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Finite Element Spectral Approximation with Numerical Integration for the Biharmonic Eigenvalue Problem

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Abstract

We investigate the convergence properties of a mixed finite element method approximation to the Biharmonic eigenvalue problem under the presence of numerical integration. We give a brief overview of the results obtained when exact integration is used in a finite element method, then develop related theories and obtained the convergence rates when numerical quadrature is taken into account. The standard approach to obtaining error estimate of variational eigenvalue problems is based on the error estimate of the solution operators of the source problems. The important issues are the rate of convergence of the solution operators and the conditions required for convergence. Paralleling the work of Babuška, Osborn and Pikäranta to overcome some technical difficulties, we will use mesh dependent norms to obtain error estimates between the solutions operators. We then use these estimates to get errors estimates between the approximate and the actual eigenvalues and eigenvectors.

1 Introduction

In broad terms, a finite element method, FEM for short, is a Ritz-Galerkin approximation using special basis functions. When used to approximate the solution of a PDE or a variational problem, a FEM reduces the differential or variational problem to a large matrix problem. For this reason the basis functions are constructed so that the matrices are sparse and if possible banded.

In this paper, we are interested in establishing the order of convergence of the approximate eigenvalues and eigenvectors obtained from a finite element method with numerical integration when applied to the following fourth order eigenvalue problem:

Problem 1.1. *Find λ and $\psi(x,y) \neq 0$ satisfying*

$$\begin{aligned} \Delta^2\psi &= \lambda\psi, && (x,y)\in\Omega \\ \psi &= \partial\psi/\partial n = 0, && (x,y)\in\partial\Omega, \end{aligned}$$

where Ω is a convex polygon in $\mathbb{R}^2$ and where $\partial\psi/\partial n$ denotes the exterior normal derivative of ψ. This eigenvalue problem arises in connection with the small, transverse vibration of a clamped plate. In order to use a FEM to get an approximation, the eigenvalue problem is put into a variational form.

To avoid the difficult task of constructing continuously differentiable functions, the mixed variational formulation is often used for Problem **??**. The mixed formulation introduces an auxiliary variable and uses it to reduce the given PDE to a system of lower order PDEs. The FEM used to approximate the solution of such mixed variational formulations is called a mixed FEM, or simply a mixed method. Since the mixed method approximates the solutions of lower order PDEs, the basis functions are not required to be as smooth as when approximating the original problem and continuous functions, C^0-elements, can be used. In fact, because the mixed method permits the use of relatively simple FEM spaces, it is often used to approximate the solutions of higher order PDEs.

The mixed variational formulation of Problem **??** is stated as follows:

Problem 1.2. *Find $\lambda \in \mathbb{R}$ and $(u, \psi) \in H^1(\Omega) \times H_0^1(\Omega)$, both nonzero, such that*

$$\int_\Omega u\sigma\ dx - \int_\Omega \nabla\sigma \cdot \nabla\psi\ dx = 0 \qquad \forall\sigma \in H^1(\Omega),$$
$$-\int_\Omega \nabla u \cdot \nabla v\ dx = -\lambda \int_\Omega \psi v\ dx \qquad \forall v \in H_0^1(\Omega).$$

A finite element method requires that Ω be divided into smaller regions, usually 'triangles', 'squares', or both. For simplicity, we will only consider methods that divide Ω into triangles. Using the triangulation, finite dimensional subspaces of $H^1(\Omega)$ and $H_0^1(\Omega)$ are constructed from C^0-functions which are polynomials or 'near' polynomial functions when restricted to each triangle. Let these subspaces respectively be denoted by V_h and W_h , where the parameter h is the maximum diameter of the triangles. The mixed finite element method approximation to Problem **??** is formulated as follows:

Problem 1.3. *Find $\lambda_h \in \mathbb{R}$ and $(u_h, \psi_h) \in V_h \times W_h$, both nonzero, such that*

$$\int_\Omega u_h\sigma\ dx - \int_\Omega \nabla\sigma \cdot \nabla\psi_h\ dx = 0 \qquad \forall\sigma \in V_h,$$
$$-\int_\Omega \nabla u_h \cdot \nabla v\ dx = -\lambda_h \int_\Omega \psi_h v\ dx \qquad \forall v \in W_h,$$

Once the bases are chosen for V_h and W_h, Problem **??** reduces to a generalized matrix eigenvalue problem, where the matrix entries are the inner products of the basis functions with each other.

The papers of Canuto and Ishihara, [**?**] and [**?**], are probably the first papers on eigenvalue approximation for the Biharmonic problem in a convex polygonal domain by mixed finite element methods. Later contributors include Descloux, Nassif, and Rappaz in [**?**] and Mercier, Osborn, Rappaz and Raviart in [**?**]. An informative survey, where many other references can be found, is the paper by Babuška and Osborn, [**?**]. Many researchers using a variety of methods show that when the eigenvector ψ lies in $H^{k+1}(\Omega)$, λ_h converges to λ at the rate of h^{2k-2} and ψ_h converges to ψ at the rate of h^k.

In actual practice, a quadrature scheme is used to evaluate the integrals of the matrix entries. Since numerical integration does not always compute integrals exactly, one usually solves another problem that is a perturbation of Problem **??**. We will study the effect of this perturbation on the rate of convergence of the approximate eigenvalues and eigenvectors. In particular, we want to know if the eigenvalues and eigenfunctions obtained by using a quadrature scheme converge to the actual values at the same rates as the case when exact integration is used. And if so, what conditions are required on the quadrature schemes to get the same orders of convergence as are obtained from using exact integration. Such problems were not addressed in the previous works, [**?**] and [**?**], with respect to the Ciarlet-Raviart method.

The effect of numerical integration on eigenvalue approximation for second order problems were first studied by Fix in [**?**], and then systematically by Banerjee and Osborn in [**?**], for convex polygonal domains. None of these incorporated curved boundaries. This was done by Vanmaele and Ženíšek in [**?**], and by Lebaud in [**?**], under different conditions. All of these works, however do not have a general theory to study problems incorporating numerical integration. For example, the method of Banerjee and Osborn could only be applied to self-adjoint problems. We have developed a general approach in an abstract setting, which could be used to study the effect of numerical integration or other perturbations such as the use of non-conformal elements. We briefly present the ideas below.

The standard approach to obtain error estimate of variationally formulated eigenvalue approximation problems is based on the error estimate of the solution operators of the source problems. For instance, if T denotes the solution operator of the source problem corresponding to Problem **??**, and if T_h denotes the solution operator of the source problem corresponding to Problem **??**, then the error between λ and λ_h is related to the error between T and T_h. Such an approach was used by Bramble and Osborn in [**?**], by Osborn in [**?**] and by Babuška and Osborn in [**?**]. When the source problem corresponding to T_h is perturbed in some way, we introduce the solution operator to the perturbed source problem, $\tilde{T}_h$. We then analyze the convergence of the eigenvalues corresponding to $\tilde{T}_h$ to those corresponding to T_h and hence to those of T. In this paper the analysis covers the case of multiple eigenvalues and the perturbation of using numerical integration.

We briefly remark that the source problems have been extensively studied by many researchers, using a wide variety of approaches. Babuška and Brezzi in their respective papers, [**?**] and [**?**], gave abstract error estimates for approximations obtained by using a Mixed Ritz-Galerkin Method approximation. Falk and Osborn in [**?**] gave a systematic method for obtaining error estimates for those problems that do not fit within the general framework of Babuška and Brezzi. Kesavan and Vanninathan studied sources problems with isoparametric element and numerical quadrature in [**?**]. Babuška, Osborn and

Pikăranta in [?] used mesh dependent spaces and norms. Paralleling their work, we will use mesh dependent norms when we study how the solutions of the perturbed source problem converge to the solutions of the source problems corresponding to Problems ?? and ??.

In the next section, we establish some notations and review such concepts as Sobolev spaces, numerical integration, triangulation of domains, and finite element spaces. In section 3, we quote and prove some results on source problems that will be used in the analysis of the eigenvalue problems. In section 4, we give a brief summary of spectral approximation of compact operators, and prove some extensions that will provide the foundations for later sections. We will also analyze eigenvalue approximation by mixed methods, in an abstract setting. In section 5, we establish the order of convergence for the eigenvalue approximations of the Biharmonic Problem. The main results in each subsection and their relevance with respect to the other subsections are given at the beginning of every section.

2 Preliminaries and Notations

When not specifically stated, the letter C with or without some accent will represent a positive constant that is independent of h and whose value may change from one occurrence to the next. Sometimes C will be followed by a list of parameters upon which it depends, enclosed in parethesis.

Let Ω be an open convex polygon in the plane and let $\Gamma = \partial\Omega$. For $m \geq 0$ an integer, we denote by $W^{m,p}(\Omega)$ the usual Sobolev space of functions u for which the norm

$$\|u\|_{m,p,\Omega} = \begin{cases} (\sum_{|\alpha|\leq m} \|D^\alpha u\|^p_{L^p(\Omega)})^{1/p}, & 1 \leq p < \infty \\ \max_{|\alpha|\leq m} \|D^\alpha u\|_{L^\infty(\Omega)}, & p = \infty, \end{cases}$$

is finite. We will also use the semi-norms

$$|u|_{m,p,\Omega} = \begin{cases} (\sum_{|\alpha|= m} \|D^\alpha u\|^p_{L^p(\Omega)})^{1/p}, & 1 \leq p < \infty \\ \max_{|\alpha|= m} \|D^\alpha u\|_{L^\infty(\Omega)}, & p = \infty. \end{cases}$$

We will usually take $p = 2$ and use the standard notation

$$H^m(\Omega) = W^{m,2}(\Omega), \qquad \|u\|_{m,\Omega} = \|u\|_{m,2,\Omega}, \qquad |u|_{m,\Omega} = |u|_{m,2,\Omega}.$$

$$H_0^m(\Omega) = \text{ Closure of } C_0^\infty(\Omega) \text{ in } H^m(\Omega)\ .$$

We denote the dual space of $H_0^m(\Omega)$ by $H^{-m}(\Omega)$, i.e.

$$H^{-m}(\Omega) = \{u\colon \|u\|_{-m,\Omega} \overset{def}{=} \sup_{v\in H_0^m(\Omega)} \frac{\int_\Omega uv\ dx}{\|v\|_{m,\Omega}} < \infty\}.$$

Finally, $H^0(\Omega)$ is more commonly denoted by $L^2(\Omega)$.

2.1 Finite Element Spaces

Triangulation

We now present the finite element (FE) spaces that will play a large part in later development. To construct the required FE spaces we need to divide $\bar{\Omega}$ into smaller regions. For $0 < h < 1$, we let $\mathcal{T}_h$ be a triangulation of $\bar{\Omega}$ by triangles T of diameter less than or equal to h . We impose the following conditions on $\mathcal{T}_h$:

1. $\mathcal{T}_h$ is a finite set.
2. $\bar{\Omega} = \bigcup_{T\in\mathcal{T}_h} T$.

3. For each distinct $T_1, T_2 \in \mathcal{T}_h$, one has $\mathring{T}_1 \cap \mathring{T}_2 = \emptyset$.
4. Any edge of a triangle T_1 in $\mathcal{T}_h$ is either a subset of the boundary Γ, or an edge of another triangle T_2 in the triangulation.

Each triangle T can be considered as the image of an invertible affine map $F_T(\hat{x}) = B_T\hat{x} + b$ which maps a reference triangle $\hat{T}$ onto T. The reference triangle is usually taken to be the triangle with vertices at the points $(0,0), (1,0), (0,1)$. The map F_T set up a correspondence not only between points of $\hat{T}$ and T but also between functions defined on the two triangles:

$$(\hat{x} \in \hat{T}) \to x = F_T(\hat{x}) \in T,$$
$$(\hat{v} \colon \hat{T} \to \mathbb{R}) \to (v = \hat{v} \circ F^{-1} \colon T \to \mathbb{R}).$$

With this correspondence, we have the identity $\hat{v}(\hat{x}) = v(x)$. The relation between each triangle T and $\hat{T}$ is an example of an affine-equivalence. Formally, two open subsets Ω and $\hat{\Omega}$ of $\mathbb{R}^n$ are affine-equivalent if there exists an invertible affine mapping

$$F \colon \hat{x} \in \mathbb{R}^n \to F(\hat{x}) = B\hat{x} + b \in \mathbb{R}^n$$

such that

$$\Omega = F(\hat{\Omega}).$$

We will need the following two theorems, quoted from [**?**], relating affine-equivalent sets.

Theorem 2.1. *Let Ω and $\hat{\Omega}$ be two affine-equivalent open domains in $\mathbb{R}^n$. If a function v belongs to the space $W^{m,p}(\Omega)$ for some integer $m \geq 0$ and some number $p \in [1,\infty]$, then the function $\hat{v} = v \circ F$ belongs to the space $W^{m,p}(\hat{\Omega})$; in addition, there exists a constant $C = C(m,n)$ such that*

$$|\hat{v}|_{m,p,\hat{\Omega}} \leq C \; \|B\|^m |\det B|^{-1/p} |v|_{m,p,\Omega} \qquad \forall v \in W^{m,p}(\Omega).$$

Analogously, one has

$$|v|_{m,p,\Omega} \leq C \; \|B^{-1}\|^m |\det B|^{1/p} |\hat{v}|_{m,p,\Omega} \qquad \forall \hat{v} \in W^{m,p}(\hat{\Omega}).$$

A demonstration of this theorem can be found in [**?**] on pages 117-118.

Theorem 2.2. *Let $\hat{h}$ and h be the diameters respectively of $\hat{\Omega}$, and Ω. Let $\hat{\rho}$ and ρ be the diameters, respectively, of the largest ball contained in $\hat{\Omega}$ and Ω. If $F(\hat{x}) = B\hat{x} + b$ is the affine mapping such that $F(\hat{\Omega}) = \Omega$, then the upper bounds*

$$\|B\| \leq \frac{h}{\hat{\rho}}, \qquad \|B^{-1}\| \leq \frac{\hat{h}}{\rho},$$

hold.

This theorem is proven in [**?**] on page 120.

We also assume that the family of triangulations $\{\mathcal{T}_h\}$ satisfies the following hypotheses.

Hypothesis 1. A family of triangulations $\{\mathcal{T}_h\}$ is *regular* if (a) there is a constant σ such that

$$\max_{T \in \mathcal{T}_h} \frac{h_T}{\rho_T} \leq \sigma \quad \forall \; h, \tag{2.1}$$

where h_T is the diameter of T and ρ_T is the diameter of the largest circle contained in T, and (b) if the quantity $h = \max_{T \in \mathcal{T}_h} h_T$ approaches zero.

Hypothesis *1a*, condition (**??**), is widely known as the minimal angle condition.

Hypothesis 2. There is a constant $\tau > 0$ such that

$$\frac{h}{h_T} \leq \tau \quad \forall\, T \in \mathcal{T}_h \text{ and } \forall\, h.$$

We say that the family of triangulations satisfies an inverse assumption, when this Hypothesis is satisfied.

The minimal angle condition together with Theorem **??** and Theorem **??** give us an important set of inequalities called the 'scaling inequalities', which are given below:

$$|\hat{v}|_{m,p,\hat{T}} \leq C\ h_T^m |\det B_T|^{-1/p} |v|_{m,p,T} \qquad \forall v \in W^{m,p}(T), \tag{2.2}$$

$$|v|_{m,p,T} \leq C\ h_T^{-m} |\det B_T|^{1/p} |\hat{v}|_{m,p,\hat{T}} \qquad \forall v \in W^{m,p}(\hat{T}). \tag{2.3}$$

FEM Spaces
Let $V_h = \{v \in C^0(\bar{\Omega})\colon v|_T \in P_k(T),\ \forall T \in \mathcal{T}_h\}$ and $W_h = V_h \cap H_0^1(\Omega)$, where $P_k(T)$ is the space of polynomials over T of degree k or less. Imposing condition (**??**) on the triangulation implies that V_h is nonempty, as proven in [**?** , page 54]. By construction, we see that V_h is a subset of $H^1(\Omega)$. With a triangulation that is regular, satisfies the inverse assumption, and for which each $T \in \mathcal{T}_h$ is affine equivalent to $\hat{T}$, the following inverse inequality hold:

Proposition 2.3.

$$(\sum_{T\in\mathcal{T}_h} |v|^2_{m,T})^{1/2} \leq Ch^{l-m}(\sum_{T\in\mathcal{T}_h} |v|^2_{l,T})^{1/2}, \quad \forall v \in V_h \textit{ with } l, m \geq 0. \tag{2.4}$$

In particular, with $m = 1$ and $l = 0$ and the fact that $V_h \subset H^1(\Omega)$, the following inequality is true

$$|v|_{1,\Omega} \leq \frac{C}{h}|v|_{0,\Omega}, \quad \forall v \in V_h.$$

The semi-norms can be replaced by the norms, thus

$$(\sum_{T\in\mathcal{T}_h} \|v\|^2_{m,T})^{1/2} \leq Ch^{l-m}(\sum_{T\in\mathcal{T}_h} \|v\|^2_{l,T})^{1/2}, \quad \forall v \in V_h \textit{ with } l, m \geq 0. \tag{2.5}$$

This is a special case of a result proven by Ciarlet in [**?** , pages 140-142]

Interpolation
We now define two interpolation operators that we will use later. For $u \in H^2(T)$ let $\mathcal{I}_T u \in P_k$ be defined by

$$\int_T (u - \mathcal{I}_T u)f\ dx = 0 \quad \forall f \in P_{k-3},$$

$$\int_{T'} (u - \mathcal{I}_T u)f\ dx = 0 \quad \forall f \in P_{k-2} \text{ and } \forall \text{ sides } T' \text{ of } T,$$

and

$$(u - \mathcal{I}_T u)(a) = 0 \quad \forall \text{ vertices } a \text{ of } T.$$

Then, for $u \in H^2(\Omega)$, we let $\mathcal{I}_h u \in V_h$ be defined by

$$(\mathcal{I}_h u)|_T = \mathcal{I}_T(u|_T).$$

For $u \in H^1(\Omega)$, we define the interpolant in a different manner, but we still use the same notation. Here we consider only the case $k = 1$. Let the vertices of $\mathcal{T}_h$ be denoted by $z_1, \ldots, z_m$ and let $w_1, \ldots, w_m$ be the basis for S_h defined by $w_i(z_j) = \delta_{ij}$. Set $S_j = (\text{supp } w_j) \cap \Omega$ and let $|S_j|$ be the area of S_j. Following Clément [**?**], we define $\mathcal{I}_h u$ by

$$\mathcal{I}_h u = \sum_{j=1}^{m} \frac{\int_{S_j} u\ dx}{|S_j|} w_j.$$

This construction is sometimes referred to as Clément interpolation. The regularity of the triangulation $\mathcal{T}_h$ and the affine equivalent of the triangles T to $\hat{T}$ together with the fact that $V_h \subset C^0(\bar{\Omega})$ give us the standard interpolation result

$$(\sum_{T\in\mathcal{T}_h} \|v - \mathcal{I}_h v\|^2_{m,T})^{1/2} \leq h^{k+1-m}|v|_{k+1,\Omega}, \quad 0 \leq m \leq k+1 \text{ and } \forall v \in H^{k+1}(\Omega), \tag{2.6}$$

(cf. [? , page 121]).

