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Yared Abayneh Abebe

Solomon D. Seyoum

Zoran Vojinović

Roland K. Price

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COMPARISON OF 2D NUMERICAL SCHEMES FOR MODELLING SUPERCRITICAL AND TRANSCRITICAL FLOWS ALONG URBAN FLOODPLAINS

YARED A. ABEBE (1), SOLOMON D. SEYOUM (1), ZORAN VOJINOVIĆ (1), ROLAND K. PRICE (1)
(1): UNESCO-IHE Institute for Water Education, Westvest 7 - P.O.Box 3015, 2601 DA, Delft, The Netherlands

Urban floodplains usually have irregular geometry due to different obstacles, urban infrastructures and slope conditions. This may change the flow regime from subcritical to supercritical flow conditions, and vice versa. Implementation of the full momentum equation in 2D shallow water equations (SWEs) is not trivial in mixed flow conditions as subcritical and supercritical flows require different boundary conditions and hence different solution algorithms. Some models ignore the convective acceleration term (CAT) to simplify implementation of the momentum equation for mixed flow conditions. This work tried to investigate the effect of neglecting CATs by testing two 2D models which implement - full SWEs and completely reduced CAT. The models' performances were then tested by setting up hypothetical case studies with changing flow regimes. Simulations results were compared to each other by setting the solutions of the method that solve the full equations as a reference. Findings of the numerical tests showed that, in the cases, results of the model which ignore CATs fully were very similar compared to solutions of the model which implement full SWEs. Hence, simplified models which ignore CATs may be used to model urban flood plains without significant loss of accuracy.

INTRODUCTION

The governing equations in 2D models are the so-called shallow water equations (SWEs). The system of 2D SWEs is obtained by integrating the Navier Stokes equations over depth and replacing the bed stress by a velocity squared resistance term in the two orthogonal directions. The assumptions used in this case are: uniform velocity distribution in the vertical direction, incompressible fluid, hydrostatic pressure distribution, and small bottom slope (Yoon and Kang [10]). This system of equations consists of three equations: one equation for continuity and two equations for the conservation of momentum in the two orthogonal directions (Mignot *et al* [6]).

The advantages of 2D models include: more accurate solution of the governing equations; two or three orders of magnitude higher resolution output; flowpaths do not have to be pre-defined; vastly more accurate mapping of flood inundation, flood levels and flood hazard. (Verwey [9]; Syme [8]). However, the primary disadvantages of 2D models are the longer simulation times (Evans [2]; Syme [8]) and complicated computation of 2D unsteady flows due to need for efficient solver routine and the inclusions of proper boundary conditions (Fennema and Chaudhry [3]).

Various methods can be introduced in discretizing SWEs to better simulate urban flooding and to reduce computational time so as to use the modeling tools for real time application. For instance, explicit finite difference schemes which capture shocks can be used to discretize SWEs (Liang *et al* [5]) though they suffer a conditional stability problem in the use of larger time steps. As a result, for example, a number of commercial software products solve the full flow equations using implicit finite difference schemes which are unconditionally stable. However, discretizing the full flow equations using implicit finite difference schemes is not easy to implement theoretically as it incurs complications in the application of boundary conditions in case of transcritical flows.

One way of tackling the boundary condition problem is ignoring the convective acceleration terms (CATs) in the momentum equations of the flow equations. The basic argument for this assumption is that these terms are small compared to the other terms in urban floodplain flows, which means a subcritical flow condition is assumed. Hence, one boundary condition at each end (i.e., upstream and downstream) is provided and the same solution algorithm is used in both subcritical and supercritical flows. Ignoring the CATs also favors the model by reducing complexity of the equations and hence the simulation time.

With this background, this work mainly tried to investigate the merits and drawbacks associated with ignoring the CATs in 2D supercritical and transcritical flow conditions. Two methods were used for this purpose: MIKE21 flow model and a Non-Inertia 2D model. Finally, numerical experiments using hypothetical case studies which somehow represent urban floodplains were carried out to test the methods.

