_n(t)}\right)=\mathbb{E}\left(\frac{\prod_{j=1}^{l}\left\{\exp\left[\sum_{k=1}^{\vartheta_j(n)}\ln\left(1+\frac{B_n^j}{\nu(I_n)}\right)\right]\right\}\cdot\exp\left[\sum_{k=1}^{\bar{\vartheta}_{l+1}(n)}\ln\left(1+\frac{B_n^{l+1}}{\nu(I_n)}\right)\right]}{e^{\int_0^t\int_{I_n}\left(e^{f_n(s)x}-1\right)\nu(dx)ds}}\right). \tag{13}$$

Since $\{\vartheta_j(n)\}$'s and $\{B_n^j\}$'s are independent and $\forall j$, $\vartheta_j(n)$ is a Poisson rv with expectation $\frac{\nu(I_n)}{N(n)}$, the moment generating function of the compound Poisson distribution,

$$\mathbb{E}_{\vartheta_j(n)}\left(\exp\left[\sum_{k=1}^{\vartheta_j(n)}\ln\left(1+\frac{B_n^j}{\nu(I_n)}\right)\right]\right)=e^{\frac{B_n^j}{N(n)}}, \tag{14}$$

where $\mathbb{E}_{\vartheta_j(n)}$ is the expectation with respect to the Poisson rv $\vartheta_j(n)$. Similarly, $\bar{\vartheta}_{l+1}(n)$ follows Poisson distribution with $\mathbb{E}_{\bar{\vartheta}_j(n)}=\nu(I_n)(t-t_l(n))$, and so

$$\mathbb{E}_{\bar{\vartheta}_j(n)}\left(\exp\left[\sum_{k=1}^{\bar{\vartheta}_{l+1}(n)}\ln\left(1+\frac{B_n^{l+1}}{\nu(I_n)}\right)\right]\right)=e^{B_n^{l+1}(t-t_l(n))}. \tag{15}$$

Also, by (7), the step random variable $f_n(s)=f_n(t_{j-1}(n))$ when $s\in[,t_{j-1}(n),\ t_j(n)]$. Then, the denominator of (13) can be rewritten as

$$e^{\int_0^t\int_{I_n}\left(e^{f_n(s)x}-1\right)\nu(dx)ds}=\left[\prod_{j=1}^{l}e^{\frac{B_n^j}{N(n)}}\right]\cdot e^{B_n^{l+1}(t-t_l(n))}. \tag{16}$$

By (13), (14),(15),(16), $\mathbb{E}\left(e^{H_n(t)}\right)\equiv 1$. Hence, $\left\{e^{H_n(t)}\right\}_{t\in[0,\ 1]}$ is a martingale.

Furthermore, by Doob's martingale inequality,

$$\mathbb{E}^2\left(\sup_{t\in[0,\ 1]}\left|e^{H_n(t)}-1\right|\right)\le\mathbb{E}\left(\sup_{t\in[0,\ 1]}\left|e^{H_n(t)}-1\right|^2\right)\le 4\left[\mathbb{E}\left(e^{2H_n(1)}-1\right)\right].$$

Therefore, to show this lemma, it suffices to show $\mathbb{E}\left(e^{2H_n(1)}-1\right)\to 0$ as $n\to\infty$. Now,

$$\begin{aligned}\mathbb{E}\left[e^{2H_n(1)}\right] &= \mathbb{E}\left\{\frac{\prod_{j=1}^{N(n)}\exp\left[\sum_{j=1}^{\vartheta_j(n)}\ln\left(1+\frac{B_n^j}{N(n)}\right)^2\right]}{\prod_{j=1}^{N(n)}\exp\left[\frac{2B_n^j}{N(n)}\right]}\right\}\\ &= \mathbb{E}\left\{\frac{\prod_{j=1}^{N(n)}\exp\left[\frac{\nu(I_n)}{N(n)}\left(\left(1+\frac{B_n^j}{N(n)}\right)^2-1\right)\right]}{\prod_{j=1}^{N(n)}\exp\left[\frac{2B_n^j}{N(n)}\right]}\right\}\\ &= \mathbb{E}\left\{\prod_{j=1}^{N(n)}\frac{e^{\frac{2B_n^j\nu(I_n)}{N(n)^2}+\frac{(B_n^j)^2\nu(I_n)}{N(n)^3}}}{e^{\frac{2B_n^j}{N(n)}}}\right\}\\ &= \mathbb{E}\left(e^{-\frac{2B_n^j}{N(n)}}\right)=1+O\left(\frac{1}{N(n)}\right)\to 0\end{aligned}$$