2.2 Mesh Dependent Spaces and Norms

We next discuss the mesh dependent spaces and norms which were studied and used in [?]. As before we let $\mathcal{T}_h$ be a partition of Ω by triangles,(see page **??** for details). If $T' = \partial T^1 \cap \partial T^2$ is an interior edge of the triangulation $\mathcal{T}_h$, we define $J\dfrac{\partial u}{\partial n}$ on T' by

$$J\frac{\partial u}{\partial n}\bigg|_{T'} = \frac{\partial u}{\partial n^1} + \frac{\partial u}{\partial n^2}$$

where n^j is the unit normal to T' exterior to T^j, and if T' is a boundary edge of T_h, we set

$$J\frac{\partial u}{\partial n}\bigg|_{T'} = \frac{\partial u}{\partial n}.$$

Now let $\Gamma_h = \bigcup_{T\in\mathcal{T}_h} \partial T$. We define

$$H^2_h = \{u \in H^1(\Omega) \colon u|_T \in H^2(T),\ \forall T \in \mathcal{T}_h\},$$

and on H^2_h define the norm

$$\|u\|^2_{2,h} = \sum_{T\in\mathcal{T}_h} \|u\|^2_{2,T} + h^{-1}\int_{\Gamma_h} \left|J\frac{\partial u}{\partial n}\right|^2 \ ds.$$

On $H^1(\Omega)$ we define the norm

$$\|u\|^2_{0,h} = \int_\Omega |u|^2 \ dx + h\int_{\Gamma_h} |u|^2 \ ds,$$

and then define H^0_h to be the completion of $H^1(\Omega)$ with respect to $\|\cdot\|_{0,h}$. H^0_h can be identified with $L^2(\Omega) \bigoplus L^2(\Gamma_h)$. It is clear from their definitions that H^0_h and H^2_h contain V_h, the finite element space constructed on page **??**, and that

$$H^2_h \subset H^1(\Omega) \subset H^0_h.$$

We need the following lemmas, of which the last two give estimates on the error between a function and its interpolant as defined on page **??**. The estimates are in terms of the two mesh dependent norms and are the analogues of the standard interpolation result (**??**). Their proofs are found in [?].

Lemma 2.4. *There is a constant C such that*

$$\|u\|_{0,h} \leq C\ \|u\|_{0,\Omega} \quad \forall u \in V_h.$$

Lemma 2.5. *If $\mathcal{I}_h u|_T \in P_k(T)$ and $u \in H^r(\Omega)$ with $r \geq 1$, then there is a constant C such that for all h and $1 \leq l \leq \min(r, k+1)$,*

$$\|u - \mathcal{I}_h u\|_{0,h} \leq C\ h^l\ |u|_{l,\Omega}.$$

Lemma 2.6. *If $\mathcal{I}_h u|_T \in P_k(T)$ and $u \in H^r(\Omega) \cap H^1_0(\Omega)$ with $r \geq 2$, then there is a constant C such that for all all h and $2 \leq l \leq \min(r, k+1)$,*

$$\|u - \mathcal{I}_h u\|_{2,h} \leq C\ h^{l-2}|u|_{l,\Omega}.$$

2.3 Numerical Quadrature

A numerical quadrature scheme over the set T consists in approximating the integral $\int_T \phi(x)\, dx$ by a finite sum of the form $\sum_{l=1}^{L} \omega_l \phi(b_l)$. The numbers ω_l are called the weights, and the points b_l are called the nodes of the quadrature scheme. For simplicity, we shall only consider schemes in which the nodes belong to the set T and the weights are positive. Let P be a space of function, $\{b_l\}_{l=1}^{L}$ is said to be P-unisolvent if $f \in P$ and $f(b_l) = 0$ for every l implies that $f \equiv 0$.

Let $\{\hat{b}_l, \hat{\omega}_l\}_{l=1}^{L}$ determines a quadrature scheme over the reference triangle, $\hat{T}$. Then $\{b_{l,T}, \omega_{l,T}\}_{l=1}^{L}$ determines a quadrature scheme over the triangle T, where

$$b_{l,T} = F_T(\hat{b}_l), \text{ and } \omega_{l,T} = (\det B_T)\hat{\omega}_l \qquad 1 \le l \le L,$$

and where F_T is the affine map taking $\hat{T}$ onto T and B_T is the linear part of F_T. Associated with the quadrature schemes are the error functionals

$$E_T(\phi) = \int_T \phi(x)dx - \sum_{l=1}^{L} \omega_{l,T}\phi(b_{l,T}), \tag{2.1}$$

$$\hat{E}(\hat{\phi}) = \int_{\hat{T}} \hat{\phi}(\hat{x})d\hat{x} - \sum_{l=1}^{L} \hat{\omega}_l\hat{\phi}(\hat{b}_l), \tag{2.2}$$

which are related by

$$E_T(\phi) = (\det B_T)\hat{E}(\hat{\phi}).$$

Given a space of functions P, we say that a quadrature scheme is exact for the space P if $E(\phi) = 0$ for all $\phi \in P$. If $P = P_k$ is the space of polynomials of degree at most k, then we say the quadrature scheme has *degree of precision k*, when it is exact on P_k.

Hypothesis. We assume that the quadrature scheme $\{\hat{b}_l, \hat{\omega}_l\}_{l=1}^{L}$ contains a P_k unisolvent set and has degree of precision $2k-1$.

A basic overview of numerical quadratures with many examples can be found in [**?** , pages 178-190].

The next two lemmas will be useful in our work. The first one is essentially a restatement of Theorem 4.1.4 in [**?**] and can be proven by modifying the proof given there.

Lemma 2.7. *Suppose* $\hat{E}(\hat{\phi}) = 0 \quad \forall\, \hat{\phi} \in P_{2k-1}(\hat{T})$ *with* $k \ge 1$*, then there is a constant,* C*, independent of* h *such that:*

$$|E_T(\phi w)| \le C h_T^k |\phi|_{k,T}\ |w|_{0,T} \quad \forall \phi, w \in P_k(\hat{T}).$$

The next lemma is similar to Lemma 3.1 of [**?**], but cannot be obtained by a direct application of that result.

Lemma 2.8. *Suppose* $\hat{E}(\hat{\phi}) = 0 \quad \forall\, \hat{\phi} \in P_{2k-1}(\hat{T})$ *with* $k \ge 1$*, then there exist a constant,* C*, independent of* h *such that*

$$|E_T(\phi w)| \le C h_T^{2k} |\phi|_{k,T}\ |w|_{k,T} \qquad \forall \phi,\ w \in P_k(\hat{T}).$$

Proof. For $k \ge 1$, $W^{2k,\infty}(\hat{T}) \hookrightarrow C^0(\hat{T})$, by a Kondrasov theorem, (see [**?** , Page 114]), thus the functional $\hat{E}$ is defined on $W^{2k,\infty}(\hat{T})$. From the definition of $\hat{E}$, it is clear that

$$|\hat{E}(\hat{g})| \le\ C(\hat{T})\ |\hat{g}|_{0,\infty,\hat{T}} \le C\|\hat{g}\|_{2k,\infty,\hat{T}}, \quad \forall \hat{g} \in W^{2k,\infty}(\hat{T}),$$

which implies that the functional $\hat{E}$ is continuous on $W^{2k,\infty}(\hat{T})$ with norm less than C. Since $\hat{E}$ vanish on $P_{2k-1}(\hat{T})$, the Bramble-Hilbert Lemma says that there is a constant $\hat{C}(\hat{T})$ such that

$$|\hat{E}(\hat{g})| \le \hat{C}(\hat{T})\ C\ |\hat{g}|_{2k,\infty,\hat{T}} \quad \forall \hat{g} \in W^{2k,\infty}(\hat{T}).$$

Letting $\hat{g} = \hat{v}\hat{w}$, with $\hat{v}, \hat{w} \in P_k(\hat{T})$, in the above inequality we get

$$|\hat{E}(\hat{v}\hat{w})| \leq C|\hat{v}\hat{w}|_{2k,\infty,\hat{T}}.$$

Using Leibnitz's rule we get that

$$|\hat{v}\hat{w}|_{2k,\infty,\hat{T}} \leq C(2k,2)\sum_{l=0}^{2k}|\hat{v}|_{2k-l,\infty,\hat{T}}|\hat{w}|_{l,\infty,\hat{T}}.$$

Since $\hat{v}$ has degree at most k, $|\hat{v}|_{2k-l} = 0$ for $2k - l > k$. Since $\hat{w}$ has degree at most k, $|\hat{w}|_l = 0$ for $l > k$. Thus all terms with index $l \neq k$ is zero, so we have

$$|\hat{v}\hat{w}|_{2k,\infty,\hat{T}} \leq C|\hat{v}|_{k,\infty,\hat{T}}|\hat{w}|_{k,\infty,\hat{T}}.$$

Using the equivalent of norm over the finite dimensional space $P_k(\hat{T})$, we get

$$|\hat{v}\hat{w}|_{2k,\infty,\hat{T}} \leq C|\hat{v}|_{k,\hat{T}}|\hat{w}|_{k,\hat{T}}.$$

Using the scaling inequality (**??**) we get

$$|\hat{v}\hat{w}|_{2k,\infty,\hat{T}} \leq C\ h_T^{2k}(\det B_T)^{-1}|v|_{k,T}|w|_{k,T}.$$

Thus $|\hat{E}(\hat{v}\hat{w})| \leq C\ h^{2k}(\det B_T)^{-1}|v|_{k,T}|w|_{k,T}$. Since $E_T(vw) = (\det B_T)\hat{E}(\hat{v}\hat{w})$ we have

$$|E_T(vw)| \leq C\ h^{2k}|v|_{k,T}|w|_{k,T}.$$

□

3 Source Problems

In this section we discuss mixed variational source problems and the approximation of solutions of such problems by mixed methods, with and without the presence of numerical integration. We quote and derive certain error estimates for the source problems in order to obtain the error estimates for the corresponding eigenvalue problems in the next section.

In subsection 3.1, we give a short discussion on mixed variational source problems and mixed methods at an abstract level and present the so called 'Babuška-Brezzi' conditions.

In subsection 3.2, we state the mixed formulation for the Biharmonic problem, and an associated mixed method. The mixed method that we will use is called the Ciarlet-Raviart method, but in the context of mesh-dependent norms as was done in [**?**]. We proved an estimate in Proposition **??** which was also obtained in [**?**] using a different approach. In Propositions **??**, **??**, and **??**, we proved that the mixed method under numerical integration satisfy the conditions required by the Babuška-Brezzi theory with respect to the mesh-dependent norms. Lemma **??** is an important result of this section from which we have derived error estimates given in Corollaries **??** and **??** which will be used in the next section. We then proved, using a duality argument, a sharper estimate stated in Lemma **??**, which we will also use later on.

3.1 Abstract Theory

Our approach in obtaining eigenvalue error estimates requires the error estimate of the solution operators of source problems, so we turn to the study of these source problems. Consider problems of the form:

Problem 3.1. *Given $g \in W'$ find $(u, \psi) \in V \times W$ satisfying*

$$a(u,\sigma) + b(\sigma,\psi) = 0 \qquad \forall \sigma \in V,$$
$$b(u,v) = \langle g, v\rangle \qquad \forall v \in W.$$

where V and W are real Hilbert spaces and $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous bilinear forms on $V\times V$ and $V\times W$, respectively.

Babuška studied these kind of variational problems and gave the following necessary and sufficient conditions for Problem **??** to have a unique solution:

$$\sup_{(v,\phi)\in V\times W-\{0\}} \frac{|a(u,v)+b(v,\psi)+b(u,\phi)|}{\|v\|_V+\|\phi\|_W} \geq \tau_1(\|u\|_V+\|\psi\|_W) \quad \forall(u,\psi)\in V\times W,$$

from some positive constant τ_1, and

$$\sup_{(u,\psi)\in V\times W-\{0\}} \frac{|a(u,v)+b(v,\psi)+b(u,\phi)|}{\|u\|_V+\|\psi\|_W} \geq \tau_2(\|v\|_V+\|\phi\|_W) \quad \forall(v,\phi)\in V\times W,$$

for some positive constant τ_2. A detailed proof is given by Babuška in [**?**] and Babuška and Aziz in [**?** , Theorem 6.2.1, p186].

Brezzi studied these problems and prove the next theorem in [**?**]; a complete demonstration may also be found in [**?** , section I, Theorem 4.1]. In the same paper, Brezzi also showed that his conditions are equivalent to those of Babuška.

Theorem 3.1. *Let $Z=\{\psi\in V: \forall\phi\in W,\ b(\psi,\phi)=0\}$. If there are positive constants $k,\bar{k}$ and γ such that*

$$\sup_{v\in Z-\{0\}} \frac{|a(\psi,v)|}{\|v\|_V} > k\ \|\psi\|_V \quad \forall\psi\in Z, \tag{3.1}$$

$$\sup_{\psi\in Z-\{0\}} \frac{|a(\psi,v)|}{\|\psi\|_V} > \bar{k}\ \|v\|_V \quad \forall v\in Z, \tag{3.2}$$

$$\sup_{\psi\in V-\{0\}} \frac{|b(\psi,\phi)|}{\|\psi\|_V} > \gamma\ \|\phi\|_W \quad \forall\phi\in W, \tag{3.3}$$

then for each $g\in W'$ Problem **??** *has a unique solution.*

Remark 1. If $a(\cdot,\cdot)$ is symmetric, both of the aforementioned conditions are the same. Conditions (**??**) is often called the *inf-sup condition*; some authors refer to it as the *Brezzi condition*, others as the *Babuška-Brezzi condition.* The current trend seems to be the expression *LBB condition*, where LBB stands for Ladyzhenskaya, Babuška, Brezzi.

Given finite dimensional spaces $V_h\subset V$ and $W_h\subset W$, indexed by the parameter $0<h<1$, the Ritz-Galerkin approximation (u_h,ψ_h) to (u,ψ) is defined as the solution of the problem:

Problem 3.2. *Find $(u_h,\psi_h)\in V_h\times W_h$ satisfying*

$$a(u_h,v)+b(v,\psi_h)=0 \qquad \forall v\in V_h, \tag{3.4}$$

$$b(u_h,\phi)=\langle g,\phi\rangle \qquad \forall\phi\in W_h. \tag{3.5}$$

Finally, with numerical integration in mind we consider the following problem:

Problem 3.3. *Find $(\tilde{u}_h,\tilde{\psi}_h)\in V_h\times W_h$ satisfying*

$$a_h(\tilde{u}_h,v)+b_h(v,\tilde{\psi}_h)=0 \qquad \forall v\in V_h, \tag{3.6}$$

$$b_h(\tilde{u}_h,\phi)=\langle g_h,\phi\rangle \qquad \forall\phi\in W_h. \tag{3.7}$$

where the continuous bilinear forms $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ are defined on $V_h\times V_h$ and $V_h\times W_h$ respectively, and the continuous linear form g_h may only be defined on W_h. For polygonal domains, if the numerical quadrature scheme has enough degree of precision then the forms b and b_h are the same, (see Proposition **??** below). For more general domains, they will often be different no matter the degree of precision. When the forms b_h and b are the same then Problem **??** is just a restatement of Problem $(\tilde{P}_h)$ on page 36.

Babuška and Brezzi applied their theorems to several classes of problems and established abstract estimates of the approximation error. We will just give Brezzi results, since it is the one we will use in our work. In his 1974 paper [**?**], Brezzi proved the next two theorems.

Theorem 3.2. *Let $Z_h = \{v \in V_h \colon b(v,\phi) = 0,\ \forall \phi \in W_h\}$. Suppose there are positive constants $k_1, \bar{k}_1$ and γ_1 such that*

$$\sup_{v \in Z_h - \{0\}} \frac{|a(\psi, v)|}{\|v\|_V} > k_1 \ \|\psi\|_V \quad \forall \psi \in Z_h, \tag{3.8}$$

$$\sup_{\psi \in Z_h - \{0\}} \frac{|a(\psi, v)|}{\|\psi\|_V} > \bar{k}_1 \ \|v\|_V \quad \forall v \in Z_h, \tag{3.9}$$

$$\sup_{\psi \in V_h - \{0\}} \frac{|b(\psi, \phi)|}{\|\psi\|_V} > \gamma_1 \ \|\phi\|_W \quad \forall \phi \in W_h, \tag{3.10}$$

then for each $g \in W'$, Problem ?? has a unique solution, (u_h, ψ_h). If (u, ψ) is the solution of Problem ??, then there is a constant C, independent of h, for which the following estimate holds

$$\|u - u_h\|_V + \|\psi - \psi_h\|_W \le C \left\{ \inf_{v \in V_h} \|u - v\|_V + \inf_{\phi \in W_h} \|\psi - \phi\|_W \right\}.$$

The above theorem tells us that in choosing finite-dimensional spaces for the Ritz-Galerkin approximation we should choose V_h and W_h so that the three conditions (??), (??) and (??) are satisfied. We note that the conditions (??) and (??) may hold without conditions (??) and (??) being satisfied, since Z_h may not be contained in Z. The condition (??) is often called the finite *inf-sup* condition, or the finite Babuška-Brezzi condition, etc.

A good discussion on the uses of the next theorem can be found in [? , page 67].

Theorem 3.3. *Let $\tilde{Z}_h = \{\psi \in V_h \colon b_h(\psi,\phi) = 0,\ \forall \phi \in W_h\}$. Suppose there are positive constants $k_2, \bar{k}_2$ and γ_2 such that*

$$\sup_{v \in \tilde{Z}_h - \{0\}} \frac{|a_h(\psi, v)|}{\|v\|_V} > k_2 \ \|\psi\|_V \quad \forall \psi \in \tilde{Z}_h, \tag{3.11}$$

$$\sup_{\psi \in \tilde{Z}_h - \{0\}} \frac{|a_h(\psi, v)|}{\|\psi\|_V} > \bar{k}_2 \ \|v\|_V \quad \forall v \in \tilde{Z}_h, \tag{3.12}$$

$$\sup_{\psi \in V_h - \{0\}} \frac{|b_h(\psi, \phi)|}{\|\psi\|_V} > \gamma_2 \ \|\phi\|_W \quad \forall \psi \in W_h, \tag{3.13}$$

then for each $g_h \in W_h'$ Problem ?? has a unique solution, $(\tilde{u}_h, \tilde{\psi}_h)$. If (u_h, ψ_h) is the solution of Problem ??, then there is a constant C, independent of h, for which the following estimate holds

$$\|u_h - \tilde{u}_h\|_V + \|\psi_h - \tilde{\psi}_h\|_W \le C \left\{ \sup_{v \in V_h - \{0\}} \frac{|a(u_h, v) - a_h(u_h, v)|}{\|v\|_V} + \sup_{\phi \in W_h - \{0\}} \frac{|b(u_h, \phi) - b_h(u_h, \phi)|}{\|\phi\|_W} \right.$$
$$\left. + \sup_{v \in V_h - \{0\}} \frac{|b(v, \phi_h) - b_h(v, \phi_h)|}{\|v\|_V} + \sup_{v \in V_h - \{0\}} \frac{|\langle g - g_h, v\rangle|}{\|v\|_V} \right\}.$$

The above theorem tells us that in choosing finite-dimensional spaces for the Ritz-Galerkin approximation with numerical integration we should choose V_h and W_h and numerical quadrature schemes such that the conditions (??), (??), and (??) are true. Needless to say, the conditions (??), (??) and (??) may be satisfied without the conditions (??), (??), and (??) or the conditions (??), (??) and (??) being satisfied.

Theorems ?? and ?? give the optimum error estimates in the product space $V \times W$. This does not mean that the estimation of a single term, $\|\psi - \psi_h\|_V$ or $\|\psi_h - \tilde{\psi}_h\|_V$ for example, is necessarily optimal. We will obtain sharper estimates later in this section in the context of mixed variational formulation of the Biharmonic Problem.

Not every problems of the form ??,?? and ?? will satisfy all of the appropriate conditions of the Brezzi theory, in the standard norm of the problem, so these theorems can not be used in the error analysis of the approximations. Alternative analysis is needed to address such problems. Falk and Osborn in [?] gave a systematic approach to handle source problems that does not fit within the framework of the Babuška or Brezzi theories. Another approach is to construct norms that depend upon the spaces V_h and W_h so that the three conditions required by the Brezzi theory are satisfied with respect to these norms. This 'mesh' dependent approach was used by Babuška, Osborn and Pitkäranta in [?]. In both of these approaches, a further analysis is needed to obtain the optimal error estimate for $\|\psi - \psi_h\|_V$.