MODELS DESCRIPTION

MIKE21

The following description of MIKE21 commercial software package is based on MIKE21 Flow Model Scientific Documentation (DHI [1]).

The hydrodynamic module in the MIKE21 Flow Model (MIKE21 HD) is a general numerical modeling system for the simulation of water levels and flows in estuaries, bays and coastal areas. It simulates unsteady 2D flows in one layer (vertically homogeneous) fluids and has been applied in a large number of studies.

The conservation of mass and momentum equations that describe flow and water level variations are given as:

$$\frac{\partial \zeta}{\partial t}+\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}=\frac{\partial d}{\partial t} \tag{1}$$

$$\frac{\partial p}{\partial t}+\frac{\partial}{\partial x}\left(\frac{p^2}{h}\right)+\frac{\partial}{\partial y}\left(\frac{pq}{h}\right)+gh\frac{\partial \zeta}{\partial x}+\frac{gp\sqrt{p^2+q^2}}{C^2h^2}-\frac{1}{\rho_w}\left[\frac{\partial}{\partial x}(h\tau_{xx})+\frac{\partial}{\partial y}(h\tau_{xy})\right]-\Omega_q - fVV_x+\frac{h}{\rho_w}\frac{\partial}{\partial x}(p_a)=0 \tag{2}$$

$$\frac{\partial q}{\partial t}+\frac{\partial}{\partial y}\left(\frac{q^2}{h}\right)+\frac{\partial}{\partial x}\left(\frac{pq}{h}\right)+gh\frac{\partial \zeta}{\partial y}+\frac{gq\sqrt{p^2+q^2}}{C^2h^2}-\frac{1}{\rho_w}\left[\frac{\partial}{\partial y}(h\tau_{yy})+\frac{\partial}{\partial x}(h\tau_{xy})\right]+\Omega_p - fVV_y+\frac{h}{\rho_w}\frac{\partial}{\partial y}(p_a)=0 \quad (3)$$

where $h(x,y,t)$ is water depth; $d(x,y,t)$ is time varying water depth; $\zeta(x,y,t)$ is surface elevation; $p,q(x,y,t)$ are flux densities in x and y directions; $C(x,y)$ is Chézy resistance; $f(V)$ is wind friction factor; $V,V_x,V_y(x,y,t)$ is wind speed and components in x and y directions; $\Omega(x,y)$ is Coriolis parameter, latitude dependent; $p_a(x,y,t)$ is atmospheric pressure; ρ_w is density of water; and $\tau_{xx},\tau_{xy},\tau_{yy}$ are components of effective shear stress.

In area of high velocity gradients, that is, for flow at high Froude numbers, selective introduction of numerical dissipation has been used to improve the robustness of the numerical solution, and to provide MIKE21 with the capability to simulate locally super-critical flows. The numerical dissipation is introduced through selective "up-winding" of the CATs, as Froude number increases.

To ensure that the dissipative effects of up-winding are only included when necessary, a Froude number dependent weighing factor α has been introduced where:

$$\begin{aligned}&\alpha=0, Fr\le 0.25\\&\alpha=\frac{4}{3}(Fr-0.25), 0.25<Fr<1.0\\&\alpha=1, Fr\ge 1.0\end{aligned} \quad (4)$$

The weighing factor α is applied to the convective momentum terms, such that:

$$\frac{\partial}{\partial x}\left(\frac{p^2}{h}\right)_j\approx(1-\alpha)\frac{\partial}{\partial x}\left(\frac{p^2}{h}\right)_j+\alpha\frac{\partial}{\partial x}\left(\frac{p^2}{h}\right)_{j-1/2} \quad (5)$$

This brings the effects of up-winding in gradually as the Froude number increases from 0.25 to 1.0. For Froude numbers of one or more, the CAT is fully up-winded.

MIKE21 HD makes use of the so-called ADI technique to integrate the equations for mass and momentum conservation in the space-time domain. The equation matrices that result for each direction and each individual grid line are resolved by a double sweep algorithm.