as $n \to \infty$, where the fourth equality is from Assumption 1(b) and the last equality is from Assumption 1(a) and term (7). Hence, $e^{H_n(t)} \xrightarrow{L^1} 1$ uniformly and so $H_n(t) \xrightarrow{P} 0$ uniformly in $[0,\ 1]$. Thus, the lemma is proved.

Lemma 4 *Assume that* $\lim_{n\to\infty} \nu(I_n)\Delta(n) = 0$. *Let*

$$\tilde{\varepsilon}_n(t) = \sum_{j=1}^{N(n)} 1_{\{t_j^1(n)\le t\}} \left\lceil \frac{\Delta\check{\varepsilon}_n(t_j^1(n))}{\Delta(n)} \right\rceil \Delta(n), \tag{17}$$

where $\Delta\check{\varepsilon}_n(t_j^1(n))$ *is the jump of* $\check{\varepsilon}_n(\cdot)$ *at* $t = t_j^1(n)$. *Then,* $\mathbb{P}\left(\sup_{t\in[0,\ 1]} |\tilde{\varepsilon}_n(t) - \check{\varepsilon}_n(t)| > 0\right) \longrightarrow 0$ *as* $n \to \infty$.

Proof. Consider that

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,\ 1]} |\tilde{\varepsilon}_n(t) - \check{\varepsilon}_n(t)| > 0\right) &\le \sum_{j=1}^{N(n)} \mathbb{P}(\vartheta_j(n) \ge 1)\Delta(n) \\ &= \sum_{j=1}^{N(n)} \left[1 - e^{-\frac{\nu(I_n)}{N(n)}}\right]\Delta(n) = \nu(I_n)\Delta(n) + O\left(\frac{\nu(I_n)^2\Delta(n)}{N(n)}\right) \to 0\end{aligned}$$

as $n \to \infty$ by Assumption 1(b). Thus, the lemma follows directly.

Lemma 5 *Assume that* $\lim_{n\to\infty} \frac{\bar{\nu}(m(n))}{N(n)} = 0$ *as* $n \to \infty$. *Let* $\varepsilon_n(t) = \tilde{\varepsilon}_n(t_j(n))$, *if* $t \in [t_j(n),\ t_{j+1}(n))$, $j = 1,\ 2,\ \cdots,\ N(n)$. *Then,* $\rho(\varepsilon_n(\cdot),\ \tilde{\varepsilon}_n(\cdot)) \xrightarrow{P} 0$ *as* $n \to \infty$.

Proof. Firstly, we recall the definition of the Skorokhod distance on $\mathbb{D}[0,\ 1]$, $\rho(\cdot,\ \cdot)$: $\forall U_t,\ 0 \le t \le 1,\ W_t,\ 0 \le t \le 1 \in \mathbb{D}[0,\ 1]$,

$$\rho(U,\ W) = \inf_{\theta\in\Lambda}\left\{\sup_{0\le t\le 1}\left|U_t - W_{\theta(t)}\right| + \sup_{0\le t\le 1}|\theta(t) - t|\right\},$$

where Λ is the set of strictly increasing continuous functions θ on $[0,\ 1]$ with $\theta(0) = 0$ and $\theta(1) = 1$.