3.2 The Biharmonic Problem

Consider the following fourth order problem:

Problem 3.4. *Given g, find $\psi(x, y)$ satisfying*

$$\begin{aligned} \Delta^2\psi &= g, && (x, y) \in \Omega, \\ \psi &= \partial\psi/\partial n = 0, && (x, y) \in \partial\Omega. \end{aligned}$$

In order to use the finite element method to find an approximation, the problem is put into a variational form. Introducing the auxiliary variable $u = -\Delta\psi$, Problem **??** can be written as a second-order system:

$$\begin{aligned} u + \Delta\psi &= 0 && \text{in } \Omega, \\ \Delta u &= -g && \text{in } \Omega, \\ \psi &= \partial\psi/\partial n = 0 && \text{on } \Gamma. \end{aligned}$$

Remark 2. We briefly note that the solution to Problem **??** has the following regularity: If Ω is a convex polygon and $g \in H^{-1}(\Omega)$ then $\psi \in H^3(\Omega) \cap H_0^2(\Omega)$, $u = -\Delta\psi \in H^1(\Omega)$ and there is a constant C such that

$$\|u\|_{1,\Omega} + \|\psi\|_{3,\Omega} \le C\|g\|_{-1,\Omega}. \tag{3.1}$$

A proof of the above regularity result can be found in [**?**].

To obtain a variational formulation of Problem **??**, we multiply the first equation of the second order system by $\sigma \in H^1(\Omega)$, the second by $v \in H_0^1(\Omega)$, integrate over Ω, and use one of Green's formula, to get the variational formulation:

Problem 3.5. *Given $g \in H^{-1}(\Omega)$, find $(u, \psi) \in H^1(\Omega) \times H_0^1(\Omega)$, such that*

$$\begin{aligned} \int_\Omega u\sigma\ dx - \int_\Omega \nabla\sigma \cdot \nabla\psi\ dx &= 0 && \forall \sigma \in H^1(\Omega), \\ -\int_\Omega \nabla u \cdot \nabla v\ dx &= -\int_\Omega gv\ dx && \forall v \in H_0^1(\Omega). \end{aligned}$$

Unfortunately this straight forward variational formulations do not satisfy all of the conditions needed to use the Brezzi theory. In particular, the forms do not satisfy Condition **??** or Condition **??** with respect to the $H^1(\Omega)$-norm. Falk and Osborn in [**?**] gave a systematic approach to handle source problems that does not fit within the framework of the Babuška or Brezzi theories. We tried but could not extend the Falk-Osborn theory to get the required estimate when numerical integration was taken into account. So we turned to another method developed in [**?**] that use mesh dependent spaces and norms and the Brezzi theory to obtain error estimates in the standard Sobolev norms.

We recall the definition of the mesh dependent spaces, H_h^0 and H_h^2, and their norms on page **??**. We now give a formulation of the Source Problem **??** that we can analyze by using the systematic approach of Brezzi. Starting from the second order system as given after Problem **??**, we multiply the first equation by $\sigma \in H_h^0$, integrate the resulting equation over Ω and write the integral involving the Laplacian as integrals over the triangles. By this process we get,

$$0 = \int_\Omega u\sigma\ dx + \sum_{T \in \mathcal{T}_h} \int_T \sigma\Delta\psi\ dx - \int_{\Gamma_h} \sigma J\left(\frac{\partial\psi}{\partial n}\right)\ ds, \qquad \sigma \in H_h^0.$$

We then multiply the second equation of the second order system by $v \in H_h^2 \cap H_0^1(\Omega)$, and integrate the resulting equation over Ω and integrate by parts the term involving the Laplacian. By this process, we get

$$-\int_\Omega gv\ dx = \sum_{T \in \mathcal{T}_h} \int_T u\Delta v\ dx - \int_{\Gamma_h} uJ\left(\frac{\partial v}{\partial n}\right)\ ds \qquad v \in H_h^2 \cap H_0^1(\Omega).$$

Thus we arrive at the variational formulation:

Problem 3.6. *Given* $g \in H^{-1}(\Omega)$ *find* $(u,\psi) \in H_h^0 \times (H_h^2 \cap H_0^1(\Omega))$*, satisfying*

$$\int_\Omega u\sigma \; dx + \sum_{T\in\mathcal{T}_h} \int_T \sigma\Delta\psi \; dx - \int_{\Gamma_h} \sigma J\left(\frac{\partial\psi}{\partial n}\right) \; ds = 0 \qquad \forall\sigma \in H_h^0,$$

$$\sum_{T\in\mathcal{T}_h} \int_T u\Delta v \; dx - \int_{\Gamma_h} uJ\left(\frac{\partial v}{\partial n}\right) \; ds = -\int_\Omega gv \; dx \qquad \forall v \in H_h^2 \cap H_0^1(\Omega).$$

It is important to note that for $u \in H^1(\Omega)$ and $v \in H_h^2$, we have

$$\sum_{T\in\mathcal{T}_h} \int_T u\Delta v \; dx - \int_{\Gamma_h} uJ\left(\frac{\partial v}{\partial n}\right) \; ds = -\int_\Omega \nabla u \cdot \nabla v \; dx.$$

Since V_h and W_h are contained in both of those spaces, the finite element approximation with exact integration is then formulated as follow:

Problem 3.7. *Given* $g \in H^{-1}(\Omega)$ *find* $(u_h,\psi_h) \in V_h \times W_h$*, satisfying*

$$\int_\Omega u_h\sigma \; dx - \int_\Omega \nabla\sigma \cdot \nabla\psi_h \; dx = 0 \qquad \forall\sigma \in V_h,$$

$$-\int_\Omega \nabla u_h \cdot \nabla v \; dx = -\int_\Omega gv \; dx \qquad \forall v \in W_h.$$

Problems **??** and **??** are examples of Problems **??** and **??**, with g replaced by $-g$, $V = H_h^0$, $W = H_h^2 \cap H_0^1(\Omega)$, $\|\cdot\|_V = \|\cdot\|_{0,h}$, $\|\cdot\|_W = \|\cdot\|_{2,h}$, $a(u,v) = \int_\Omega uv \; dx$, and

$$b(u,v) = \sum_{T\in\mathcal{T}_h} \int_T u\Delta v \; dx - \int_{\Gamma_h} uJ\left(\frac{\partial v}{\partial n}\right) \; ds.$$

Babuška, Osborn and Pitkäranta in [**?**], verified that the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ satisfy all the hypotheses of Theorem **??** with respect to the norms $\|\cdot\|_{0,h}$ and $\|\cdot\|_{2,h}$. To be concise, they proved that the two forms are continuous and that there are constants k_1 and γ_1 independent of h for which the following conditions hold:

$$\sup_{v\in Z_h-\{0\}} \frac{|a(\psi,v)|}{\|v\|_{0,h}} > k_1 \; \|\psi\|_{0,h} \quad \forall\psi \in Z_h, \tag{3.2}$$

$$\sup_{\psi\in V_h-\{0\}} \frac{|b(\psi,\phi)|}{\|\psi\|_{0,h}} > \gamma_1 \; \|\phi\|_{2,h} \quad \forall\phi \in W_h. \tag{3.3}$$

Since the form $a(\cdot,\cdot)$ is symmetric Condition (**??**) implies the remaining condition required by the Brezzi theory. Let (u,ψ) and (u_h,ψ_h) be the solutions of Problems **??** and **??** respectively. Assuming that $\psi \in H^r(\Omega), r \geq 3$ and $k \geq 2$, they obtained the estimate

$$\|u - u_h\|_{0,h} + \|\psi - \psi_h\|_{2,h} \leq C \; h^{s-2} \; \|\psi\|_{s,\Omega}, \tag{3.4}$$

where $s = \min(r, k+1)$ and k is the maximum degree of the polynomial used in constructing the finite element space.

Remark 3. For a convex polygon, the regularity result (**??**) implies that $\psi \in H^3(\Omega)$, so $s \leq 3$. Thus in constructing the spaces V_h and W_h we only need to use polynomials of degree $k = 2$. Higher degree polynomials could be used, but would be more costly and would not improve the order of convergence. From now on, we work under the assumption that $k = 2$, which in light of the regularity result implies that $s = 3$. If ψ has more regularity then the methods in [**?**] show that the approximation obtained with exact integration will converge with a higher order of h when higher order polynomials are used.

From (??), Babuška, Osborn and Pitkäranta in [?] derived the estimate

$$\|u - u_h\|_{0,\Omega} \leq C\ h^{s-2}\ \|\psi\|_{s,\Omega}, \tag{3.5}$$

$$(\sum_{T\in\mathcal{T}_h} \|\psi - \psi_h\|_{2,T}^2)^{1/2} \leq C\ h^{s-2}\ \|\psi\|_{s,\Omega}. \tag{3.6}$$

Since $\psi - \psi_h \in H^1(\Omega)$, the last inequality implies that

$$\|\psi - \psi_h\|_{1,\Omega} = (\sum_{T\in\mathcal{T}_h} \|\psi - \psi_h\|_{1,T}^2)^{1/2} \leq C\ h^{s-2}\ \|\psi\|_{s,\Omega},$$

but using a duality argument they got a sharper estimate

$$\|\psi - \psi_h\|_{1,\Omega} \leq C\ h^{s-1}\ \|\psi\|_{s,\Omega}. \tag{3.7}$$

The above inequalities and the regularity result (??) yield the following,

Proposition 3.4.

$$\|u - u_h\|_{0,\Omega} \leq Ch\|g\|_{0,\Omega},$$
$$\|\psi - \psi_h\|_{1,\Omega} \leq Ch^2\|g\|_{0,\Omega}.$$

Proof. Using $s = 3$ (see remark **??**), the regularity result (??) and the fact that $\|g\|_{-1,\Omega} \leq C\|g\|_{0,\Omega}$ in the inequalities (??) and (??) give the desired bounds. □

We next establish an estimate for the error $\|u - u_h\|_{1,\Omega}$, which will be needed in estimating the error of the eigenvalue approximation. This result was not stated or proven in [?], but was proven by Falk-Osborn in [?] using a different approach. Since we are working in the framework of mesh-dependent norms of [?], we prove the result within this framework.

Proposition 3.5.

$$\|u - u_h\|_{1,\Omega} \leq C\|g\|_{0,\Omega}.$$

Proof. The triangle inequality gives

$$\|u - u_h\|_{1,\Omega} \leq \|u - \mathcal{I}_h u\|_{1,\Omega} + \|\mathcal{I}_h u - u_h\|_{1,\Omega}.$$

Using the standard interpolation result (??) and the inverse inequality (??) with $v = u_h - \mathcal{I}_h u$ we get

$$\|u - u_h\|_{1,\Omega} \leq C\|u\|_{1,\Omega} + \frac{C}{h}\|\mathcal{I}_h u - u_h\|_{0,\Omega}. \tag{3.8}$$

Since u is the second component of the solution to Problem **??**, the regularity result (??) gives

$$\|u\|_{1,\Omega} \leq C\|g\|_{-1,\Omega} \leq C\|g\|_{0,\Omega}. \tag{3.9}$$

We have

$$\|\mathcal{I}_h u - u_h\|_{0,\Omega} \leq \|\mathcal{I}_h u - u\|_{0,\Omega} + \|u - u_h\|_{0,\Omega}.$$

Using the standard interpolation result (??) and Proposition **??**, we get

$$\|\mathcal{I}_h u - u_h\|_{0,\Omega} \leq Ch|u|_{1,\Omega} + Ch\|g\|_{0,\Omega} \qquad \leq Ch\{|u|_{1,\Omega} + \|g\|_{0,\Omega}\}.$$

Using (??) in the above inequality we get

$$\|\mathcal{I}_h u - u_h\|_{0,\Omega} \leq Ch\|g\|_{0,\Omega} \tag{3.10}$$

Using (??) and (??) in (??) yields the desired result. □

We now consider the source problem in which the integrals are approximated by numerical quadrature. We examine the following

Problem 3.8. *Given* $g \in V_h$*, find* $(\tilde{u}_h, \tilde{\psi}_h) \in V_h \times W_h$*, satisfying*

$$a_h(\tilde{\psi}_h, \phi) + b_h(\phi, \tilde{\psi}_h) = 0 \qquad \forall \phi \in V_h,$$
$$b_h(\tilde{u}_h, \phi) = \langle g, v\rangle_h \qquad \forall v \in W_h,$$

where

$$a_h(u, v) = \sum_{T\in\mathcal{T}_h} \sum_l \omega_{l,T}(uv)(b_{l,T}),$$
$$b_h(u, \phi) = -\sum_{T\in\mathcal{T}_h} \sum_l \omega_{l,T}(\nabla u \cdot \nabla \phi)(b_{l,T}),$$
$$\langle g, \phi\rangle_h = -\sum_{T\in\mathcal{T}_h} \sum_l \omega_{l,T}(g\phi)(b_{l,T}).$$

Problem **??** can be defined for $g \in H^{-1}(\Omega) \cap C^0(\Omega)$, but since we will use Problem **??** in the context of eigenvalue approximation, where g will be replaced by an approximate eigenvector which is in V_h, we consider Problem **??** only with $g \in V_h$. Moreover, this will simplify our analysis.

We now establish an error estimate between $(\tilde{u}_h, \tilde{\psi}_h)$, the solution of Problem **??**, and (u_h, ψ_h), the solution of Problem **??** with $g \in V_h$, by using Theorem **??**. We first show that there are positive constants k_2 and γ_2 for which

$$\sup_{v\in Z_h-\{0\}} \frac{|a_h(\psi, v)|}{\|v\|_{0,h}} > k_2 \ \|\psi\|_{0,h} \quad \forall \psi \in Z_h,$$
$$\sup_{\psi\in V_h-\{0\}} \frac{|b_h(\psi, \phi)|}{\|\psi\|_{0,h}} > \gamma_2 \ \|\phi\|_{2,h} \quad \forall \phi \in W_h,$$

then prove the continuity of the two forms.

We recall the concepts of unisolvent sets and degree of precision and how they are related in subsection 2.3 on page **??**. The proof of the following proposition is in the lines of the proof of Theorem 4.1.2 in [**?**].

Proposition 3.6. *If* $\{\hat{b}_l\}_{l=1}^L$ *contains a* $P_k(\hat{T})$ *unisolvent set and* $\hat{\omega}_l > 0$ *then there is a constant* C *independent of* h *such that*

$$a_h(u, u) \geq C\|u\|_{0,h}^2 \qquad \forall u \in V_h. \tag{3.11}$$

Proof. Because $\{\hat{b}_l\}_{l=1}^L$ contains a $P_k(\hat{T})$ unisolvent set and $\hat{\omega}_l > 0$, if $\hat{u} \in P_k(\hat{T})$ then $\sum_l \hat{\omega}_l \hat{u}^2(\hat{b}_l) = 0$ iff $\hat{u} \equiv 0$. So the mapping taking $\hat{u}$ to $(\sum_l \hat{\omega}_l \hat{u}^2(\hat{b}_l))^{1/2}$ defines a norm over $P_k(\hat{T})$. Since the mapping $|\cdot|_{0,\hat{T}}$ is also a norm over the finite dimensional space $P_k(\hat{T})$, there is a constant $\hat{C} > 0$ such that

$$\hat{C}|\hat{u}|_{0,\hat{T}}^2 \leq \sum_l \hat{\omega}_l \hat{u}^2(\hat{b}_l).$$

By construction $u(b_{l,T}) = \hat{u}(\hat{b}_l)$ and $\omega_{l,T} = (\det\ B_T)\hat{\omega}_l$, thus we have

$$\sum_l \omega_{l,T} u^2(b_{l,T}) = (\det\ B_T) \sum_l \hat{\omega}_l \hat{u}^2(\hat{b}_l) \geq \hat{C}(\det\ B_T)|\hat{u}|_{0,\hat{T}}^2 \geq \hat{C}|u|_{0,T}^2,$$

where the last inequality follows from Theorem **??**. Since

$$a_h(u, u) = \sum_{T\in\mathcal{T}_h} \sum_l \omega_{l,T}(u^2)(b_{l,T}) \geq \hat{C} \sum_{T\in\mathcal{T}_h} |u|_{0,T}^2 \geq \hat{C}\ |u|_{0,\Omega}^2,$$

by Lemma **??** we have

$$a_h(u, u) \geq \frac{\tilde{C}}{C}\|u\|_{0,h}^2 \qquad u \in V_h.$$

□

Proposition 3.7. *If the quadrature scheme* $\{\hat{b}_l, \hat{\omega}_l\}_{l=1}^L$ *is exact for* $P_{2k-2}(\hat{T})$ *then there is a positive constant* γ_2 *such that*

$$\sup_{\psi \in V_h - \{0\}} \frac{|b_h(\psi, \phi)|}{\|\psi\|_{0,h}} > \gamma_2 \ \|\phi\|_{2,h} \quad \forall \phi \in W_h. \tag{3.12}$$

Proof. The spaces V_h and W_h are constructed by using function whose restriction to any triangle T is a polynomial of degree at most k, (see page **??**). The derivative of these functions when restricted to T is a polynomial of degree at most $k-1$. The product of two derivatives of these functions have degree at most $2k-2$. Thus if the quadrature scheme is exact for $P_{2k-2}(\hat{T})$ then $b_h(\cdot,\cdot) = b(\cdot,\cdot)$ on $V_h \times W_h$ and (**??**) will be satisfied by virtue of (**??**). □

Since the forms $b(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ are the same, the continuity of the latter is due to the continuity of the former. Thus we just need to show the continuity of the form $a_h(\cdot,\cdot)$.

Proposition 3.8. *There is a constant* $C > 0$*, independent of* h*, such that*

$$|a_h(\phi, v)| \le C\|\phi\|_{0,\Omega}\|v\|_{0,\Omega} \le C\|\phi\|_{0,h}\|v\|_{0,h}.$$

Proof. We begin by noting

$$|a_h(\phi, v)| = |\sum_{T\in\mathcal{T}_h}\sum_{l}\omega_{l,T}(\phi v)(b_{l,T})| \le C\sum_{T\in\mathcal{T}_h}|\phi|_{0,\infty,T}|v|_{0,\infty,T}.$$

Using the scaling inequality (**??**) and the equivalence of norms over the finite dimensional space $P_k(\hat{T})$, the above inequality becomes

$$|a_h(\phi, v)| \le C\sum_{T\in\mathcal{T}_h}|\hat{\phi}|_{0,\infty,\hat{T}}|\hat{v}|_{0,\infty,\hat{T}} \le C\sum_{T\in\mathcal{T}_h}|\hat{\phi}|_{0,\hat{T}}|\hat{v}|_{0,\hat{T}}.$$

Using the scaling inequality (**??**), we arrive at the inequality

$$|a_h(\phi, v)| \le C\sum_{T\in\mathcal{T}_h}|\phi|_{0,T}|v|_{0,T} \le C(\sum_{T\in\mathcal{T}_h}|\phi|_{0,T}^2)^{1/2}(\sum_{T\in\mathcal{T}_h}|v|_{0,T}^2)^{1/2} \le C|\phi|_{0,\Omega}|v|_{0,\Omega} \le C\|\phi\|_{0,h}\|v\|_{0,h}.$$

□

We come to the important lemma of this section which will be needed in estimating the eigenvalue errors.