Non-Inertia 2D Model

This modeling software was developed in UNESCO-IHE as part of a PhD research. The description of the model is based on Seyoum *et al* [7].

The system of 2D SWEs is obtained by integrating the Navier Stokes equations over depth and replacing the bed stress by a velocity squared resistance term in the two orthogonal directions. The continuity equation for the 2D flood plain flows is formulated as

$$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}+\frac{\partial (hv)}{\partial y}=0 \quad (6)$$

Neglecting eddy losses, Coriolis force, atmospheric pressure, wind shear effect and lateral inflow, the momentum equations in x and y directions can be written as

$$\frac{\partial(hu)}{\partial t}+\frac{\partial(hu^2)}{\partial x}+\frac{\partial(huv)}{\partial y}+gh\frac{\partial H}{\partial x}+gC_f u\sqrt{u^2+v^2}=0 \tag{7}$$

$$\frac{\partial(hv)}{\partial t}+\frac{\partial(hv^2)}{\partial y}+\frac{\partial(huv)}{\partial x}+gh\frac{\partial H}{\partial y}+gC_f v\sqrt{u^2+v^2}=0 \tag{8}$$

where H is the water level; u and v are the velocities in the directions of the two orthogonal axes (the x and y directions); and the coefficient C_f appearing in the friction terms is normally expressed in terms of the Manning n or Chézy roughness factor C.

Two-dimensional flow over inundated urban flood plain is assumed to be a slow, shallow phenomenon and the CATs can be assumed to be small compared with the other terms; and therefore, can be ignored.

Expressing the velocities in terms of the discharges and using Chézy roughness factor, the simplified momentum equations in x and y directions can be written as:

$$\frac{\partial}{\partial t}\left(\frac{Q}{Z_Q}\right)+\Delta Yg\frac{\partial h}{\partial x}+g\frac{Q}{C^2{Z_Q}^2}\left[\left(\frac{1}{\Delta Y}\frac{Q}{Z_Q}\right)^2+\left(\frac{1}{\Delta X}\frac{R}{Z_R}\right)^2\right]^{0.5}=0 \tag{9}$$

$$\frac{\partial}{\partial t}\left(\frac{R}{Z_R}\right)+\Delta Xg\frac{\partial h}{\partial y}+g\frac{R}{C^2{Z_R}^2}\left[\left(\frac{1}{\Delta Y}\frac{Q}{Z_Q}\right)^2+\left(\frac{1}{\Delta X}\frac{R}{Z_R}\right)^2\right]^{0.5}=0 \tag{10}$$

where Q and R are the discharges in the directions of the two orthogonal axes (the x and y directions); Δx and Δy are the grid spacing in the x and y directions; and Z_Q and Z_R are the water depths at the cell boundaries.

The ADI finite difference method is implemented for the numerical solution of the governing equations. The PDEs of the governing equations are transformed to difference equations on a regular Cartesian grid.

NUMERICAL EXPERIMENTS

The 2D numerical experiments were conducted based on steady flow tests for supercritical and transcritical flow conditions. The tests were conducted on prismatic channels with constant and variable slopes. Each test was conducted for three different channel bed resistances defined by Chézy coefficients of $C_1=10$, $C_2=28$ and $C_3=45$.

For the MIKE21 model set up, an initial condition of 8 cm water depth was used, whereas, the non-inertia model started the computation from a dry bed. The upstream boundary condition used in all the 2D experiments was a steady flow of 5 m^3/s which flows for a period of one hour. As a downstream boundary condition, a normal depth boundary for subcritical flows and a critical depth boundary for critical and supercritical flows were used. In addition, a 1 m by 1 m DTM and a time step of $\Delta t=0.1$ second were used in all experiments.

Test 1 – Prismatic channel with constant slope

This experiment was conducted on a 1200 m long and 10 m wide hypothetical rectangular channel as shown in Figure 1. The bed slope of this channel was $S = 0.02$ which was constant along the channel length. The channel bed was made horizontal across the channel width.