Notice that the paths of both $\varepsilon_n(\cdot)$ and $\tilde{\varepsilon}_n(\cdot)$ have at most one jump in each time subinterval. Actually, $\varepsilon_n(\cdot)$ is obtained by delaying its jump in a time subinterval to the following time grid point. Let $\theta(t) \in \Lambda$ be such a function that is obtained by interpolating piecewise linearly and $\theta(0) = 0$, $\theta(1) = 1$, $\theta(t_j^1(n)) = t_j(n)$, $\forall j = 1,\ 2,\ \cdots,\ N(n) - 1$. So, $\varepsilon_n(t) - \tilde{\varepsilon}_n(\theta(t)) = 0$ when $t < t^1_{N(n)}(n)$. Hence,

$$\begin{aligned}\rho(\tilde{\varepsilon}_n,\ \varepsilon_n) &\le \left|\tilde{\varepsilon}_n(t^1_{N(n)}(n)) - \varepsilon_n(\theta(t^1_{N(n)}(n)))\right| + \Delta t(n) \\ &= \left|\tilde{\varepsilon}_n(1) - \varepsilon_n(t_{N(n)-1}(n)))\right| + \Delta t(n).\end{aligned}$$

Therefore, $\forall a > 0$, $\mathbb{P}\left[\rho(\tilde{\varepsilon}_n,\ \varepsilon_n) > a\right] \le \mathbb{P}(\vartheta_{N(n)}(n) > 0) = 1 - \exp\left(-\frac{\nu(I_n)}{N(n)}\right) \to 0$ as $n \to \infty$, as required.

Lemma 6 *Let* $S_n(t) = s_0 e^{\varepsilon_n(t)}$, $\forall t \in [0,\ 1]$. *Then,* $S_n(\cdot) \xrightarrow{P} S(\cdot)$ *as* $n \to \infty$ *under the Skorokhod* J_1 *topology.*

Proof. From Lemma 2, 3, 4 and 5, it follows that $\varepsilon_n \xrightarrow{P} \varepsilon$ under the Skorokhod J_1 topology. Then, this Lemma is obtained directly by the continuous mapping theorem.

Lemma 7 *Assume that Assumption 1(a) holds and that* $\lim_{n\to\infty} \nu(I_n)\Delta(n) = 0$. *Then,* $\max_{n\ge1} \mathbb{E}\left(\max_{0\le k\le N(n)} S_n(k)\right) < \infty$ *where* $S_n(k) = S_n(t_k(n))$, $k = 0,\ 1,\ 2,\ \cdots,\ N(n)$. *That is,* $\{S_n(k),\ 0 \le k \le N(n)\}$ *is uniformly bounded.*

Proof. Let $\tilde{S}_n(t) = s_0 e^{\tilde{\varepsilon}_n(t)}$ and $\check{S}_n(t) = s_0 e^{\check{\varepsilon}_n(t)}$, $\forall t \in [0,\ 1]$. Notice that $\mathbb{E}\left(\max_{0\le k\le N(n)} S_n(k)\right) = \mathbb{E}\left(\sup_{0\le t\le 1} \tilde{S}_n(t)\right)$. Recall (17),

$$\begin{aligned}&\max_{n\ge1}\mathbb{E}\left(\sup_{0\le t\le 1}\tilde{S}_n(t)\right) \\ \le\ &\max_{n\ge1}\mathbb{E}\left(\sup_{0\le t\le 1}|\tilde{S}_n(t) - \check{S}_n(t)| + \sup_{0\le t\le 1}|\check{S}_n(t)|\right) \\ \le\ &\max_{n\ge1}\mathbb{E}\left(\sup_{0\le t\le 1}|\check{S}_n(t)|\right)\cdot\max_{n\ge1}\mathbb{E}\left[\sup_{0\le t\le 1}\left|\exp\left(\sum_{j=1}^{N(n)} 1_{\{t_j^1(n)\le t\}}\left\{\left\lceil\frac{\Delta\check{\varepsilon}_n(t_j^1(n))}{\Delta(n)}\right\rceil\Delta(n) - \Delta\check{\varepsilon}_n(t_j^1(n))\right\}\right) - 1\right|\right] + \max_{n\ge1}\mathbb{E}\left(\sup_{0\le t\le 1}|\check{S}_n(t)|\right).\end{aligned}$$

By Assumption 1(a), the boundedness of $\max_{n\geq 1}\mathbb{E}\left(\sup_{0\leq t\leq 1}|\check{S}_n(t)|\right)$ was showed in the proof of Lemma 3(i) of (Xing, 2015).