Lemma 3.9.

$$\|u_h - \tilde{u}_h\|_{0,h} + \|\psi_h - \tilde{\psi}_h\|_{2,h} \le C \ h\|g\|_{0,\Omega}.$$

Proof. From Theorem **??** with the forms b and b_h equal, we get the inequality

$$\|u_h - \tilde{u}_h\|_{0,h} + \|\psi_h - \tilde{\psi}_h\|_{2,h} \le C\left\{\sup_{\phi\in V_h}\frac{|a(u_h,\phi) - a_h(u_h,\phi)|}{\|\phi\|_{0,h}} + \sup_{\mu\in W_h}\frac{|\langle g,\mu\rangle - \langle g,\mu\rangle_h|}{\|\mu\|_{2,h}}\right\}. \tag{3.13}$$

Now

$$|\langle g,\mu\rangle - \langle g,\mu\rangle_h| = \left|\sum_{T\in\mathcal{T}_h}E_T(g\mu)\right| \le \sum_{T\in\mathcal{T}_h}|E_T(g\mu)|. \tag{3.14}$$

Using Lemma **??** with $\phi = \mu, w = g$ we have

$$|E_T(g\mu)| \le C \ h^k \ |g|_{0,T} \ |\mu|_{k,T} \qquad \forall \mu \in W_h. \tag{3.15}$$

The inequalities (**??**) and (**??**) combine to give

$$\begin{aligned}|\langle g,\mu\rangle - \langle g,\mu\rangle_h| &\le C \ h^k\sum_{T\in\mathcal{T}_h}|g|_{0,T}|\mu|_{k,T} \le C \ h^k(\sum_{T\in\mathcal{T}_h}|g|_{0,T}^2)^{1/2}(\sum_{T\in\mathcal{T}_h}|\mu|_{k,T}^2)^{1/2}\\ &\le C \ h^k\|g\|_{0,\Omega}(\sum_{T\in\mathcal{T}_h}|\mu|_{k,T}^2)^{1/2} \qquad \text{since } g \in V_h \subset H^1(\Omega).\end{aligned}$$

Using the inverse inequality (**??**) with $m = k$ and $l = 2$, we get

$$|\langle g, \mu\rangle - \langle g, \mu\rangle_h| \leq C\ h^2 \|g\|_{0,\Omega} (\sum_{T\in\mathcal{T}_h} |\mu|^2_{2,T})^{1/2}.$$

Now

$$\|\mu\|^2_{2,h} = \sum_{T\in\mathcal{T}_h} \|\mu\|^2_{2,T} + h^{-1} \int_{\Gamma_h} \left| J\frac{\partial u}{\partial n}\right|^2 ds \geq \sum_{T\in\mathcal{T}_h} |\mu|^2_{2,T},$$

$$\therefore\ \frac{|\langle g, \mu\rangle - \langle g, \mu\rangle_h|}{\|\mu\|_{2,h}} \leq C\ h^2 \|g\|_{0,\Omega} \qquad \forall \mu \in W_h.$$

Hence

$$\sup_{\mu\in W_h} \frac{|\langle g, \mu\rangle - \langle g, \mu\rangle_h|}{\|\mu\|_{2,h}} \leq C\ h^2 \|g\|_{0,\Omega}. \tag{3.16}$$

Now

$$|a(u_h, \phi) - a_h(u_h, \phi)| = \left|\sum_{T\in\mathcal{T}_h} E_T(u_h\phi)\right| \leq \sum_{T\in\mathcal{T}_h} |E_T(u_h\phi)|. \tag{3.17}$$

By Lemma **??**, we get

$$\begin{aligned}|a(u_h, \phi) - a_h(u_h, \phi)| &\leq Ch^k \sum_{T\in\mathcal{T}_h} |u_h|_{k,T} |\phi|_{0,T} \leq Ch^k (\sum_{T\in\mathcal{T}_h} |u_h|^2_{k,T})^{1/2} (\sum_{T\in\mathcal{T}_h} |\phi|^2_{0,\Omega})^{1/2} \\ &\leq Ch^k (\sum_{T\in\mathcal{T}_h} |u_h|^2_{k,T})^{1/2} \|\phi\|_{0,\Omega} \qquad \text{since } \phi \in V_h \subset H^1(\Omega).\end{aligned}$$

Since

$$\|\phi\|^2_{0,h} = \int_\Omega |\phi|^2\ dx + h \int_{\Gamma_h} |\phi|^2\ ds \geq \|\phi\|^2_{0,\Omega},$$

we get

$$\sup_{\phi\in V_h} \frac{|a(u_h, \phi) - a_h(u_h, \phi)|}{\|\phi\|_{0,h}} \leq Ch^k (\sum_{T\in\mathcal{T}_h} |u_h|^2_{k,T})^{1/2} \qquad \forall\ \phi \in V_h.$$

Using the inverse inequality (**??**) with $m = k$ and $l = 1$, and the fact that $(\sum_{T\in\mathcal{T}_h} |u_h|^2_{1,T})^{1/2} = |u_h|_{1,\Omega}$, we obtain

$$\sup_{\phi\in V_h} \frac{|a(u_h, \phi) - a_h(u_h, \phi)|}{\|\phi\|_{0,h}} \leq Ch|u_h|_{1,\Omega} \leq C\|u_h\|_{1,\Omega}.$$

Using the regularity result (**??**) and Proposition **??** we obtain

$$\|u_h\|_{1,\Omega} \leq \|u\|_{1,\Omega} + \|u - u_h\|_{1,\Omega} \leq C\|g\|_{-1,\Omega} + C\|g\|_{0,\Omega} \leq C\|g\|_{0,\Omega},$$

thus

$$\sup_{\phi\in V_h} \frac{|a(u_h, \phi) - a_h(u_h, \phi)|}{\|\phi\|_{0,h}} \leq Ch\|g\|_{0,\Omega} \qquad \forall\ \phi \in V_h. \tag{3.18}$$

Combining (**??**), (**??**) and (**??**), we get the desired result. □

Lemma **??** has the following three immediate corollaries, which will be needed later.

Corollary 3.10.

$$\|u_h - \tilde{u}_h\|_{0,\Omega} \le Ch\|g\|_{0,\Omega}.$$

Proof. The definition of the norm $\|\cdot\|_{0,h}$, (see page **??**), and Lemma **??** give the desired result. □

Corollary 3.11.

$$\|u_h - \tilde{u}_h\|_{1,\Omega} < C\|g\|_{0,\Omega}.$$

Proof. To get an estimate on the error $\|u_h - \tilde{u}_h\|_{1,\Omega}$, we use the inverse inequality (**??**) with $\phi = u_h - \tilde{u}_h \in V_h$. We have

$$|u_h - \tilde{u}_h|_{1,\Omega} < \frac{C}{h}|u_h - \tilde{u}_h|_{0,\Omega},$$

which together with Corollary **??** implies

$$\|u_h - \tilde{u}_h\|_{1,\Omega} < C\|g\|_{0,\Omega}.$$

□

Corollary 3.12.

$$(\sum_{T\in\mathcal{T}_h} \|\psi_h - \tilde{\psi}_h\|^2_{2,T})^{1/2} \le Ch\|g\|_{0,\Omega}.$$

Proof. The definition of the norm $\|\cdot\|_{2,h}$,(see page **??**), with Lemma **??** give the desired estimate. □

The above corollary in turn implies

$$\|\psi_h - \tilde{\psi}_h\|_{1,\Omega} = (\sum_{T\in\mathcal{T}_h} \|\psi_h - \tilde{\psi}_h\|^2_{1,T})^{1/2} \le Ch\|g\|_{0,\Omega}.$$

This estimate is not optimal. In [**?**], Bakuska, Osborn and Pitkäranta got a sharper estimate on the error $\|\psi - \psi_h\|_{1,\Omega}$ by using a duality argument. We use similar ideas to get a sharper estimate on $\|\psi_h - \tilde{\psi}_h\|_{1,\Omega}$, which will be used later.

To use the duality argument we need to consider the adjoint problem that corresponds to Problem **??** :

Given $d \in H^{-1}(\Omega)$ find $(y, \lambda) \in H^1(\Omega) \times H^1_0(\Omega)$ satisfying

$$(P') \qquad a(\phi, y) + b(\phi, \lambda) = 0 \qquad \forall \phi \in H^1(\Omega), \tag{3.19}$$

$$b(y, v) = \langle d, v\rangle \qquad \forall v \in H^1_0(\Omega). \tag{3.20}$$

Remark 4. In the case when the forms $a(\cdot,\cdot)$ is not symmetric, the adjoint Problem (P') will not be the same as the original problem. In our case, the form $a(\cdot,\cdot)$ is symmetric and so we have the following regularity result: If Ω is a convex polygon and $d \in H^{-1}(\Omega)$ then $\lambda \in H^3(\Omega) \cap H^2_0(\Omega)$, $y = -\Delta\lambda \in H^1(\Omega)$ and there is a constant C such that

$$\|y\|_{1,\Omega} + \|\lambda\|_{3,\Omega} \le C\ \|d\|_{-1,\Omega}. \tag{3.21}$$

Even though the adjoint and original problems in our situation are the same, we continue to work with the adjoint problem in the duality argument. We do this to show how the duality argument depends upon the adjoint problem.

To get to our main error estimate we will need the following:

Lemma 3.13.

$$\|\psi_h - \tilde{\psi}_h\|_{1,\Omega} = \sup_{d\in H^{-1}(\Omega)} \frac{|\langle d, \psi_h - \tilde{\psi}_h\rangle|}{\|d\|_{-1,\Omega}} \le C\ h^2\|g\|_{0,\Omega}.$$

The proof of Lemma **??** will be reduced to proving two propositions.

Proposition 3.14. *Let (y,λ) be the solution of Problem (P') corresponding to d, and let $\mathcal{I}_h y$ and $\mathcal{I}_h\lambda$ be their interpolants, then*

$$\begin{aligned}|\langle d,\psi_h-\tilde{\psi}_h\rangle| \le C\ \{&(\|u_h-\tilde{u}_h\|_{0,h}+\|\psi_h-\tilde{\psi}_h\|_{2,h})\ \|y-\mathcal{I}_h y\|_{0,h}\\ &+|a(\tilde{u}_h,\mathcal{I}_h y)-a_h(\tilde{u}_h,\mathcal{I}_h y)|+\|u_h-\tilde{u}_h\|_{0,h}\ \|\lambda-\mathcal{I}_h\lambda\|_{2,h}\\ &+|\langle g,\mathcal{I}_h\lambda\rangle-\langle g,\mathcal{I}_h\lambda\rangle_h|\}.\end{aligned}$$

Proposition 3.15. *Let (y,λ) be the solution of Problem (P') corresponding to d, and let $\mathcal{I}_h y$ and $\mathcal{I}_h\lambda$ be their interpolants, then*

$$|a(\tilde{u}_h,\mathcal{I}_h y)-a_h(\tilde{u}_h,\mathcal{I}_h y)| \le Ch^2\|g\|_{0,\Omega}\ \|d\|_{-1,\Omega}, \tag{3.22}$$

$$|\langle g,\mathcal{I}_h\lambda\rangle-\langle g,\mathcal{I}_h\lambda\rangle_h| \le Ch^2\|g\|_{0,\Omega}\ \|d\|_{-1,\Omega}. \tag{3.23}$$

Supposing that the two propositions are true, we prove Lemma **??**.

Proof of Lemma **??**. Using Lemma **??** together with the regularity result (**??**) we get

$$\|y-\mathcal{I}_h y\|_{0,h} \le Ch\ |y|_{1,\Omega} \le Ch\|d\|_{-1,\Omega}.$$

Using Lemma **??** together with the regularity result (**??**), we get

$$\|\lambda-\mathcal{I}_h\lambda\|_{2,h} \le Ch|\lambda|_{3,\Omega} \le Ch\|d\|_{-1,\Omega}.$$

Using Proposition **??**, the above inequalities and Lemma **??**, we get

$$\begin{aligned}|\langle d,\psi_h-\tilde{\psi}_h\rangle| \le C\ \{&h^2\ \|g\|_{0,\Omega}\ \|d\|_{-1,\Omega} \quad +|a(\tilde{u}_h,\mathcal{I}_h y)-a_h(\tilde{u}_h,\mathcal{I}_h y)|\\ &+h^2\ \|g\|_{0,\Omega}\ \|d\|_{-1,\Omega}+|\langle g,\mathcal{I}_h\lambda\rangle-\langle g,\mathcal{I}_h\lambda\rangle_h|\}.\end{aligned} \tag{3.24}$$

Using Proposition **??** in the inequality (**??**), we obtain

$$\|\psi_h-\tilde{\psi}_h\|_{1,\Omega} = \sup_{d\in H^{-1}(\Omega)} \frac{|\langle d,\psi_h-\tilde{\psi}_h\rangle}{\|d\|_{-1,\Omega}} \le C\ h^2\|g\|_{0,\Omega},$$

which is the desired result of Lemma **??**. □

All that remains now are the proofs of the two propositions.

Proof of Proposition **??**. From Problems **??** and **??**, by subtracting (**??**) from (**??**), we get

$$a(u_h,\phi)-a_h(\tilde{u}_h,\phi)+b(\phi,\psi_h-\tilde{\psi}_h)=0 \quad \forall\ \phi\in V_h,$$

which can be written as

$$a(u_h-\tilde{u}_h,\phi)+a(\tilde{u}_h,\phi)-a_h(\tilde{u}_h,\phi)+b(\phi,\psi_h-\tilde{\psi}_h)=0 \quad \forall\phi\in V_h. \tag{3.25}$$

From Problems **??** and **??**, by subtracting (**??**) from (**??**), we get

$$b(u_h-\tilde{u}_h,v)=\langle g,v\rangle-\langle g,v\rangle_h \qquad \forall v\in W_h,$$

which can be written as

$$b(u_h-\tilde{u}_h,v)-\langle g,v\rangle+\langle g,v\rangle_h=0 \qquad \forall v\in W_h. \tag{3.26}$$

Adding (**??**) and (**??**) together, we get

$$\langle d, v\rangle = a(\phi, y) + b(\phi, \lambda) + b(y, v) \qquad \forall(\phi, v) \in V \times W. \tag{3.27}$$

Let $\phi = u_h - \tilde{u}_h$ and $v = \psi_h - \tilde{\psi}_h$ in (**??**), we get

$$\langle d, \psi_h - \tilde{\psi}_h\rangle = a(u_h - \tilde{u}_h, y) + b(y, \psi_h - \tilde{\psi}_h) + b(u_h - \tilde{u}_h, \lambda). \tag{3.28}$$

Subtracting (**??**) from (**??**), we get

$$\begin{aligned}\langle d, \psi_h - \tilde{\psi}_h\rangle &= a(u_h - \tilde{u}_h, y-\phi) + a_h(\tilde{u}_h, \phi) - a(\tilde{u}_h, \phi)\\ &\quad + b(y-\phi, \psi_h - \tilde{\psi}_h) + b(u_h - \tilde{u}_h, \lambda) \qquad \forall\phi \in V_h.\end{aligned} \tag{3.29}$$

Subtracting (**??**) from (**??**), we get

$$\begin{aligned}\langle d, \psi_h - \tilde{\psi}_h\rangle &= a(u_h - \tilde{u}_h, y-\phi) + a_h(\tilde{u}_h, \phi) - a(\tilde{u}_h, \phi)\\ &\quad + b(y-\phi, \psi_h - \tilde{\psi}_h) + b(u_h - \tilde{u}_h, \lambda - v)\\ &\quad + \langle g, v\rangle - \langle g, v\rangle_h \qquad \forall(\phi, v) \in V_h \times W_h.\end{aligned} \tag{3.30}$$

Using the continuity of the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, we get

$$\begin{aligned}|\langle d, \psi_h - \tilde{\psi}_h\rangle| &\le C_1\ \|u_h - \tilde{u}_h\|_{0,h}\|y-\phi\|_{0,h} + |a_h(\tilde{u}_h, \phi) - a(\tilde{u}_h, \phi)|\\ &\quad + C_2\ \|y-\phi\|_{0,h}\|\psi_h - \tilde{\psi}_h\|_{2,h} + C_2\ \|u_h - \tilde{u}_h\|_{0,h}\|\lambda - v\|_{2,h}\\ &\quad + |\langle g, v\rangle - \langle g, v\rangle_h| \qquad \forall(\phi, v) \in V_h \times W_h\\ &\le C\ \{(\|u_h - \tilde{u}_h\|_{0,h} + \|\psi_h - \tilde{\psi}_h\|_{2,h})\|y-\phi\|_{0,h} + |a(\tilde{u}_h, \phi) - a_h(\tilde{u}_h, \phi)|\\ &\quad + \|u_h - \tilde{u}_h\|_{0,h}\|\lambda - v\|_{2,h} + |\langle g, v\rangle - \langle g, v\rangle_h|\} \qquad \forall(\phi, v) \in V_h \times W_h.\end{aligned}$$

Let $\phi = \mathcal{I}_h y$ and $v = \mathcal{I}_h\lambda$ be the interpolants of y and λ respectively, then

$$\begin{aligned}|\langle d, \psi_h - \tilde{\psi}_h\rangle| &\le C\ \{(\|u_h - \tilde{u}_h\|_{0,h} + \|\psi_h - \tilde{\psi}_h\|_{2,h})\|y - \mathcal{I}_h y\|_{0,h}\\ &\quad + |a(\tilde{u}_h, \mathcal{I}_h y) - a_h(\tilde{u}_h, \mathcal{I}_h y)|\\ &\quad + \|u_h - \tilde{u}_h\|_{0,h}\|\lambda - \mathcal{I}_h\lambda\|_{2,h} + |\langle g, \mathcal{I}_h\lambda\rangle - \langle g, \mathcal{I}_h\lambda\rangle_h|\}.\end{aligned}$$

□

Proof of Proposition **??**. We prove the inequality (**??**) first. We have

$$|\langle g, \mathcal{I}_h\lambda\rangle - \langle g, \mathcal{I}_h\lambda\rangle_h| = |\sum_{T\in\mathcal{T}_h} E_T(g\mathcal{I}_h\lambda)| \le \sum_{T\in\mathcal{T}_h} |E_T(g\mathcal{I}_h\lambda)|.$$

Using Lemma **??** with $k = 2$, we get

$$\begin{aligned}|\langle g, \mathcal{I}_h\lambda\rangle - \langle g, \mathcal{I}_h\lambda\rangle_h| &\le C\ h^2 \sum_{T\in\mathcal{T}_h} |g|_{0,T}|\mathcal{I}_h\lambda|_{2,T} \le C\ h^2 \sum_{T\in\mathcal{T}_h} |g|_{0,T}\|\mathcal{I}_h\lambda\|_{2,T},\\ &\le C\ h^2\ (\sum_{T\in\mathcal{T}_h} |g|_{0,T}^2)^{1/2}(\sum_{T\in\mathcal{T}_h} \|\mathcal{I}_h\lambda\|_{2,T}^2)^{1/2}.\end{aligned} \tag{3.31}$$

Now

$$(\sum_{T\in\mathcal{T}_h} \|\mathcal{I}_h\lambda\|_{2,T}^2)^{1/2} \le (\sum_{T\in\mathcal{T}_h} \|\lambda\|_{2,T}^2)^{1/2} + (\sum_{T\in tri} \|\lambda - \mathcal{I}_h\lambda\|_{2,T}^2)^{1/2}. \tag{3.32}$$

Using the standard interpolation result (**??**) with $m = 2$ and $k + 1 = 2$, we have that

$$(\sum_{T \in \mathcal{T}_h} \|\lambda - \mathcal{I}_h \lambda\|^2_{2,T})^{1/2} \leq C|\lambda|_{2,T}. \tag{3.33}$$

The inequalities (**??**) and (**??**) implies that there exist a constant, C, independent of h such that

$$(\sum_{T \in \mathcal{T}_h} \|\mathcal{I}_h \lambda\|^2_{2,T})^{1/2} \leq C\|\lambda\|_{2,\Omega}.$$

Using this result in (**??**), and the fact that $g \in H^0(\Omega)$ we get

$$|\langle g, \mathcal{I}_h \lambda\rangle - \langle g, \mathcal{I}_h \lambda\rangle_h| \leq C\ h^2 \|g\|_0 \|\lambda\|_{2,\Omega}.$$