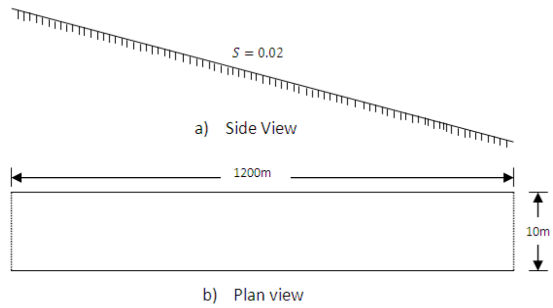

Figure 1. a) Side view and b) Plan view of a rectangular hypothetical channel used for numerical experiment

The first simulation was for a channel bed resistance of $C_1 = 10$. In this test, the Froude number was 0.452 such that the flow is subcritical. The second and third simulations were for $C_2 = 28$ and a Froude number of 1.264 and $C_3 = 45$ and a Froude number of 2.032. Since the Froude numbers were greater than one, the latter two flows were supercritical.

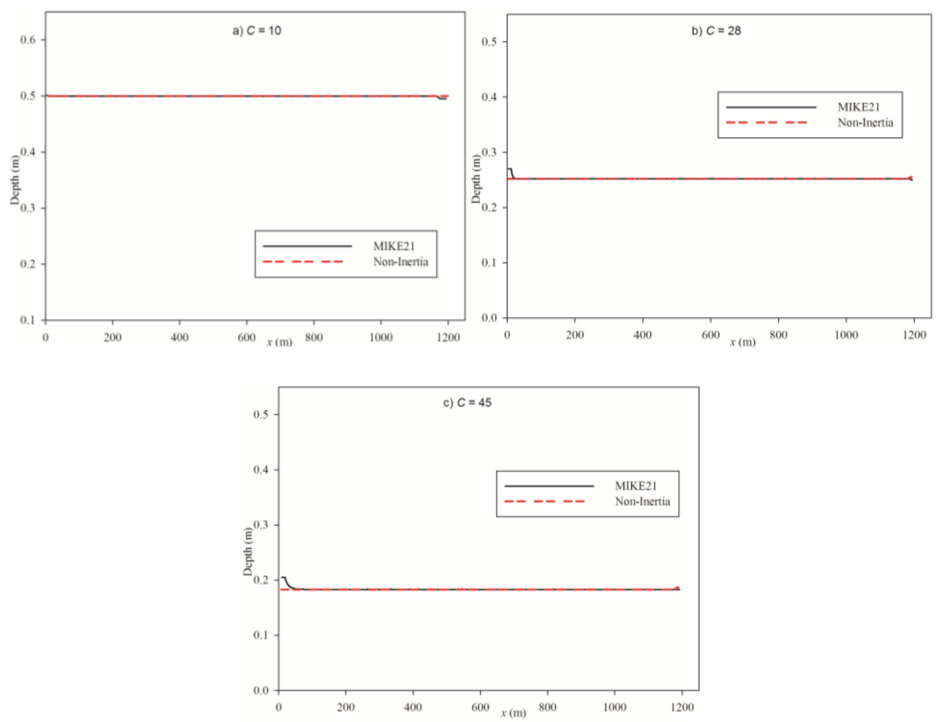

Figure 2. Longitudinal water depths 30 minutes after simulation started

Figure 2 shows simulation results of the two model setups half an hour after simulation started. The flow depths shown in the figures are taken along the longitudinal profile. Since the channel bed is horizontal across the channel, the flow in that direction is insignificant. In the figures, it is shown that the non-inertia model results are quite similar to the MIKE21 model results even in those flows which are characterized by very high Froude numbers. Though the effects did not propagate further, there were discrepancies in the model results at the upstream and downstream ends. Those mismatches were due to the implementation of the modeling tools while treating the given boundary conditions. Consequently, in this case, solving the 2D SWEs which completely ignore the CATs give almost the same result as the full 2D SWEs in simulating pure supercritical flows.