Now,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq 1}\left|\exp\left(\sum_{j=1}^{N(n)}1_{\{t_j^1(n)\leq t\}}\left\{\left\lfloor\frac{\Delta\check{\varepsilon}_n(t_j^1(n))}{\Delta(n)}\right\rfloor\Delta(n)-\Delta\check{\varepsilon}_n(t_j^1(n))\right\}\right)-1\right|\right]\\
\leq\ &\mathbb{E}\left[\exp\left(\sum_{j=1}^{N(n)}1_{\{t_j^1(n)\leq 1\}}\left|\left\lfloor\frac{\Delta\check{\varepsilon}_n(t_j^1(n))}{\Delta(n)}\right\rfloor\Delta(n)-\Delta\check{\varepsilon}_n(t_j^1(n))\right|\right)-1\right]\\
\leq\ &\prod_{j=1}^{N(n)}\left[e^{\Delta(n)}\left(1-e^{-\frac{\nu(I_n)}{N(n)}}\right)+e^{-\frac{\nu(I_n)}{N(n)}}\right]-1\\
=\ &\prod_{j=1}^{N(n)}\left[1+\frac{\nu(I_n)\Delta(n)}{N(n)}+o\left(\frac{\nu(I_n)\Delta(n)}{N(n)}\right)\right]-1\\
=\ &\left[1+\frac{\nu(I_n)\Delta(n)}{N(n)}+o\left(\frac{\nu(I_n)\Delta(n)}{N(n)}\right)\right]^{N(n)}-1\\
=\ &\left[1+\nu(I_n)\Delta(n)+o\left(\nu(I_n)\Delta(n)\right)\right]-1=\nu(I_n)\Delta(n)+o\left(\nu(I_n)\Delta(n)\right)\to 0.
\end{aligned}$$

Above all, the lemma is proved.

Lemma 8 *Let $M_n(k)=\mathbb{E}(S_n(N(n)|\mathcal{F}_k(n)))$, $k=1,2,\cdots,N(n)$, where $\mathcal{F}_k(n)$ is the natural filtration generated by $\varepsilon_n(k)$. Then,*

$$M_n(k)-S_n(k)=\left(1-\frac{k}{N(n)}\right)\nu(I_n)O(\Delta(n))S_n(k).$$

Proof. By the mutually independence of $\{t_j^1(n)\}_j$ and $\{Y_j^1(n)\}_j$, $M_n(k)$ can be rewritten as

$$M_n(k)=\mathbb{E}(S_n(N(n))|\mathcal{F}_k(n))=S_n(k)\mathbb{E}\left(\exp\left(\sum_{j=k+1}^{N(n)}1_{\{\vartheta_j(n)\geq 1\}}\left\lfloor\frac{\Delta\check{\varepsilon}_n(t_j^1(n))}{\Delta(n)}\right\rfloor\Delta(n)\right)|\mathcal{F}_k(n)\right).$$

Consider the expectation on right is equivalent to

$$\begin{aligned}
&\prod_{j=k+1}^{N(n)}\mathbb{E}\left[\exp\left(1_{\{\vartheta_j(n)\geq 1\}}\left\lfloor\frac{\Delta\check{\varepsilon}_n(t_j^1(n))}{\Delta(n)}\right\rfloor\Delta(n)\right)\right]\\
=\ &\prod_{j=k+1}^{N(n)}\left[\mathbb{P}(\vartheta_j(n)=0)+\mathbb{P}(\vartheta_j(n)\geq 1)\mathbb{E}\left(e^{\left\lfloor\frac{\Delta\check{\varepsilon}_n(t_j^1(n))}{\Delta(n)}\right\rfloor\Delta(n)}|\mathcal{F}_k^n\right)\right]\\
=\ &\prod_{j=k+1}^{N(n)}\left[e^{-\bar{\nu}(I_n)/N(n)}+\left(1-e^{-\bar{\nu}(I_n)/N(n)}\right)(1+O(\Delta(n)))\right]\\
=\ &\prod_{j=k+1}^{N(n)}\left[1+\frac{\bar{\nu}(I_n)}{N(n)}O(\Delta(n))\right]\\
=\ &1+\frac{(N(n)-k)\bar{\nu}(I_n)O(\Delta(n))}{N(n)}.
\end{aligned}$$

Then, the lemma follows directly.

Remark 3 By Lemma 6, 7, 8 and Assumption 1, Theorem 1 will be obtained. The proof is almost the same as that of Theorem 1 in Xing (2015).