The regularity result (**??**) gives the desired inequality

$$|\langle g, \mathcal{I}_h \lambda\rangle - \langle g, \mathcal{I}_h \lambda\rangle_h| \leq C\ h^2 \|g\|_{0,\Omega} \|d\|_{-1,\Omega}.$$

We next prove the inequality (**??**). We have

$$|a(\tilde{u}_h, \mathcal{I}_h y) - a_h(\tilde{u}_h, \mathcal{I}_h y)| \leq \sum_{T \in \mathcal{T}_h} |E_T(\tilde{u}_h\ \mathcal{I}_h y)|.$$

Using Lemma **??**, we get

$$|a(\tilde{u}_h, \mathcal{I}_h y) - a_h(\tilde{u}_h, \mathcal{I}_h y)| \leq C\ h^{2k} \sum_{T \in \mathcal{T}_h} |\tilde{u}_h|_{k,T} |\mathcal{I}_h y|_{k,T} \leq C\ h^{2k} (\sum_{T \in \mathcal{T}_h} |\tilde{u}_h|^2_{k,T})^{1/2} (\sum_{T \in \mathcal{T}_h} |\mathcal{I}_h y|^2_{k,T})^{1/2}.$$

Using the inverse inequality (**??**) with $m = k$ and $l = 1$ on both term, we get

$$|a(\tilde{u}_h, \mathcal{I}_h y) - a_h(\tilde{u}_h, \mathcal{I}_h y)| \leq C\ h^2\ (\sum_{T \in \mathcal{T}_h} |\tilde{u}_h|^2_{1,T})^{1/2} (\sum_{T \in \mathcal{T}_h} |\mathcal{I}_h y|^2_{1,T})^{1/2} \tag{3.34}$$

$$\leq C\ h^2\ |\tilde{u}_h|_{1,\Omega}\ |\mathcal{I}_h y|_{1,\Omega}. \tag{3.35}$$

We now need to bound the quantities $|\tilde{u}_h|_{1,\Omega}$ and $|\mathcal{I}_h y|_{1,\Omega}$ independently of h.
Using the familiar trick, we get

$$|\tilde{u}_h|_{1,\Omega} \leq |u_h|_{1,\Omega} + |u_h - \tilde{u}_h|_{1,\Omega} \leq |u|_{1,\Omega} + |u - u_h|_{1,\Omega} + |u_h - \tilde{u}_h|_{1,\Omega}.$$

Using the regularity result (**??**), Proposition **??**, and Corollary **??**, the above inequality becomes

$$|\tilde{u}_h|_{1,\Omega} \leq C\|g\|_{0,\Omega}. \tag{3.36}$$

Similarly, we have

$$|\mathcal{I}_h y|_{1,\Omega} \leq C|y|_{1,\Omega} + |y - \mathcal{I}_h y|_{1,\Omega}.$$

Using the interpolation error (**??**) and the regularity result (**??**) we get

$$|\mathcal{I}_h y|_{1,\Omega} \leq C|y|_{1,\Omega} \leq C\|d\|_{-1,\Omega}. \tag{3.37}$$

Using (**??**) and (**??**) in (**??**) we get the desired result,

$$|a(\tilde{u}_h, \mathcal{I}_h y) - a_h(\tilde{u}_h, \mathcal{I}_h y)| \leq C\ h^2 \|g\|_{0,\Omega} \|d\|_{-1,\Omega}.$$

□

4 Abstract Operator Theory

In this section, we present a brief summary of operator theory and prove some abstract results in spectral approximation of compact operators. We will use these results to obtain the desired spectral error estimates in section 5. 5.

In subsection 4.1, we give a brief survey of spectral theory. We also provide fairly detailed proofs of certain results of [?] and [?] which we will need later.

In subsection 4.2, we discuss abstract error estimates related to spectral approximation of compact operators. We prove Theorems **??** and **??** which are two of the main technical results of this paper. We further note that, even though these results will be used to prove an important theorem in the next section, Theorems **??** and **??** are important in their own right and could be used in the analysis of other problems.

In subsection 4.3, we discuss abstract error estimates related to the approximation of eigenvalues and eigenvectors of a class of mixed variational problems where certain bilinear forms have been 'perturbed' during approximation. We use the results obtained in subsection 3.2 to prove Theorem **??**, which is another important technical result of this paper. We will use Theorem **??** later in section 5 to prove error estimates related to the approximation of eigenvectors and eigenvalues of the Biharmonic problem under numerical integration.

4.1 Spectral Theory

We begin with a brief summary of spectral theory for compact operators, concentrating on what is relevant to our work. Those readers who are interested in the details are referred to Dunford and Schwartz, [?], in which a complete development is given.

Let $T\colon V \to V$ be a compact operator on a complex Banach space V with norm $\|\cdot\|$. We denote by $\rho(T)$ the resolvent set of T, i.e. the set

$$\rho(T) = \{z \in \mathbb{C}\colon (z-T)^{-1} \text{ exists as a bounded operator on } V\},$$

and by $\sigma(T)$ the spectrum of T, i.e., $\sigma(T) \overset{def}{=} \mathbb{C} - \rho(T)$. The set $\sigma(T)$ is countable with no nonzero limit points; nonzero numbers in $\sigma(T)$ are eigenvalues; and if zero is in $\sigma(T)$, it may or may not be an eigenvalue.

Let $\mu \in \sigma(T)$ be nonzero. There is a smallest integer α, called the ascent of $\mu - T$, such that $N((\mu-T)^{\alpha}) = N((\mu-T)^{\alpha+1})$, where N denotes the null space. $N((\mu-T)^{\alpha})$ is finite-dimensional and $m = \dim N((\mu-T)^{\alpha})$ is called the *algebraic multiplicity* of μ. The vectors in $N((\mu-T)^{\alpha})$ are called the *generalized eigenvectors* of T corresponding to μ. The *order of a generalized eigenvector* u is the smallest integer j such that $u \in N((\mu-T)^{j})$. The generalized eigenvectors of order 1, i.e., the vectors in $N(\mu - T)$, are the *eigenvectors* of T corresponding to μ. The *geometric multiplicity* of μ is equal to $\dim N(\mu - T)$, and is less than or equal to the algebraic multiplicity. The ascent of $\mu - T$ is one, and the two multiplicities are equal if V is a Hilbert space and T is self-adjoint; in this case the eigenvalues are real. If μ is an eigenvalue of T and v is a corresponding eigenvector, we will refer to (μ, v) as an *eigenpair* of T. Given two closed subspaces M and N of V, we define

$$\delta(M,N) = \sup_{\{x\in M:\|x\|=1\}} \operatorname{dist}(x,N) \quad \text{and} \quad \hat{\delta}(M,N) = \max(\delta(M,N), \delta(N,M)).$$

The quantity $\hat{\delta}(M,N)$ is called the *gap between M and N*, and is often used to formulate results on approximations of generalized eigenvectors. If V is a Hilbert space then $\delta(N,M) = \delta(M,N)$ and $\hat{\delta}(M,N) < 1$. For a discussion on the gap, the above fact and the next two theorems we refer to [?] and [? , pages 197-200].

Theorem 4.1. *If dim M = dim $N < \infty$, then $\delta(N,M) \le \delta(M,N)[1-\delta(M,N)]^{-1}$.*

Theorem 4.2. *Let M, N be closed linear subspaces of V. $\hat{\delta}(M,N) < 1$ implies* $\dim M = \dim N$.

Let W be a subspace of V, we will use the notation,

$$\|T\|_W = \sup_{w\in W} \frac{\|Tw\|}{\|w\|}.$$

For a Banach space V, we denote by V_h a family of finite dimensional subspaces of V that depend upon the real valued parameter h.

Suppose we want to approximate the eigenvalues of a compact operator $T\colon V \to V$, by using the eigenvalues of an operator $T_h\colon V \to V_h$, that approximates T in some sense. In many real world applications, T_h is expensive to compute, so it is approximated by another operator, $\tilde{T}_h\colon V_h \to V_h$. We then use the eigenvalues of $\tilde{T}_h$ to approximate those of T. When we apply our abstract theories to a concrete problem, T will be the solution operator of a variational problem, T_h will be a Ritz-Galerkin approximation to T, and $\tilde{T}_h$ will be a perturbation of T_h. In this paper we have considered the perturbation due to numerical integration. Some questions that comes immediately to mind are 'What conditions are required on the operators so that the approximate eigenvalues and eigenvectors converge to the actual values?' and 'How good is our approximation?'

The answer to the first question: if the spaces 'V_h approach V' and if the differences between T, T_h, and $\tilde{T}_h$ approach zero, in a sense defined below, then the approximate eigenvalues and eigenvectors will converge to the actual values. For the remainder of this section we assume the following three conditions:

$$\lim_{h\to 0} \operatorname{dist}(u, V_h) = 0 \qquad \forall u \in V, \tag{4.1}$$

$$\lim_{h\to 0} \|T - T_h\|_V = 0, \tag{4.2}$$

$$\lim_{h\to 0} \|T_h - \tilde{T}_h\|_{V_h} = 0. \tag{4.3}$$

With the use of the triangle inequality, (**??**) and (**??**) imply that

$$\lim_{h\to 0} \|T - \tilde{T}_h\|_{V_h} = 0. \tag{4.4}$$

We now proceed to show that these conditions imply that the approximate eigenpairs of $\tilde{T}_h$ converge to the corresponding eigenpairs of T. Our presentation combines the approaches of [**?**] and [**?**].

As a first step, we derive some results needed later. The first result is stated without proof in [**?**].

Proposition 4.3. *Let $F \subset \rho(T)$ be a closed set. There is a constant $C > 0$ such that*

$$C\|u\| \leq \|(z - T)u\| \quad \forall u \in V \text{ and } \forall z \in F,$$

$$\|R_z(T)\| \leq \frac{1}{C} \quad \forall z \in F. \tag{4.5}$$

The next proposition is essentially Lemma 1 of [**?**].

Proposition 4.4. *Let $F \subset \rho(T)$ be a closed set. There is a constant $C(T)$ independent of h such that*

$$\|R_z(T_h)\|_{V_h} \leq C(T) \qquad \forall z \in F \text{ and } \forall \text{ small } h. \tag{4.6}$$

and a constant $\tilde{C}(T)$, independent of h, such that

$$\|R_z(\tilde{T}_h)\|_{V_h} \leq \tilde{C}(T) \qquad \forall z \in F \text{ and } \forall \text{ small } h. \tag{4.7}$$

Let μ be a nonzero eigenvalue of T and let Γ be a circle centered at μ that lies in $\rho(T)$ and encloses no other points of $\sigma(T)$. We can define the spectral projection associated with T and μ by

$$E\colon V \to V, \qquad E = \frac{1}{2\pi\imath}\int_\Gamma R_z(T)\, dz. \tag{4.8}$$

The operator E is a projection onto the space of generalized eigenvectors associated with μ and T, i.e. $R(E) = N((\mu - T)^\alpha)$, where R denotes the range. Since its range is finite dimensional, E is bounded. For h sufficiently small, Γ will be a subset of $\rho(T_h) \cap \rho(\tilde{T}_h)$ and the projections

$$E_h\colon V \to V_h \qquad E_h = \frac{1}{2\pi\imath}\int_\Gamma R_z(T_h)\, dz, \tag{4.9}$$

$$\tilde{E}_h,\colon V_h \to V_h \qquad \tilde{E}_h = \frac{1}{2\pi\imath}\int_\Gamma R_z(\tilde{T}_h)\, dz, \tag{4.10}$$

exist. The operator E_h is the spectral projection associated with T_h and the eigenvalues of T_h which lie inside Γ and is a projection onto the direct sum of the spaces of generalized eigenvectors corresponding to these eigenvalues, i.e. if B is the open disk bounded by Γ,

$$R(E_h) = \sum_{\mu_h \in \ \sigma(T_h)\cap B} N((\mu_h - T_h)^{\alpha_{\mu_h}}),$$

where α_{μ_h} is the ascent of $\mu_h - T_h$. Similarly

$$R(\tilde{E}_h) = \sum_{\tilde{\mu}_h \in \ \sigma(\tilde{T}_h)\cap B} N((\tilde{\mu}_h - \tilde{T}_h)^{\alpha_{\tilde{\mu}_h}}),$$

where $\alpha_{\tilde{\mu}_h}$ is the ascent of $\tilde{\mu}_h - \tilde{T}_h$.

Using (**??**), (**??**) and (**??**) one can show that

$$\|E - E_h\|_V \le C\ \|T - T_h\|_V, \tag{4.11}$$

which together with (**??**) implies

$$\lim_{h\to 0} \|E - E_h\|_V = 0. \tag{4.12}$$

Using (**??**), (**??**), (**??**) and (**??**) one can show that

$$\|E_h - \tilde{E}_h\|_{V_h} \le C\ \|T_h - \tilde{T}_h\|_{V_h}, \tag{4.13}$$

which together with (**??**) implies

Proposition 4.5.

$$\lim_{h\to 0} \|E_h - \tilde{E}_h\|_{V_h} = 0.$$

Instead of proving these implications, we will prove a more general result which implies the inequalities (**??**) and (**??**). The first part of this result is similar to a statement in [**?** , page 115], which they say is immediate by means of Dunford integrals.

Lemma 4.6. *If $M \subset V$ and $R_z(T)(M) \subset M$, i.e M is an $R_z(T)$-invariant subset of V, then there is a constant C, independent of h, for which*

$$\|E - E_h\|_M \le C\ \|T - T_h\|_M. \tag{4.14}$$

If $M_h \subset V_h$ and $R_z(T_h)(M_h) \subset M_h$, i.e M_h is an $R_z(T_h)$-invariant subset of V_h, then there is a constant C, independent of h, for which

$$\|E_h - \tilde{E}_h\|_{M_h} \le C\ \|T_h - \tilde{T}_h\|_{M_h}. \tag{4.15}$$

Proof of (**??**). Let $v \in M_h$,

$$\|(E_h - \tilde{E}_h)v\| \le \frac{1}{2\pi}\left\|\int_\Gamma (R_z(T_h) - R_z(\tilde{T}_h))v\ dz\right\| \le \frac{1}{2\pi}\left\|\int_\Gamma R_z(\tilde{T}_h)(T_h - \tilde{T}_h)R_z(T_h)v\ dz\right\|,$$

since $R_z(T_h)(M_h) \subset M_h$, we have

$$\|(E_h - \tilde{E}_h)v\| \le \frac{1}{2\pi}\,length(\Gamma)\left(\sup_{z\in\Gamma}\|R_z(\tilde{T}_h)\|_{V_h}\right)(\|T_h - \tilde{T}_h\|_{M_h})\left(\sup_{z\in\Gamma}\|R_z(T_h)\|_{V_h}\right)\|v\|.$$

Dividing by $\|v\|$, taking sup and using (**??**) and (**??**) of Proposition **??** we obtain the bound (**??**)

$$\|E_h - \tilde{E}_h\|_{M_h} \le C\ \|T_h - \tilde{T}_h\|_{M_h}.$$

The derivation of the bound (**??**) proceeds in the same fashion, using (**??**) instead of (**??**). □

Remark 5. To get the bound (**??**), just use the above lemma and the fact that $R_z(T)(V)$ is a subset of V. Similarly to get the bound (**??**), use the above lemma and the fact that $R_z(T_h)(V_h)$ is a subset of V_h.

In turn, (**??**) and Proposition **??** and the triangle inequality implies the next lemma, which is Lemma 2 of [**?**].

Proposition 4.7.

$$\lim_{h\to 0} \|E - \tilde{E}_h\|_{V_h} = 0.$$

For the remainder of our work, we will use the following notations for the range of the projection operators.

$$M = R(E) = E(V) \subset V,$$
$$M_h = R(E_h) = E_h(V_h) \subset V_h,$$
$$\tilde{M}_h = R(\tilde{E}_h) = \tilde{E}_h(V_h) \subset V_h.$$

Suppose $\dim M = m < \infty$, it was shown by Osborn in [**?**] that $\lim_{h\to 0} \hat{\delta}(M, M_h) = 0$ and so $\hat{\delta}(M, M_h) < 1$ for h sufficiently small. From Theorem **??**, he concluded that $\dim M_h = \dim M$ for h small enough, which shows that exactly m eigenvalues of T_h lie inside Γ; denote these by $\{\mu_{h,j}\}_{j=1}^m$. If Γ' is another circle with arbitrary small radius we see that $\{\mu_{h,j}\}_{j=1}^m$ lie inside Γ', i.e.,

$$\lim_{h\to 0} \mu_{h,j} = \mu, \qquad j = 1, 2, ..., m.$$

We next show that $\lim_{h\to 0} \hat{\delta}(M, \tilde{M}_h) = 0$, which implies that $\dim \tilde{M}_h = \dim M$, and that there are exactly m, counting according to algebraic multiplicity, eigenvalues of $\tilde{T}_h$ inside of Γ which converge to μ. This result was proved and stated after Theorem 3 in [**?**].

We conclude that there are exactly m eigenvalues of $\tilde{T}_h$, counting according to algebraic multiplicity, which converge to μ, the eigenvalue of T under consideration.

To finish up with our summary of spectral theory, we list some facts that will be used in later development. The subspaces $R(E)$, $R(E_h)$ and $R(\tilde{E}_h)$ are invariant subspaces for T, T_h and $\tilde{T}_h$ respectively, being the eigenspaces associated with the eigenvalues used to define the spectral projection operators. Also $TE = ET$, $T_hE_h = E_hT_h$ and $\tilde{T}_h\tilde{E}_h = \tilde{E}_h\tilde{T}_h$. If μ is an eigenvalue of T with algebraic multiplicity m, then μ is an eigenvalue with algebraic multiplicity m of the adjoint operator T' on the dual space V'. The ascent of $\mu - T'$ will be α. E' will be the projection operator associated with T' and μ; likewise E'_h and $\tilde{E}'_h$ will be the projection operators associated respectively with T'_h and $\tilde{T}'_h$ and their corresponding eigenvalues that converge to μ. If $v \in V$ and $v' \in V'$, we denote the value of the linear functional v' at v by $\langle v', v\rangle$. Here T' is the Banach adjoint. If V is a Hilbert space, we would work with the Hilbert adjoint T^*. Then μ would be an eigenvalue of T if and only if $\bar{\mu}$ is an eigenvalue of T^*.

4.2 Abstract Estimate of the Errors

In his paper [**?**], Osborn proved the following two theorems. Let μ be a nonzero eigenvalue of T with algebraic multiplicity m and let α be the ascent of $\mu - T$. Let $\mu_{h,1}, ..., \mu_{h,m}$ be the eigenvalues of T_h that converge to μ. The first theorem gives an abstract error estimate of the generalized eigenvectors associated with $\mu_{h,1}, ..., \mu_{h,m}$ to the generalized eigenvectors associated with μ.

Theorem 4.8. *There is a constant C, independent of h, such that*

$$\hat{\delta}(R(E), R(E_h)) \leq C \ \|(T - T_h)|_{R(E)}\|.$$

for h small.

It is known that each of the eigenvalues $\mu_{h,1}, ..., \mu_{h,m}$ is close to μ for small h, but their arithmetic mean is generally a closer approximation to μ (cf. [**?**]). The second theorem gives an estimate between the arithmetic mean of the $\mu_{h,i}$'s and μ.

Theorem 4.9. *Let $\{\phi_i\}_{i=1}^m$ be a basis for $R(E)$ and let $\{\phi_i'\}_{i=1}^m$ be the corresponding dual basis in $R(E')$. Then there is a constant C, independent of h, such that*

$$|\mu - \frac{1}{m}\sum_{i=1}^{m}\mu_{h,i}| \le C\{\sum_{i,j=1}^{m}|\langle (T-T_h)\phi_j, \phi_i'\rangle| + \|(T-T_h)|_{R(E)}\| \ \|(T'-T_h')|_{R(E')}\|\}.$$

Proofs of these theorems can be found in [**?**] and [**?** , pages 685-687].