Test 2 – Prismatic channel with variable bed gradient

This case study was designed to test the capability of the models to handle changes in flow regimes. Similar to the previous experiment, this experiment was conducted on a 1200 m long and 10m wide hypothetical rectangular channel. Nevertheless, as illustrated in Figure 3, the channel was divided into three reaches of length $L_1 = 300$ m, $L_2 = 600$ m and $L_3 = 300$ m with a bed slope of $S_1 = 0.01$, $S_2 = 0.02$ and $S_3 = 0.01$ respectively. Once again, the channel bed was made horizontal across the width.

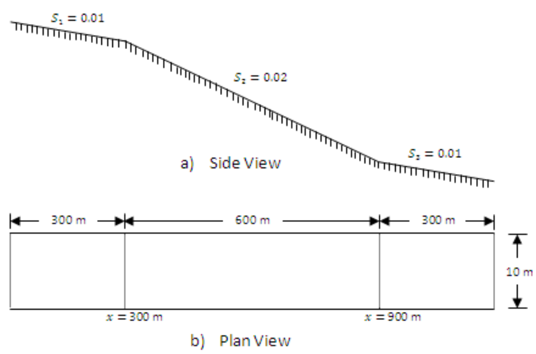

Figure 3. a) Side view and b) Plan view of a rectangular non-prismatic channel used for numerical experiment

Three tests were conducted in this case also. The first one was with $C_1 = 10$ and Froude number range of 0.319 to 0.452; the second one was with $C_2 = 28$ and Froude number range of 0.894 to 1.264; and the last one was with $C_3 = 45$ and Froude number range of 1.436 to 2.032. The higher Froude numbers were registered in the relatively steeper second reach. In the first test, the flow was subcritical in the three reaches. In the second test, the flow was subcritical in the first and third reaches while it was supercritical in the second reach. Whereas in the third test, the flow was supercritical in all reaches.

The results of the models are presented graphically in Figure 4 a, b and c. The figures show flow depths along the channel half an hour after simulation started and are taken at the centre of the channel width. Since the channel bed is horizontal across the channel, the flow in that direction is insignificant. The figures demonstrate that, in depth basis, the non-inertia model results are quite similar to the MIKE21 model results in all flow conditions – pure subcritical, transcritical and pure supercritical flows. In this case again, the discrepancies at the two ends were due to the implementation of boundary conditions by the modeling tools.

However, the depth results from the two models showed slight variation at/around the critical and shock points. As shown in Figure 3, $x = 300$ m and $x = 900$ m mark the breaks in the bed slope. It is observed in Figure 4 that the depth outputs from the two models were almost the same except at the critical and shock points. In addition, the discrepancy between the two model outputs increased when the flow became more supercritical. For instance, the difference between the two model outputs at the shock point was 0.05%, 12.40% and 19.00% for the respective roughness of $C_1 = 10$, $C_2 = 28$ and $C_3 = 45$. This finding is in line with findings of Hunter *et al* [4]. The authors concluded that even though flows in urban environments are characterized by transitions to supercritical flow and numerical shocks, the effects are localized and they did not appear to affect overall wave propagation.