3.3 Discussion on Existence Condition of Liquidity Premium

Recall that Theorem 1 gives a lower bound of the continuous time super-replication cost:

$$\lim_{n\to\infty} V_n \geq \sup_{f\in\mathcal{A}_\alpha} \mathbb{E}\left\{F(S^f) - \int_0^1 \widehat{G}(t,\ S^f,\ (t-1)S^f(t))dt\right\},$$

where $\mathcal{A}_\alpha := \{f(t) \in \mathcal{H}_2(1) :\ |f| < \alpha \text{ and } \mathbb{E}(S^f(t) = 1) \text{ for any } t \in [0,\ 1]\}$.

If the optimal expected value on right is proved to be greater than $\mathbb{E}\left(F(S^f)\right)$ with $f \equiv 1$, then the continuous time super-replication cost contains a liquidity premium. In this section, we use an example to discuss the conditions under which continuous time super-replication cost and liquidity premium exist.

Example 1: Let the trading function be

$$g_\beta(\lambda) = \frac{1}{\beta}|\lambda|^\beta,$$

where $\beta > 1$. By the definition of dual function, the dual function of g is

$$G_\beta(y) = \frac{1}{\beta^*}|\lambda|^{\beta^*},$$

where $\beta^* = \frac{\beta}{\beta-1}$. Notice that $G_\beta(0) = 0$ and so $\hat{G}_\beta(0) = 0$ by Assumption 1(b). Also, from Assumption 1(b), $\lim_{n\to\infty} \Delta(n)\nu(I_n) = 0$, and for $y \neq 0$,

$$\hat{G}_\beta(y) = \lim_{n\to\infty} N(n)G_\beta(y\Delta(n)\nu(I_n)O(1)) = \lim_{n\to\infty} N(n)(\Delta(n)\nu(I_n))^{\beta^*} G_\beta(yO(1)).$$

Suppose that $\Delta(n)\nu(I_n) = \frac{1}{N(n)^\delta}$, where $\delta > 0$. Obviously, when $\delta \geq 1$, the limit above is 0. For $\delta \in (0,\ 1)$,

$$\begin{aligned}
\hat{G}_\beta(y) &= \lim_{n\to\infty} N(n)^{1-\delta\beta^*} G_\beta(yO(1)) \\
&= \lim_{n\to\infty} N(n)^{1-\delta\beta/(\beta-1)} G_\beta(yO(1)) \\
&= \begin{cases} G_\beta(y),\ if\ \beta = \frac{1}{1-\delta} \\ 0,\ if\ 1 < \beta < \frac{1}{1-\delta} \\ \infty,\ if\ \beta > \frac{1}{1-\delta}. \end{cases}
\end{aligned}$$

Above all, under the condition that $\delta \geq 1$ or $1 < \beta < \frac{1}{1-\delta}$ $(0 < \delta < 1)$, $\hat{G}_\beta(y) = 0$ and hence the continuous time super-replication cost contains a liquidity premium. For the case of $\beta = \frac{1}{1-\delta}$, $0 < \delta < 1$, the existence of a liquidity premium is not determined.

The super-replication costs of European put options with respect to the trading function as that in Example 1 will be calculated in the following section.

4. Examples and Comparison

In this present section, super-replication costs of European put options under multinomial models from the Variance Gamma (VG) process and the Normal Inverse Gaussian (NIG) process will be calculated and compared but the procedure is applicable to a much wider class of derivatives.

Let the $s_0 = \$1$, $r = 0.10$ and the times to maturity is $T = 1$ year. The put option prices were computed for exercise prices $K = \$0.90,\ 0.95,\ 1.00,\ 1.05,\ 1.10$. The parameters for the multinomial scheme were chosen as $N = 1000$, $\Delta = \sqrt{1/N}$. Then, from Example 1 in last section, it follows that $\hat{G}_\beta(y) = 0$ when $1 \leq \beta < 2$. So, both the super-replication cost and liquidity Premium exist. For $\beta = 2$, when $f = 1$, the discrete approximation proposed in Maller (2006) can be taken to replace the $\varepsilon_n(k)$ used in Lemma 8 in last section. And under that scheme, $M_n(k) = S_n(k)$. So, $G = 0$. Therefore, a super-replication cost exists and so does a liquidity Premium.