We will now show how the estimates in Theorems **??** and **??** change when T is approximated by $\tilde{T}_h$. When $\tilde{T}_h$ approximates T, we mean that T is first approximated by T_h, which is then approximated by $\tilde{T}_h$. Let $\tilde{\mu}_{h,1}, ..., \tilde{\mu}_{h,m}$ be the eigenvalues of $\tilde{T}_h$ which converge to μ. The next theorem gives an error estimate for the associated eigenvectors and should be compared with Theorem **??**. Our proof was suggested by the work of Banerjee and Osborn in [**?**].

Theorem 4.10. *There is a constant C independent of h, for h sufficiently small, such that*

$$\hat{\delta}(R(\tilde{E}_h), R(E)) \le C \ \{\|(T-T_h)|_{R(E)}\| + \|(T_h - \tilde{T}_h)|_{R(E_h)}\|\}.$$

Proof. As before, let $M = R(E)$ and $\tilde{M}_h = R(\tilde{E}_h)$. Using Theorem **??** with M as itself and $N = \tilde{M}_h$, we have

$$\begin{aligned}\delta(\tilde{M}_h, M) &\le C \ \delta(M, \tilde{M}_h)[1-\delta(M,\tilde{M}_h)]^{-1}\\ &\le 2 \ \delta(M, \tilde{M}_h),\end{aligned}$$

where the last inequality follows from the fact that $\lim_{h\to 0}\delta(M, \tilde{M}_h) = 0$. From the definition of the gap we obtain

$$\hat{\delta}(\tilde{M}_h, M) \le 2 \ \delta(M, \tilde{M}_h).$$

Thus we just need to show that

$$\delta(M, \tilde{M}_h) \le C \ \{\|(T-T_h)|_M\| + \|(T_h - \tilde{T}_h)|_{M_h}\|\}.$$

Let $v \in M$, we have

$$\|v - \tilde{E}_h E_h v\| \le \|v - E_h v\| + \|E_h v - \tilde{E}_h E_h v\|. \tag{4.1}$$

We now estimate each of the terms on the right hand side of (**??**). We have by (**??**) and the fact that $Ev = v$ for $v \in M$,

$$\|v - E_h v\| = \|Ev - E_h v\| \le \|E - E_h\|_M \ \|v\|.$$

Since M is an $R_z(T)$-invariant subset of V, (**??**) implies

$$\|v - E_h v\| \le C\|T - T_h\|_M \ \|v\|. \tag{4.2}$$

Since $E_h v \in M_h$ and $E_h^2 = E_h$, we have

$$\|E_h v - \tilde{E}_h E_h v\| = \|(E_h - \tilde{E}_h)E_h v\| \le \|E_h - \tilde{E}_h\|_{M_h} \ \|E_h v\|.$$

Using the triangle inequality and (**??**) we get

$$\begin{aligned}\|E_h v\| &\le \|E_h v - Ev\| + \|Ev\|\\ &\le \|E_h - E\| \ \|v\| + \|E\| \ \|v\| \le 2\|E\| \ \|v\| \ \text{ for } h \text{ sufficiently small.}\end{aligned}$$

Thus we get

$$\|E_h v - \tilde{E}_h E_h v\| \le C \ \|E_h - \tilde{E}_h\|_{M_h} \ \|v\|.$$

Since M_h is an $R_z(T_h)$-invariant subset of V, (**??**) implies

$$\|E_h v - \tilde{E}_h E_h v\| \leq C\|T_h - \tilde{T}_h\|_{M_h} \ \|v\|. \tag{4.3}$$

Using (**??**) - (**??**) and the definition of $\delta(M, \tilde{M}_h)$, we get

$$\delta(M, \tilde{M}_h) \leq C \ \{\|T - T_h\|_M + \|T_h - \tilde{T}_h\|_{M_h}\}.$$

□

We will now establish one of our main results, an estimate of the error between μ and the arithmetic mean of the $\tilde{\mu}_{h,i}$'s, which as far as we know has never been done. Let $\bar{\mu}_h = \dfrac{1}{m}\sum_{i=1}^m \mu_{h,i}$ and $\hat{\mu}_h = \dfrac{1}{m}\sum_{i=1}^m \tilde{\mu}_{h,i}$. Theorem **??** gives an estimate for $|\mu - \bar{\mu}_h|$. We will estimate $|\bar{\mu}_h - \hat{\mu}_h|$, then use the triangle inequality and Theorem **??** to get an estimate for $|\mu - \hat{\mu}_h|$. We use this round about approach to estimate $|\mu - \hat{\mu}_h|$, since we did not see an easy way to get the estimate directly. The main difficulty lies in the fact that the operators $\tilde{T}_h$ and $\tilde{E}_h$ may not be defined on the whole space V. Our proof is based upon the techniques found in [**?**].

Theorem 4.11. *Let $\{\phi_{h,i}\}_{i=1}^m$ be a normal basis for $R(E_h)$ and let $\{\phi'_{h,i}\}_{i=1}^m$ be the dual basis in $R(E'_h)$ as defined in the proof to follow. Then there is a constant C, independent of h, such that*

$$|\bar{\mu}_h - \hat{\mu}_h| \leq C\{\sum_{i,j=1}^m |\langle (T_h - \tilde{T}_h)\phi_{h,j}, \phi'_{h,i}\rangle| + \|(T_h - \tilde{T}_h)|_{R(E_h)}\| \ \|(T'_h - \tilde{T}'_h)|_{R(E'_h)}\|\}.$$

Proof. We first show that for small h, $\tilde{E}_h|_{M_h} : M_h \to \tilde{M}_h$ is one to one. From Proposition **??**, we have for h small enough $\|E_h - \tilde{E}_h\|_{V_h} < \frac{1}{2}$. Let $f \in M_h - \{0\}$. Since the projection operator E_h acts like the identity on its range, $f = E_h f$. If $\tilde{E}_h f = 0$ then

$$\|f\| = \|E_h f\| = \|E_h f - \tilde{E}_h f\| \leq \|E_h - \tilde{E}_h\|_{V_h} \ \|f\| < \frac{1}{2}\|f\|,$$

which implies that one is less than one-half. Hence $\tilde{E}_h f \neq 0$.

Since M_h is finite dimensional and $\tilde{E}_h|_{M_h}$ is one-to-one, we have that $\tilde{E}_h|_{M_h}$ is onto.

Thus $(\tilde{E}_h|_{M_h})^{-1} : \tilde{M}_h \to M_h$ is defined. Write $\tilde{E}_h^{-1}$ for $(\tilde{E}_h|_{M_h})^{-1}$. For h sufficiently small and $f \in M_h$ with $\|f\| = 1$, we have

$$\begin{aligned} 1 - \|\tilde{E}_h f\| &= \|E_h f\| - \|\tilde{E}_h f\| \\ &\leq \|(E_h - \tilde{E}_h) f\| \leq \|E_h - \tilde{E}_h\|_{V_h} \ \|f\| = \|E_h - \tilde{E}_h\|_{V_h} \leq \frac{1}{2}, \end{aligned}$$

and hence $\|\tilde{E}_h f\| \geq \dfrac{1}{2} \ \|f\|$. This implies $\|\tilde{E}_h^{-1}\|_{\tilde{M}_h}$ is bounded independently of h, for small h. We note that $\tilde{E}_h \tilde{E}_h^{-1}$ is the identity on $\tilde{M}_h$ and $\tilde{E}_h^{-1}\tilde{E}_h$ is the identity on M_h. We define $\hat{\tilde{T}}_h = \tilde{E}_h^{-1}\tilde{T}_h\tilde{E}_h|_{M_h} : M_h \to M_h$. Since $\tilde{M}_h = R(\tilde{E}_h)$ is the eigenspace which correspond to the eigenvalues $\{\tilde{\mu}_{h,i}\}_{i=1}^m$, it is $\tilde{T}_h$-invariant. We see that $\sigma(\hat{\tilde{T}}_h) = \{\tilde{\mu}_{h,i}\}_{i=1}^m$ and that the algebraic (geometric) multiplicity of any $\tilde{\mu}_{h,i}$ as an eigenvalue of $\hat{\tilde{T}}_h$ is equal to its algebraic (geometric) multiplicity as an eigenvalue of $\tilde{T}_h$. Letting $\hat{T}_h = T_h|_{M_h}$ we likewise see that $\sigma(\hat{T}_h) = \{\mu_{h,i}\}_{i=1}^m$. Thus trace $\hat{T}_h = m\bar{\mu}_h$ and trace $\hat{\tilde{T}}_h = m\hat{\mu}_h$ and, since $\hat{T}_h$ & $\hat{\tilde{T}}_h$ act on the same space, we can write

$$\bar{\mu}_h - \hat{\mu}_h = \frac{1}{m}\,\mathrm{trace}(\hat{T}_h - \hat{\tilde{T}}_h). \tag{4.4}$$

Let $\{\phi_{h,i}\}_{i=1}^m$ be a normal basis for $M_h = R(E_h)$ and let $\{\phi'_{h,i}\}_{i=1}^m$ be the corresponding dual basis, then from (**??**) we get

$$\bar{\mu}_h - \hat{\mu}_h = \frac{1}{m}\,\mathrm{trace}(\hat{T}_h - \hat{\tilde{T}}_h) = \frac{1}{m}\sum_{j=1}^m \langle (\hat{T}_h - \hat{\tilde{T}}_h)\phi_{h,j}, \phi'_{h,j}\rangle.$$

Taking absolute value and adding in the off-diagonal terms we get

$$|\bar{\mu}_h - \hat{\mu}_h| \leq \frac{1}{m} \sum_{i,j=1}^{m} |\langle (\hat{T}_h - \hat{\tilde{T}}_h)\phi_{h,j}, \phi'_{h,i}\rangle|. \tag{4.5}$$

Here each $\phi'_{h,j}$ is an element of $R(E_h)'$, the dual space of $R(E_h)$, but we can extend each $\phi'_{h,j}$ to all of V_h by defining it to be zero outside of $R(E_h)$, as done below.

$$V_h = R(E_h) \oplus N(E_h), \text{ define } \langle f, \phi'_{h,j}\rangle = 0 \text{ for } f \in N(E_h) \ .$$

Clearly $\phi'_{h,j}$ is bounded, i.e. $\phi'_{h,j} \in V'_h$. Since $\langle f, \phi'_{h,j}\rangle = 0 \ \forall f \in N(E_h)$, each $\phi'_{h,j} \in N(E_h)^{\perp}$. Since V_h is finite dimensional $N(E_h)^{\perp} = R(E'_h)$, so $\phi'_{h,j} \in R(E'_h)$ for $j = 1, 2, \ldots, m$. Using the facts that $\tilde{T}_h\tilde{E}_h = \tilde{E}_h\tilde{T}_h$ and $\tilde{E}_h^{-1}\tilde{E}_h$ is the identity on $R(E_h)$, we have

$$\begin{aligned}
\langle (\hat{T}_h - \hat{\tilde{T}}_h)\phi_{h,j}, \phi'_{h,i}\rangle &= \langle T_h\phi_{h,j} - \tilde{E}_h^{-1}\tilde{T}_h\tilde{E}_h\phi_{h,j}, \phi'_{h,i}\rangle \\
&= \langle \tilde{E}_h^{-1}\tilde{E}_h T_h\phi_{h,j} - \tilde{E}_h^{-1}\tilde{E}_h\tilde{T}_h\phi_{h,j}, \phi'_{h,i}\rangle \\
&= \langle \tilde{E}_h^{-1}\tilde{E}_h(T_h - \tilde{T}_h)\phi_{h,j}, \phi'_{h,i}\rangle \\
&= \langle (T_h - \tilde{T}_h)\phi_{h,j}, \phi'_{h,i}\rangle + \langle (\tilde{E}_h^{-1}\tilde{E}_h - I)(T_h - \tilde{T}_h)\phi_{h,j}, \phi'_{h,i}\rangle.
\end{aligned} \tag{4.6}$$

Since $\tilde{E}_h\tilde{E}_h^{-1}$ is the identity on $\tilde{M}_h = R(\tilde{E}_h)$, we have

$$\tilde{E}_h(\tilde{E}_h^{-1}\tilde{E}_h - I) = (\tilde{E}_h\tilde{E}_h^{-1})\tilde{E}_h - \tilde{E}_h = (\tilde{E}_h\tilde{E}_h^{-1} - I)\tilde{E}_h = 0.$$

Thus

$$\begin{aligned}
\langle (\tilde{E}_h^{-1}\tilde{E}_h - I)(T_h - \tilde{T}_h)\phi_{h,j}, \tilde{E}'_h\phi'_{h,i}\rangle &= \langle \tilde{E}_h(\tilde{E}_h^{-1}\tilde{E}_h - I)(T_h - \tilde{T}_h)\phi_{h,j}, \phi'_{h,i}\rangle \\
&= 0.
\end{aligned} \tag{4.7}$$

Since $\phi'_{h,i} \in R(E'_h)$, $E'_h\phi'_{h,i} = \phi'_{h,i}$, which together with (**??**) yield

$$\langle (\tilde{E}_h^{-1}\tilde{E}_h - I)(T_h - \tilde{T}_h)\phi_{h,j}, \phi'_{h,i}\rangle = \langle (\tilde{E}_h^{-1}\tilde{E}_h - I)(T_h - \tilde{T}_h)\phi_{h,j}, (E'_h - \tilde{E}'_h)\phi'_{h,i}\rangle,$$

which in turn yields

$$|\langle (\tilde{E}_h^{-1}\tilde{E}_h - I)(T_h - \tilde{T}_h)\phi_{h,j}, \phi'_{h,i}\rangle| \leq \|(\tilde{E}_h^{-1}\tilde{E}_h - I)\|_{V_h} \ \|(T_h - \tilde{T}_h)|_{M_h}\| \ \|\phi_{h,j}\| \|(E'_h - \tilde{E}'_h)|_{R(E'_h)}\| \ \|\phi'_{h,i}\|.$$

Using (**??**) applied to T'_h and $\tilde{T}'_h$ and the fact that the basis vectors are normalized we get

$$|\langle (\tilde{E}_h^{-1}\tilde{E}_h - I)(T_h - \tilde{T}_h)\phi_{h,j}, \phi'_{h,i}\rangle| \leq \|(\tilde{E}_h^{-1}\tilde{E}_h - I)\|_{V_h}\|(T_h - \tilde{T}_h)|_{M_h}\|\|(T'_h - \tilde{T}'_h)|_{R(E'_h)}\|. \tag{4.8}$$

We next bound the quantity $\|(\tilde{E}_h^{-1}\tilde{E}_h - I)\|_{V_h}$ independently of h. Let $v \in V_h$

$$\|(\tilde{E}_h^{-1}\tilde{E}_h - I)v\| \leq \|\tilde{E}_h^{-1}\tilde{E}_h v\| + \|v\| \leq \|\tilde{E}_h^{-1}\|_{\tilde{M}_h}\|\tilde{E}_h\|_{V_h}\|v\| + \|v\|. \tag{4.9}$$

Early in the proof, on page **??**, we show that $\|\tilde{E}_h^{-1}\|_{\tilde{M}_h}$ is bounded independently of h, so we just need a bound for $\|\tilde{E}_h\|_{V_h}$. Since

$$\|\tilde{E}_h\|_{V_h} \leq \|E\|_{V_h} + \|E - \tilde{E}_h\|_{V_h},$$

using the fact that E is bounded on the whole space V and Proposition **??**, we get

$$\|\tilde{E}_h\|_{V_h} \leq 2 \ \|E\| \ \text{ for } h \text{ sufficiently small.} \tag{4.10}$$

From (**??**), (**??**) and the fact that $\|\tilde{E}_h^{-1}\|_{\tilde{M}_h}$ can be bounded independently of h, we conclude that there is a constant, C, independent of h such that

$$\|(\tilde{E}_h^{-1}\tilde{E}_h - I)\|_{V_h} = \sup_{v\in V_h-\{0\}} \frac{\|(\tilde{E}_h^{-1}\tilde{E}_h - I)v\|}{\|v\|} \leq C. \tag{4.11}$$

Combining (**??**), (**??**), (**??**), (**??**), the fact that $M_h = R(E_h)$ and taking absolute value, we get the desired estimate. □

For completeness, we state the following theorem, which as far as we know has never been done, and which can be obtained by using Theorems **??** and **??** and the triangle inequality. It should be compared with Theorem **??**

Theorem 4.12. *Let $\{\phi_i\}_{i=1}^m$ be a basis for $R(E)$ and let $\{\phi'_i\}_{i=1}^m$ be the corresponding dual basis of $R(E')$. Let $\{\phi_{h,i}\}_{i=1}^m$ be a normal basis for $R(E_h)$ and let $\{\phi'_{h,i}\}_{i=1}^m$ be the corresponding dual basis of $R(E'_h)$. Then there is a constant C, independent of h, such that*

$$\begin{aligned}|\mu - \frac{1}{m}\sum_{i=1}^m \tilde{\mu}_{h,i}| \leq & C\{\sum_{i,j=1}^m |\langle (T-T_h)\phi_j, \phi'_i\rangle| + \|(T-T_h)|_{R(E)}\| \ \|(T'-T'_h)|_{R(E')}\| \\ & + \sum_{i,j=1}^m |\langle (T_h-\tilde{T}_h)\phi_{h,j}, \phi'_{h,i}\rangle| + \|(T_h-\tilde{T}_h)|_{R(E_h)}\| \ \|(T'_h-\tilde{T}'_h)|_{R(E'_h)}\|\}.\end{aligned}$$

Remark 6. In the proof of Theorem **??** we show that for any $1 \leq i, j \leq m$,
$|\langle(\hat{T}_h - \hat{\tilde{T}}_h)\phi_{h,j}, \phi'_{h,i}\rangle| \leq C\tilde{\delta}_h$ where

$$\tilde{\delta}_h = \sum_{i,j=1}^m |\langle (T_h-\tilde{T}_h)\phi_{h,j}, \phi'_{h,i}\rangle| + \|(T_h-\tilde{T}_h)|_{R(E_h)}\| \ \|(T'_h-\tilde{T}'_h)|_{R(E'_h)}\|.$$

Noting that $\langle(\hat{T}_h - \hat{\tilde{T}}_h)\phi_{h,j}, \phi'_{h,i}\rangle$ is a matrix representation of $\hat{T}_h - \hat{\tilde{T}}_h$, we see that

$$\|\hat{T}_h - \hat{\tilde{T}}_h\|_{M_h} \leq C\tilde{\delta}_h \tag{4.12}$$

In their survey paper [**?**], Babuška and Osborn provided a sketch of a proof of the following result relating the reciprocals of the eigenvalues of T and T_h:

Theorem 4.13.

$$\left|\mu^{-1} - \frac{1}{m}\sum_{j=1}^m \mu_{h,j}^{-1}\right| \leq C\sum_{i,j=1}^m |\langle (T-T_h)\phi_j, \phi'_i\rangle| + \|(T-T_h)|_{R(E)}\| \ \|(T'-T'_h)|_{R(E')}\|.$$

We now prove a related estimate for the average of the reciprocals of the eigenvalues of T_h and $\tilde{T}_h$, which is our next main result. Our method of proof may also be used to verify the above theorem.