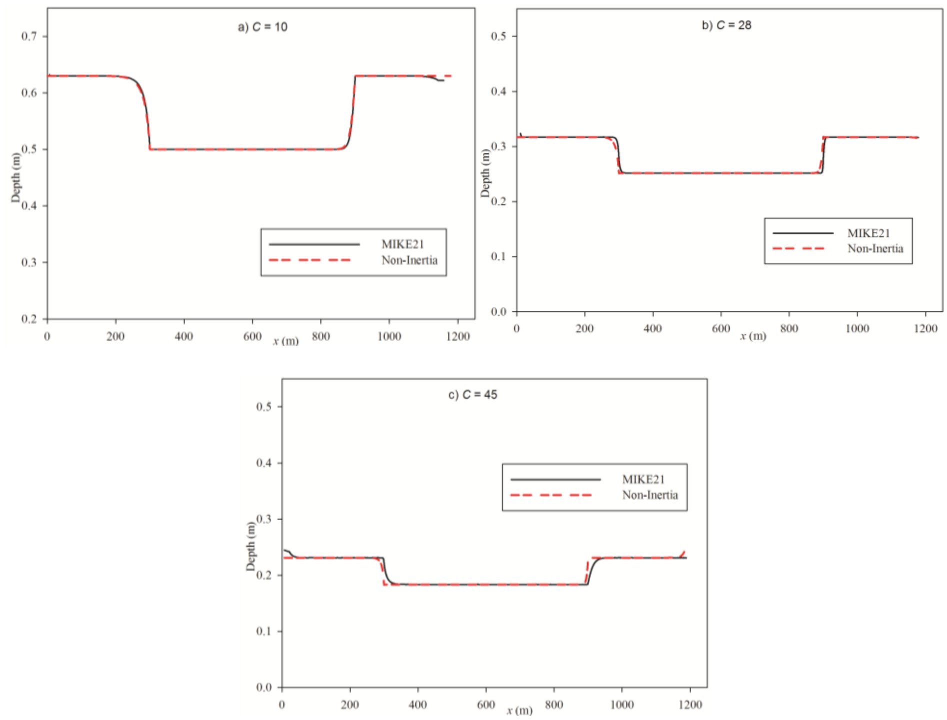

Figure 4. Longitudinal water depths 30 minutes after simulation started

Figure 5 also shows zoomed profiles of the depths around the two points for $C_2 = 28$. The figures demonstrate that the results of the two models were different around the critical and shock points.

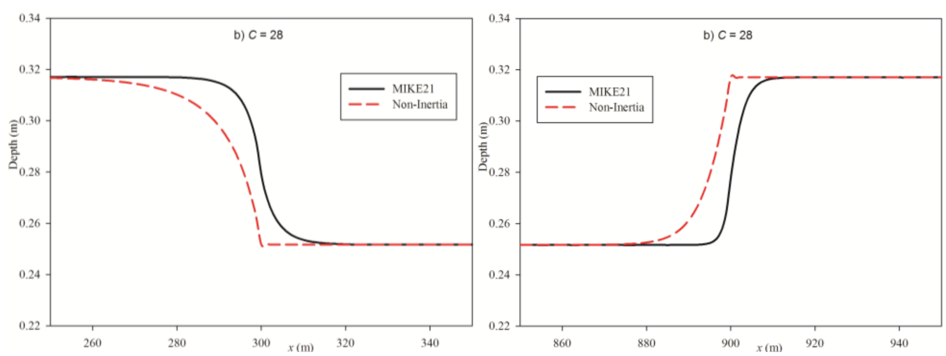

Figure 5. Depth profiles at critical and shock points (left and right pictures respectively)

As a result, in this case also, solving the 2D SWEs which completely ignore the CATs give almost the same result as the full 2D SWEs in simulating pure supercritical and transcritical flows.

CONCLUSIONS

The comparison results showed that there are cases in which the cost of completely neglecting the convective acceleration terms (CATs) from the 2D SWEs is minor. This is especially shown on the results of simulations of flows in channels with uniform bed gradient. If one is interested in simulating urban floods which are characterized by relatively flat surface, it may be enough to use those modeling tools which does not implement CATs with a possible advantage of reducing simulation time. On the other hand, at critical and shock points, the results of the non-inertia model record differences compared to the MIKE21 results. This shows that, if one is interested in designing hydraulic structures with breaks in bed gradient, it may be necessary to trace special features like hydraulic jumps; and in this case, the use of modeling packages which better treat the CATs and capture shocks are more suitable. Besides, the CATs may still have higher importance in dam break analysis, modeling tsunami wave or modeling flows characterized by reflected waves. In conclusion, it seems that the use of the full SWEs is not a strongly binding rule in modeling urban floodplains. It rather depends on the circumstances even if supercritical states dominate the flow.

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