The Lévy measure for the VG process can be expressed as

$$\nu(dx) = \frac{e^{\frac{\theta x}{\sigma^2}} e^{-\frac{x}{\sigma}\sqrt{\frac{2}{\delta}+\frac{\theta^2}{\sigma^2}}}}{\delta|x|}dx,$$

where $\delta = 0.1686$, $\theta = -0.14$ and $\sigma = 0.1213$. The parameters are the same as those estimated from S&P 500 index options data by Madan et al. (1998). The Lévy measure for the NIG process is

$$\nu(dx) = \frac{\eta\zeta e^{\mu|x|}K_1(\eta|x|)}{\pi|x|}dx,$$

where $\eta = 28.421$, $\mu = -15.086$, $\zeta = 0.317$ and $K_1(\cdot)$ is the modified Bessel function of the second kind with index 1. The parameters we use for the NIG model are the same as those used in (Maller, et al., 2006). Let the trading function, g, is as that discussed as in Example 1.

In the following table, let $K = 0.9$, $\nu(I_(n)) = N^{1/4}$ and $\beta = 4/3$. The values of f taken are those rounding to the nearest tenth.

Table 1. Super-replication costs of European put options under the VG model

f	8.6	8.7	8.8	8.9	9.0	9.1
$J(S^f)$ under VG	0.2454	0.2467	0.2474	0.2473	0.2459	0.2423

Recall that function $J(S^f)$ is as that defined in Theorem 1. Table 1 shows that the Super-replication cost of European put options under the VG model is the maximum $J(S^f)$ value \$0.2474 which is obtained when $f = 8.8$. The same method will be used to calculate Super-replication costs of European put under the VG model and NIG model at different strike prices. The results are illustrated in the following table 2.

Table 2. Super-replication costs of European put options under the VG and NIG model. Let $\beta = 4/3$, and $10/9$

K	0.9	0.95	1	1.05	1.1
VG $\beta = 4/3$	0.2474	0.2775	0.3086	0.3403	0.3727
VG $\beta = 10/9$	0.1187	0.1416	0.1664	0.1930	0.2213
NIG $\beta = 4/3$	0.2803	0.3116	0.3436	0.3762	0.4094
NIG $\beta = 10/9$	0.1313	0.1548	0.1799	0.2068	0.2068

As discussed in sectin 3.3, the liquidity premium is the difference between the super-replication cost and the value of $F(S^f)$ when $f = 1$. The following table 3 give liquidity premiums of Euroupean put under VG and NIG models when $\beta = 20/19$.

Table 3. Liquidity Premiums of European put options under the VG and NIG model. Let $\beta = 20/19$

K	0.9	0.95	1	1.05	1.1
VG	0.0295	0.0341	0.0896	0.0362	0.0330
NIG	0.0383	0.0442	0.0472	0.0467	0.0429

5. Conclusion and Discussion

In this paper, the existence problem of continuous time super-replication cost of an European option based on a multinomial model is studied. Special emphasis is placed on the multinomial scheme corresponding to those Lévy processes that have infinitely many unbounded jumps, almost surely, in any finite time interval. Under a mild assumption, the continuous time super-replication cost is proved to be greater than or equal to an optimal control problem. The conditions under which the liquidity premium exist is discussed which should have the practical importance in the real world. So, the result in this paper is strong enough to fulfill the practical need. Different cases were also discussed on the existence of liquidity premium. The super-replication costs and liquidity premiums with respect to the variance gamma process and the normal inverse Gaussian process were calculated and compared.

The main tool is a multinomial approximation scheme that is based on a discrete grid, on a finite time interval $[0, 1]$, and having a finite number of states, for a Lévy process. The approach overcomes some difficulties that can be encountered when the Lévy process has infinite activity.

This paper showed that a super-replication cost and liquidity premium exist when the trading cost $g(y) = \frac{1}{\beta}|\lambda|^\beta$ is the power $\beta \in [1, 2)$. When $\beta = 2$, this paper only showed liquidity premium is nonnegative. So, I will leave the existence of a liquidity premium for $\beta = 2$ in a future study.

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