Theorem 4.14.

$$\left|\frac{1}{m}\sum_{j=1}^m \mu_{h,j}^{-1} - \frac{1}{m}\sum_{j=1}^m \tilde{\mu}_{h,j}^{-1}\right| \leq C\tilde{\delta}_h.$$

Proof. We have

$$\begin{aligned}\left|\frac{1}{m}\sum_{j=1}^m \mu_{h,j}^{-1} - \frac{1}{m}\sum_{j=1}^m \tilde{\mu}_{h,j}^{-1}\right| &= \frac{1}{m}|\operatorname{trace}(\hat{T}_h^{-1} - \hat{\tilde{T}}_h^{-1})| \\ &\leq \|\hat{T}_h^{-1} - \hat{\tilde{T}}_h^{-1}\|_{M_h} = \|\hat{\tilde{T}}_h^{-1}(\hat{\tilde{T}}_h - \hat{T}_h)\hat{T}_h^{-1}\|_{M_h} \\ &\leq \|\hat{\tilde{T}}_h^{-1}\|_{M_h}\|\hat{T}_h - \hat{\tilde{T}}_h\|_{M_h}\|\hat{T}_h^{-1}\|_{M_h}.\end{aligned}$$

We use a result in [**?** , page 28] which is proven by using Cramer's rule and the equivalence of three different norms over a finite dimensional space and which says: For $A\colon X \to X$, nonsingular, with $N = \dim X$ finite, there is a constant γ depending upon only the norm defined on X such that

$$\|A^{-1}\| \leq \gamma \ \frac{\|A\|^{N-1}}{|\det A|}. \tag{4.13}$$

Using (**??**) with $\hat{T}_h^{-1}$ and $\hat{\tilde{T}}_h^{-1}$ in place of A, with $X = M_h$ and $N = m$, we get

$$\|\hat{T}_h^{-1}\|_{M_h} \leq \gamma \ \frac{\|\hat{T}_h\|_{M_h}^{m-1}}{|\det \hat{T}_h|}, \tag{4.14}$$

$$\|\hat{\tilde{T}}_h^{-1}\|_{M_h} \leq \gamma \ \frac{\|\hat{\tilde{T}}_h\|_{M_h}^{m-1}}{|\det \hat{\tilde{T}}_h|}. \tag{4.15}$$

Now the domain of the operator $\hat{T}_h$ is $M_h = E_h(V_h)$, the eigenspace of $\hat{T}_h$ corresponding to the m eigenvalues which converge to μ. Thus $\det \hat{T}_h$ is just the product of the eigenvalues of $\hat{T}_h$ which converges to μ. If we take h small enough we get the inequality $(|\mu|/2)^m \leq \det \hat{T}_h$. Similarly if we take h small enough we get the inequality $(|\mu|/2)^m \leq \det \hat{\tilde{T}}_h$. Using these facts in (**??**) and (**??**) we get

$$\|\hat{T}_h^{-1}\|_{M_h} \leq C \ \|\hat{T}_h\|_{M_h}^{m-1}, \tag{4.16}$$

$$\|\hat{\tilde{T}}_h^{-1}\|_{M_h} \leq C \ \|\hat{\tilde{T}}_h\|_{M_h}^{m-1}. \tag{4.17}$$

Since $\lim_{h\to 0}\|T - T_h\| = 0$, there exist a constant C independent of h, for h sufficiently small, so that the following chain of inequalities hold

$$\|\hat{T}_h\|_{M_h} = \|T_h|_{M_h}\| \leq C\|T|_{M_h}\| \leq C\|T\|.$$

Now $\hat{\tilde{T}}_h = \tilde{E}_h^{-1}\tilde{T}_h\tilde{E}_h|_{M_h}$ where $\tilde{E}_h^{-1} = (\tilde{E}_h|_{M_h})^{-1}\colon \tilde{M}_h \to M_h$, (see page **??**). Thus we have

$$\|\hat{\tilde{T}}_h\|_{M_h} \leq \|\tilde{E}_h^{-1}\|_{\tilde{M}_h} \ \|\tilde{T}_h\|_{\tilde{M}_h}\|\tilde{E}_h\|_{M_h}.$$

The condition $\lim_{h\to 0}\|E - \tilde{E}_h\|_{V_h} = 0$ implies that

$$\|\tilde{E}_h\|_{M_h} \leq \|\tilde{E}_h\|_{V_h} < 2\|E\|_{V_h} < 2\|E\|, \quad \text{for small } h.$$

Similarly $\lim_{h\to 0}\|T - \tilde{T}_h\|_{V_h} = 0$ implies that

$$\|\tilde{T}_h\|_{M_h} \leq \|\tilde{T}_h\|_{V_h} < 2\|T\|_{V_h} < 2\|T\| \quad \text{for small } h.$$

In the proof of Theorem **??**, we showed that $\|\tilde{E}_h^{-1}\|_{\tilde{M}_h}$ is bounded independently of h, for small h, (see page **??**). Hence $\|\hat{\tilde{T}}_h\|_{M_h}$ can be bounded independently of h when h is small. We conclude from (**??**) and (**??**) and the above sequence of inequalities that $\|\hat{T}_h^{-1}\|_{M_h}$ and $\|\hat{\tilde{T}}_h^{-1}\|_{M_h}$ can be bounded independently of h, for h sufficiently small. Thus

$$|\frac{1}{m}\sum_{j=1}^{m}\mu_{h,j}^{-1} - \frac{1}{m}\sum_{j=1}^{m}\tilde{\mu}_{h,j}^{-1}| \leq \|\hat{T}_h^{-1}\|\|\hat{T}_h - \hat{\tilde{T}}_h\|\|\hat{\tilde{T}}_h^{-1}\| \leq C \ \|\hat{T}_h - \hat{\tilde{T}}_h\|.$$

Combining the above with (**??**) yields the desired result. □

4.3 Spectral Approximation by Mixed Methods

In the introduction of this paper, we stated that the fourth order eigenvalue problem, Problem **??**, can be written in an equivalent mixed formulation given by Problem **??**. So we now turn to the study of problems given in mixed formulation. We give a survey of what was done by many authors among which includes the likes of Canuto in [**?**], Mercier, Osborn, Rappaz and Raviart in [**?**]. We then extend their results to Theorem **??** that we will use later in section 5.

Let V, W, H and G be four real Hilbert Spaces with inner products and norms $\langle\cdot,\cdot\rangle_V$, $\|\cdot\|_V$, $\langle\cdot,\cdot\rangle_W$, $\|\cdot\|_W$, $\langle\cdot,\cdot\rangle_H$, $\|\cdot\|_H$, and $\langle\cdot,\cdot\rangle_G$, $\|\cdot\|_G$, respectively. We impose the conditions that $V\subset H$ and $W\subset G$. Let $a(\sigma,\psi)$ and $b(\psi,u)$ be bilinear forms on $H\times H$ and $V\times W$, respectively, satisfying the following continuity conditions:

$$|a(\sigma,\psi)|\le C_1\ \|\sigma\|_H\ \|\psi\|_H \qquad \forall\sigma,\psi\in H, \tag{4.1}$$

and

$$|b(\psi,u)|\le C_2\ \|\psi\|_V\ \|u\|_W \qquad \forall\psi\in V, u\in W. \tag{4.2}$$

We assume $a(\sigma,\psi)$ is symmetric and satisfy

$$a(\sigma,\sigma)>0 \qquad \forall\sigma\in H-\{0\}.$$

Consider the eigenvalue problem

Problem 4.1. *Find $\lambda\in\mathbb{R},(u,\psi)\in V\times W$, both nonzero, satisfying*

$$\begin{aligned} a(u,v)+b(v,\psi)&=0 \qquad \forall v\in V,\\ b(u,\phi)&=-\lambda\langle\psi,\phi\rangle_G \qquad \forall\phi\in W.\end{aligned}$$

The quantity $(\lambda,(u,\psi))$ is called an eigenpair of this problem, and the functions u and ψ are said to be *components of the eigenvector or eigenfuntion.*

A discretization of Problem **??** is obtained by selecting finite dimensional spaces $V_h\subset V$ and $W_h\subset W$. We consider the approximate eigenvalue problem

Problem 4.2. *Find $\lambda_h\in\mathbb{R},(u_h,\psi_h)\in V_h\times W_h$, both nonzero, satisfying*

$$\begin{aligned} a(u_h,v)+b(v,\psi_h)&=0 \qquad \forall v\in V_h,\\ b(u_h,\phi)&=-\lambda_h\langle\psi_h,\phi\rangle_G \qquad \forall\phi\in W_h.\end{aligned}$$

The quantity $(\lambda_h,(u_h,\psi_h))$ is called an eigenpair of the above problem and is viewed as an approximation to $(\lambda,(u,\psi))$. Given bases for V_h and W_h, Problem **??** becomes a matrix eigenvalue problem. Now further suppose that we perturb the form $a(\cdot,\cdot)$ to the form $a_h(\cdot,\cdot)$, which may only be defined on $V_h\times V_h$. We required that a_h have the same properties as a, which means that a_h is symmetric and that

$$a_h(\sigma,\sigma)>0 \qquad \forall\sigma\in V_h-\{0\}.$$

And we approximate the inner product $\langle\cdot,\cdot\rangle_G$ by $\langle\cdot,\cdot\rangle_h$, which may be defined only on W_h. With these approximations, we consider the following approximate eigenvalue problem:

Problem 4.3. *Find $\tilde\lambda_h\in\mathbb{R},(\tilde u_h,\tilde\psi_h)\in V_h\times W_h$, both nonzero, satisfying*

$$\begin{aligned} a_h(\tilde u_h,v)+b(v,\tilde\psi_h)&=0 \qquad \forall v\in V_h,\\ b(\tilde u_h,\phi)&=-\tilde\lambda_h\langle\tilde\psi_h,\phi\rangle_h \qquad \forall\phi\in W_h.\end{aligned}$$

The quantity $(\tilde\lambda_h,(\tilde u_h,\tilde\psi_h))$ is called an eigenpair of the above problem and is viewed as an approximation to $(\lambda_h,(u_h,\psi_h))$ and $(\lambda,(u,\psi))$. Given bases for V_h and W_h, Problem **??** also becomes a matrix eigenvalue problem.

In order to estimate the error of our approximations, we consider the associated source problems, Problems **??**, **??**, and **??** given earlier, but restated here with g replaced by $-g$:

Given $g \in G$ find $(u, \psi) \in V \times W$ such that

$$(P) \qquad a(u, v) + b(v, \psi) = 0 \qquad \forall v \in V, \tag{4.3}$$

$$b(u, \phi) = -\langle g, \phi \rangle_G \qquad \forall \phi \in W. \tag{4.4}$$

Given $g \in G$ find $(u_h, \psi_h) \in V_h \times W_h$ such that

$$(P_h) \qquad a(u_h, v) + b(v, \psi_h) = 0 \qquad \forall v \in V_h, \tag{4.5}$$

$$b(u_h, \phi) = -\langle g, \phi \rangle_G \qquad \forall \phi \in W_h. \tag{4.6}$$

Given $g \in W_h$ find $(\tilde{u}_h, \tilde{\psi}_h) \in V_h \times W_h$ such that

$$(\tilde{P}_h) \qquad a_h(\tilde{u}_h, v) + b(v, \tilde{\psi}_h) = 0 \qquad \forall v \in V_h, \tag{4.7}$$

$$b(\tilde{u}_h, \phi) = -\langle g, \phi \rangle_h \qquad \forall \phi \in W_h. \tag{4.8}$$

The section on source problems, give conditions when these three problems are uniquely solvable for each g. Assuming they are uniquely solvable, we introduce the corresponding component solution operators:

$$\begin{aligned}
&S\colon G \to V, && Sg = u,\\
&S_h\colon G \to V_h, && S_h g = u_h,\\
&\tilde{S}_h\colon W_h \to V, && \tilde{S}_h g = \tilde{u}_h,\\
&T\colon G \to G, && Tg = \psi,\\
&T_h\colon G \to W_h, && T_h g = \psi_h,\\
&\tilde{T}_h\colon W_h \to W_h, && \tilde{T}_h g = \tilde{\psi}_h,
\end{aligned}$$

where $(u, \psi), (u_h, \psi_h), (\tilde{u}_h, \tilde{\psi}_h)$ are defined by $(P), (P_h), (\tilde{P}_h)$ respectively.

The eigenpair $(\lambda, (u, \psi))$ of Problem **??** can be characterized in terms of the operator T. To establish this, we first need to show that $\lambda > 0$. This follows from

$$\lambda = a(u, u)/\langle \psi, \psi \rangle_G,$$

which is implied by substituting $v = u$ and $\phi = \psi$ in the two equations of Problem **??**, subtracting the resultant equations, and using the fact that both components u and ψ of an eigenvector are nonzero. Now if $(\lambda, (u, \psi))$ is an eigenpair of Problem **??**, then $\lambda T\psi = \psi,\ u \neq 0$, and if $\lambda T\psi = \psi,\ \psi \neq 0$, then there is a $u \in V(u = S(\lambda\psi))$ such that $(\lambda, (u, \psi))$ is an eigenpair of Problem **??**. Thus λ is an eigenvalue of Problem **??** if and only if λ^{-1} is an eigenvalue of T. The correspondence between the eigenvectors is given by $\psi \leftrightarrow (u, \psi)$. In a similar way the approximate eigenvalues defined by Problems **??** and **??** can be characterized in terms of the eigenvalues of T_h and $\tilde{T}_h$ respectively: λ_h is an eigenvalue of Problem **??** if and only if λ_h^{-1} is an eigenvalue of T_h; $\tilde{\lambda}_h$ is an eigenvalue of Problem **??** if and only if $\tilde{\lambda}_h^{-1}$ is an eigenvalue of $\tilde{T}_h$. The correspondences between the eigenvectors are given by $\psi_h \leftrightarrow (u_h, \psi_h)$ and $\tilde{\psi}_h \leftrightarrow (\tilde{u}_h, \tilde{\psi}_h)$.

We assume that T is a bounded operator on G and that

$$\lim_{h \to 0} \|T - T_h\|_{GG} = 0, \tag{4.9}$$

where, for an operator $A\colon D(A) \subset X \to Y$, we let

$$\|A\|_{XY} = \sup_{w \in D(A)} \frac{\|Aw\|_Y}{\|w\|_X}.$$

Since dim $R(T_h)$ is finite for each h, the operators T_h are compact, which together with (**??**) implies that T is compact. We also note that T is self-adjoint on G. This can be proven as follow. Let $\phi = Tf$ in (**??**) to obtain

$$b(Sg, Tf) = -\langle g, Tf \rangle_G.$$

Again consider (P) with, with g replaced by f, and let $v = Sg$ in (**??**) to get

$$a(Sf, Sg) + b(Sg, Tf) = 0.$$

From these two equations we have

$$\langle g, Tf\rangle_G = a(Sf, Sg) \qquad \forall f, g \in G. \tag{4.10}$$

Using (**??**) and the symmetry of $a(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_G$ we get

$$\langle Tg, f\rangle_G = \langle f, Tg\rangle_G = a(Sg, Sf) = a(Sf, Sg) = \langle g, Tf\rangle_G,$$

which shows that T is self-adjoint. By similar methods we see that T_h and $\tilde{T}_h$ are self-adjoint.

We also assume that $\tilde{T}_h$ converge to T in the domain of $\tilde{T}_h$, i.e.

$$\lim_{h\to 0} \|T - \tilde{T}_h\|_{GG} = 0. \tag{4.11}$$

This Hypothesis along with (**??**) and the fact that the domain of T_h is all of G implies that $\tilde{T}_h$ converge to T_h in the domain of $\tilde{T}_h$, i.e.

$$\lim_{h\to 0} \|T_h - \tilde{T}_h\|_{GG} = 0. \tag{4.12}$$

Let λ^{-1} be an eigenvalue of T of multiplicity m. Since $\|T - T_h\|_{GG} \to 0$ we know from our survey of spectral theory that m eigenvalues of T_h converge to λ^{-1} (cf. page **??**). We denote these m eigenvalues by $\lambda_{h,1}^{-1}, \ldots, \lambda_{h,m}^{-1}$. Since T and T_h are self-adjoint all the eigenvalues have equal geometric and algebraic multiplicities. Let $M = R(E)$, the range of the spectral projection E associated with T and λ^{-1}. The following theorem gives an estimate of the error between λ and the arithmetic mean of $\{\lambda_{h,k}\}_{k=1}^m$. A proof is given in [**?**], [**?**] and [**?**]. These proofs use theorems that are equivalent to Theorem **??**.

Theorem 4.15. *Under the assumptions made above, there is a constant C such that*

$$\left|\lambda - \frac{1}{m}\sum_{k=1}^{m}\lambda_{h,k}\right| \le C\ \left\{\|(S - S_h)|_M\|_{GH}^2 + \|(S - S_h)|_M\|_{GV}\|(T - T_h)|_M\|_{GW} + \|(T - T_h)|_M\|_{GG}^2\right\}. \tag{4.13}$$

Since condition (**??**) holds, we know from our brief survey of spectral theory that m eigenvalues of $\tilde{T}_h$ converge to λ^{-1} (cf. page **??**). We denote these m eigenvalues by $\{\tilde{\lambda}_{h,i}^{-1}\}_{i=1}^m$. Since the operators T_h and $\tilde{T}_h$ are self-adjoint all eigenvalues have equal geometric and algebraic multiplicities. Let $M_h = R(E_h)$, the range of the spectral projection E_h associated with T_h and $\lambda_{h,1}^{-1}, \ldots, \lambda_{h,m}^{-1}$ and let $\{\phi_{h,i}\}_{i=1}^m$ be a basis for M_h, satisfying $\|\phi_{h,i}\|_G = 1$. We next establish one of our main results, a bound for the difference between the averages of the reciprocals of the eigenvalues of T_h and $\tilde{T}_h$, namely the values $\{\lambda_{h,i}\}_{i=1}^m$ and $\{\tilde{\lambda}_{h,i}\}_{i=1}^m$.

Theorem 4.16. *With the above conditions, there is a constant C such that*

$$\begin{aligned}\left|\frac{1}{m}\sum_{k=1}^{m}\lambda_{h,k} - \frac{1}{m}\sum_{k=1}^{m}\tilde{\lambda}_{h,k}\right| \le C\Big\{&\|(S_h - \tilde{S}_h)|_{M_h}\|_{GH}^2\\ &+ \|(S_h - \tilde{S}_h)|_{M_h}\|_{GV}\|(T_h - \tilde{T}_h)|_{M_h}\|_{GW} + \|(T_h - \tilde{T}_h)|_{M_h}\|_{GG}^2\\ &+ \sum_{i,j=1}^{m} |a(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j}) - a_h(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j})|\\ &+ \sum_{i,j=1}^{m} |\langle\phi_{h,i}, \tilde{T}_h\phi_{h,j}\rangle_G - \langle\phi_{h,i}, \tilde{T}_h\phi_{h,j}\rangle_h|\Big\}.\end{aligned}$$

Proof. Using Theorem **??**, with Hilbert spaces and the fact that $T_h - \tilde{T}_h$ is self-adjoint, we get

$$\left| \frac{1}{m} \sum_{k=1}^{m} \lambda_{h,k} - \frac{1}{m} \sum_{k=1}^{m} \tilde{\lambda}_{h,k} \right| \leq C \left\{ \sum_{i,j=1}^{m} |\langle (T_h - \tilde{T}_h)\phi_{h,j}, \phi_{h,i} \rangle_G| + \|(T_h - \tilde{T}_h)|_{M_h}\|_{GG}^2 \right\}. \tag{4.14}$$

For $g, f \in M_h$ we estimate $\langle (T_h - \tilde{T}_h)g, f \rangle_G$. Now adding (**??**) and (**??**) and using the definition of T_h and S_h, we get

$$-\langle g, \phi \rangle_G = a(S_h g, v) + b(v, T_h g) + b(S_h g, \phi) \qquad \forall\ (v, \phi) \in V_h \times W_h. \tag{4.15}$$

Letting $\phi = (T_h - \tilde{T}_h)f$ and $v = (S_h - \tilde{S}_h)f$ yields

$$-\langle g, (T_h - \tilde{T}_h)f \rangle_G = a(S_h g, (S_h - \tilde{S}_h)f) + b((S_h - \tilde{S}_h)f, T_h g) \quad + b(S_h g, (T_h - \tilde{T}_h)f). \tag{4.16}$$

Adding, (**??**) and (**??**), with g replaced by f, and using the definition of $\tilde{S}_h$, $\tilde{T}_h$, we obtain

$$-\langle f, \phi \rangle_h = a_h(\tilde{S}_h f, v) + b(v, \tilde{T}_h f) + b(\tilde{S}_h f, \phi) \qquad \forall\ (v, \phi) \in V_h \times W_h. \tag{4.17}$$

Replacing g by f in (**??**) and subtracting (**??**) from it we get

$$\begin{aligned} -\langle f, \phi \rangle_G + \langle f, \phi \rangle_h &= a((S_h - \tilde{S}_h)f, v) + a(\tilde{S}_h f, v) - a_h(\tilde{S}_h f, v) \\ &\quad + b(v, (T_h - \tilde{T}_h)f) + b((S_h - \tilde{S}_h)f, \phi) \qquad \forall\ (v, \phi) \in V_h \times W_h, \end{aligned}$$

or equivalently,

$$\begin{aligned} 0 &= a((S_h - \tilde{S}_h)f, v) + a(\tilde{S}_h f, v) - a_h(\tilde{S}_h f, v) \\ &\quad + b(v, (T_h - \tilde{T}_h)f) + b((S_h - \tilde{S}_h)f, \phi) \\ &\quad + \langle f, \phi \rangle_G - \langle f, \phi \rangle_h \qquad \forall (v, \phi) \in V_h \times W_h. \end{aligned} \tag{4.18}$$

Subtracting (**??**) from (**??**) and using the symmetry of $a(\cdot, \cdot)$ we get

$$\begin{aligned} -\langle g, (T_h - \tilde{T}_h)f \rangle_G &= a((S_h - \tilde{S}_h)f, S_h g - v) + b((S_h - \tilde{S}_h)f, T_h g - \phi) \\ &\quad + b(S_h g - v, (T_h - \tilde{T}_h)f) - a(\tilde{S}_h f, v) + a_h(\tilde{S}_h f, v) \\ &\quad - \langle f, \phi \rangle_G + \langle f, \phi \rangle_h \qquad \forall (v, \phi) \in V_h \times W_h. \end{aligned} \tag{4.19}$$

Using the continuity of $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ we get

$$\begin{aligned} |\langle g, (T_h - \tilde{T}_h)f \rangle_G| &\leq C_1\ \|(S_h - \tilde{S}_h)f\|_H \|S_h g - v\|_H + C_2 \|(S_h - \tilde{S}_h)f\|_V \|T_h g - \phi\|_W \\ &\quad + C_2 \|S_h g - v\|_V \|(T_h - \tilde{T}_h)f\|_W + |a(\tilde{S}_h f, v) - a_h(\tilde{S}_h f, v)| \\ &\quad + |\langle f, \phi \rangle_G - \langle f, \phi \rangle_h| \qquad \forall (v, \phi) \in V_h \times W_h. \end{aligned}$$

Letting $v = \tilde{S}_h g$ and $\phi = \tilde{T}_h g$ in the above inequality, we get

$$\begin{aligned} |\langle g, (T_h - \tilde{T}_h)f \rangle_G| &\leq C_1\ \|(S_h - \tilde{S}_h)f\|_H \|(S_h - \tilde{S}_h)g\|_H + C_2 \|(S_h - \tilde{S}_h)f\|_V \|(T_h - \tilde{T}_h)f\|_W \\ &\quad + C_2 \|(S_h - \tilde{S}_h)g\|_V \|(T_h - \tilde{T}_h)f\|_W \\ &\quad + |a(\tilde{S}_h f, \tilde{S}_h g) - a_h(\tilde{S}_h f, \tilde{S}_h g)| + |\langle f, \tilde{T}_h g \rangle_G - \langle f, \tilde{T}_h g \rangle_h|. \end{aligned}$$

The operator $T_h - \tilde{T}_h$ is self-adjoint, so

$$\langle g, (T_h - \tilde{T}_h)f \rangle_G = \langle (T_h - \tilde{T}_h)g, f \rangle_G.$$

Letting $g = \phi_{h,j}$, $f = \phi_{h,i}$ and using the fact that $\|\phi_{h,i}\|_G = 1$ we get

$$\begin{aligned}|\langle (T_h - \tilde{T}_h)\phi_{h,j}, \phi_{h,i}\rangle_G| \leq C_1 \ \|(S_h - \tilde{S}_h)|_{M_h}\|^2_{GH} + 2\ C_2\|(S_h - \tilde{S}_h)|_{M_h}\|_{GV}\|(T_h - \tilde{T}_h)|_{M_h}\|_{GW} \\ + |a(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j}) - a_h(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j})| + |\langle \phi_{h,i}, \tilde{T}_h\phi_{h,j}\rangle_G - \langle \phi_{h,i}, \tilde{T}_h\phi_{h,j}\rangle_h|.\end{aligned} \tag{4.20}$$

Combining (**??**) and (**??**) we get

$$\begin{aligned}\left|\frac{1}{m}\sum_{k=1}^{m}\lambda_{h,k} - \frac{1}{m}\sum_{k=1}^{m}\tilde{\lambda}_{h,k}\right| \leq C\Big\{&\|(S_h - \tilde{S}_h)|_{M_h}\|^2_{GH} \\ &+ \|(S_h - \tilde{S}_h)|_{M_h}\|_{GV}\|(T_h - \tilde{T}_h)|_{M_h}\|_{GW} + \|(T_h - \tilde{T}_h)|_{M_h}\|^2_{GG} \\ &+ \sum_{i,j=1}^{m}|a(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j}) - a_h(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j})| \\ &+ \sum_{i,j=1}^{m}|\langle \phi_{h,i}, \tilde{T}_h\phi_{h,j}\rangle_G - \langle \phi_{h,i}, \tilde{T}_h\phi_{h,j}\rangle_h|\Big\}.\end{aligned} \tag{4.21}$$

□

For completeness we state the following theorem, whose proof consists of using the triangle inequality with Theorems **??** and **??**.

Theorem 4.17.

$$\begin{aligned}\left|\lambda - \frac{1}{m}\sum_{k=1}^{m}\tilde{\lambda}_{h,k}\right| \leq C\ \Big\{&\|(S - S_h)|_M\|^2_{GH} + \|(S - S_h)|_M\|_{GV}\|(T - T_h)|_M\|_{GW} \\ &+ \|(T - T_h)|_M\|^2_{GG} + \|(S_h - \tilde{S}_h)|_{M_h}\|^2_{GH} \\ &+ \|(S_h - \tilde{S}_h)|_{M_h}\|_{GV}\|(T_h - \tilde{T}_h)|_{M_h}\|_{GW} + \|(T_h - \tilde{T}_h)|_{M_h}\|^2_{GG} \\ &+ \sum_{i,j=1}^{m}|a(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j}) - a_h(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j})| \\ &+ \sum_{i,j=1}^{m}|\langle \phi_{h,i}, \tilde{T}_h\phi_{h,j}\rangle_G - \langle \phi_{h,i}, \tilde{T}_h\phi_{h,j}\rangle_h|\Big\}.\end{aligned}$$

Remark 7. The estimate in the above theorem clearly shows how the estimate in Theorem **??** changed when $a(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_G$ are perturbed.

5 Biharmonic Eigenvalue Problems

In this section we present the error estimates for the approximate eigenvectors and eigenvalues of the Biharmonic problem, obtained from the Ciarlet-Raviart method with numerical integration. These error estimates are some of the main results of this paper and are given as Theorems **??** and **??**. We note that most of the results need to prove these theorems were derived in sections 3 & 4.

Consider the following fourth order eigenvalue problem:

Problem 5.1. *Find λ and $\psi(x, y) \neq 0$ satisfying*

$$\begin{aligned}\Delta^2\psi &= \lambda\psi & (x, y) &\in \Omega, \\ \psi &= \partial\psi/\partial n = 0 & (x, y) &\in \partial\Omega.\end{aligned}$$

This eigenvalue problem arises in connection with the small, transverse vibration of a clamped plate, (cf. [? , page 28]), and corresponds to the source Problem **??**.

In order to use the finite element method to obtain an approximation, the eigenvalue problem is put into a variational form. Introducing the auxiliary variable $u = -\Delta\psi$, Problem **??** can be written as a second-order system:

$$\begin{aligned} u + \Delta\psi &= 0 && \text{in } \Omega, \\ \Delta u &= -\lambda\psi && \text{in } \Omega, \\ \psi &= \partial\psi/\partial n = 0 && \text{on } \Gamma. \end{aligned}$$

We multiply the first equation of the preceding system by $\sigma \in H^1(\Omega)$, the second by $v \in H_0^1(\Omega)$, integrate over Ω, and use one of Green's formula, to get the variational formulations given in the introduction as Problems **??** and **??**.

Refering to these two problems, we define the two forms a, b and the inner product $< \cdot, \cdot >$ as:

$$\begin{aligned} a(\phi, \sigma) &= \int_\Omega \phi\sigma \; dx, \\ b(\sigma, \phi) &= -\int_\Omega \nabla\sigma \cdot \nabla\phi \; dx \\ \langle \phi, v \rangle &= \int_\Omega \phi v \; dx. \end{aligned}$$

In actual practice, a quadrature scheme is used to evaluate the integrals. Under numerical integration Problem **??** becomes the following problem whose convergence properties we will study:

Problem 5.2. *Find $\tilde{\lambda}_h \in \mathbb{R}$ and $(\tilde{u}_h, \tilde{\psi}_h) \in V_h \times W_h$, nonzero, such that*

$$\begin{aligned} a_h(\tilde{u}_h, \sigma) + b_h(\sigma, \psi_h) &= 0 && \forall \sigma \in V_h, \\ b_h(u_h, v) &= \tilde{\lambda}_h \langle \tilde{\psi}_h, v \rangle_h && \forall v \in W_h, \end{aligned}$$

where

$$\begin{aligned} a_h(\phi, \sigma) &= \sum_{T \in \mathcal{T}_h} \sum_l \omega_{l,T}(\phi\sigma)(b_{l,T}), \\ b_h(\sigma, \phi) &= -\sum_{T \in \mathcal{T}_h} \sum_l \omega_{l,T}(\nabla\sigma \cdot \nabla\phi)(b_{l,T}), \\ \langle \phi, v \rangle_h &= -\sum_{T \in \mathcal{T}_h} \sum_l \omega_{l,T}(\phi v)(b_{l,T}), \end{aligned}$$

and where $\{b_{l,T}, \omega_{l,T}\}_{l=1}^L$ determines a quadrature scheme on each triangle T. Since we are working under the assumption that the quadrature scheme has degree of precision $2k-1$, $b_h(v, \phi) = b(v, \phi)$ for $v, \phi \in V_h$. We note that this will also hold if we use a quadrature scheme of precision $2k-2$ in defining the form b_h .

We would like to know if the eigenvalues and eigenfunctions obtained by using numerical integration converges to the actual values at the same 'order of convergence' as the approximation with exact integration. And if so, what conditions are required on the quadrature scheme to get the same order of convergence as with exact integration.

All the effort of section 4 was motivated by the our desire to use Theorems **??**, **??**, and **??** to estimate the errors in the eigenvector and eigenvalue approximations. We use the first two theorems with $G = H = H^0(\Omega),\ W = H_0^1(\Omega),\ V = H^1(\Omega)$, (cf. page **??**). Now we need to check if all the conditions required to use these theorems are satisfied:

1. $a(\cdot, \cdot)$ and $a_h(\cdot, \cdot)$ are symmetric.
2. There is constant, C, such that

$$|a(v, \phi)| \le C\|v\|_{0,\Omega}\ \|\phi\|_{0,\Omega} \qquad \forall\ v,\ \phi \in H^0(\Omega).$$

3. There is a constant, C, such that

$$|a_h(v,\phi)| \le C\|v\|_{0,\Omega}\ \|\phi\|_{0,\Omega} \qquad \forall\ v,\ \phi \in V_h.$$

4. There is constant, C, such that

$$|b(v,\phi)| \le C\|v\|_{1,\Omega}\ \|\phi\|_{1,\Omega} \qquad \forall\ v \in H^1(\Omega),\ \phi \in H^1_0(\Omega).$$

5. The forms $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ satisfy

$$\begin{aligned} a(\sigma,\sigma) &> 0 && \forall \sigma \in H^0(\Omega) - \{0\}, \\ a_h(\sigma,\sigma) &> 0 && \forall \sigma \in V_h - \{0\}. \end{aligned}$$

Clearly $a(v,u) = \int_\Omega vu\,dx$ and $a_h(v,u)$ are symmetric. The second and third condition are the continuity conditions on the two forms and is just Proposition **??**. The fourth condition is just the continuity condition of the form $b(v,\phi) = \int_\Omega \nabla v \cdot \nabla \phi\,dx$, which is clear. The last condition is just the ellipticity conditions of the two forms $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ and is just Proposition **??**.

Using the definition of the solution operators (see page **??**), Proposition **??** and Proposition **??**, we get the following error estimates, for $g \in H^0(\Omega)$

$$\|Tg - T_h g\|_{1,\Omega} \le Ch^2\|g\|_{0,\Omega}, \tag{5.22}$$

$$\|Sg - S_h g\|_{0,\Omega} \le Ch\|g\|_{0,\Omega},$$

$$\|Sg - S_h g\|_{1,\Omega} \le C\|g\|_{0,\Omega}. \tag{5.23}$$

From which we obtained the bounds

Proposition 5.1.

$$\begin{aligned} \|(T-T_h)|_M\|_{H^0(\Omega),H^0(\Omega)} &\le Ch^2, \\ \|(T-T_h)|_M\|_{H^0(\Omega),H^1(\Omega)} &\le Ch^2, \\ \|(S-S_h)|_M\|_{H^0(\Omega),H^0(\Omega)} &\le Ch, \\ \|(S-S_h)|_M\|_{H^0(\Omega),H^1(\Omega)} &\le C. \end{aligned}$$

We note that the first inequality can easily be obtained from (**??**).

The above bounds are then used to prove the next theorem, which has been proven by many authors under different hypotheses, (cf. [**?**] and [**?**]).

Theorem 5.2.

$$\left|\lambda - \frac{1}{m}\sum_{k=1}^{m}\lambda_{h,k}\right| \le Ch^2.$$

Proof. Theorem **??** give the error bound

$$\begin{aligned} \left|\lambda - \frac{1}{m}\sum_{k=1}^{m}\lambda_{h,k}\right| \le C\{ &\|(S-S_h)|_M\|^2_{H^0(\Omega),H^0(\Omega)} \\ &+ \|(S-S_h)|_M\|_{H^0(\Omega),H^1(\Omega)}\|(T-T_h)|_M\|_{H^0(\Omega),H^1(\Omega)} \\ &+ \|(T-T_h)|_M\|^2_{H^0(\Omega),H^0(\Omega)}\}. \end{aligned}$$

From Proposition **??** we get

$$\left|\lambda - \frac{1}{m}\sum_{k=1}^{m}\lambda_{h,k}\right| \le Ch^2.$$

□

Using the definition of the solution operators and Lemma **??**, Corollary **??** and Corollary **??**, we get for $g \in V_h$

$$\|T_h g - \tilde{T}_h g\|_{1,\Omega} \leq Ch^2 \|g\|_{0,\Omega}, \tag{5.24}$$
$$\|S_h g - \tilde{S}_h g\|_{0,\Omega} \leq Ch \|g\|_{0,\Omega},$$
$$\|S_h g - \tilde{S}_h g\|_{1,\Omega} \leq C \|g\|_{0,\Omega}. \tag{5.25}$$

From which we obtained the bounds

Proposition 5.3.

$$\|(T_h - \tilde{T}_h)|_{M_h}\|_{H^0(\Omega),H^0(\Omega)} \leq Ch^2,$$
$$\|(T_h - \tilde{T}_h)|_{M_h}\|_{H^0(\Omega),H^1(\Omega)} \leq Ch^2,$$
$$\|(S_h - \tilde{S}_h)|_{M_h}\|_{H^0(\Omega),H^0(\Omega)} \leq Ch,$$
$$\|(S_h - \tilde{S}_h)|_{M_h}\|_{H^0(\Omega),H^1(\Omega)} \leq C.$$

The first inequality can easily be obtained from (**??**).

We are now in position to obtain an error estimate for the eigenvector approximation.

Theorem 5.4. *If the numerical quadrature scheme used in finding the generalized eigenvectors in the space $R(\tilde{E}_h)$ has degree of precision $2k-1$, then*

$$\hat{\delta}(R(\tilde{E}_h), R(E)) \leq Ch^2.$$

Proof. By Theorem **??** we have the estimate

$$\hat{\delta}(R(\tilde{E}_h), R(E)) \leq C \, \{\|(T - T_h)|_{R(E)}\|_{H^0(\Omega),H^0(\Omega)} + \|(T_h - \tilde{T}_h)|_{R(E_h)}\|_{H^0(\Omega),H^0(\Omega)}\}.$$

Using the first error bounds in Proposition **??** and **??** we get

$$\hat{\delta}(R(\tilde{E}_h), R(E)) \leq Ch^2.$$

□

Remark 8. We note that one gets the same order convergence, $O(h^2)$, for the approximate generalized eigenvectors obtained by using numerical integration as for the approximation obtained by using exact integration, (see [**?**]).

In order to use Theorem **??** to obtain an estimate, in terms of h, of the error $|\frac{1}{m}\sum_{i=1}^{m} \lambda_{h,i} - \frac{1}{m}\sum_{i=1}^{m} \tilde{\lambda}_{h,i}|$, we need to get bounds involving some power of h for the following two quantities,

$$\sum_{i,j=1}^{m} |a(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j}) - a_h(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j})|,$$
$$\sum_{i,j=1}^{m} |\langle \phi_{h,i}, \tilde{T}_h\phi_{h,j}\rangle - \langle \phi_{h,i}, \tilde{T}_h\phi_{h,j}\rangle_h|,$$

where $\{\phi_{h,i}\}_{i=1}^{m}$ is a basis for M_h, a subspace of V_h, such that $\|\phi_{h,i}\|_{0,\Omega} = 1$.

In our derivation of the required bounds, we will need the fact that both $\tilde{S}_h$ and $\tilde{T}_h$ are bounded independently of h with respect to certain norms.

Proposition 5.5. *There is a constant C, independent of h for which*

$$\|\tilde{S}_h\phi\|_{1,\Omega} \leq C \, \|\phi\|_{0,\Omega} \qquad \forall \phi \in V_h.$$

Proof.

$$\|\tilde{S}_h\phi\|_{1,\Omega} \le \|S\phi\|_{1,\Omega} + \|S\phi - S_h\phi\|_{1,\Omega} + \|S_h\phi - \tilde{S}_h\phi\|_{1,\Omega}$$

Using the definition of S, the regularity result (**??**), inequalities (**??**) and (**??**) we get the desired result. □

Proposition 5.6. *There is a constant C, independent of h for which*

$$\Big(\sum_{T\in\mathcal{T}_h} \|\tilde{T}_h\phi\|_{2,T}^2\Big)^{1/2} \le C\ \|\phi\|_{0,\Omega} \qquad \forall\phi \in V_h.$$

Proof. Using the triangle inequality twice, we get

$$\Big(\sum_{T\in\mathcal{T}_h} \|\tilde{T}_h\phi\|_{2,T}^2\Big)^{1/2} \le \Big(\sum_{T\in\mathcal{T}_h} \|T\phi\|_{2,T}^2\Big)^{1/2} + \Big(\sum_{T\in\mathcal{T}_h} \|T\phi - T_h\phi\|_{2,T}^2\Big)^{1/2} + \Big(\sum_{T\in\mathcal{T}_h} \|T_h\phi - \tilde{T}_h\phi\|_{2,T}^2\Big)^{1/2}.$$

Using the definition of T, the regularity result (**??**), we have

$$\|T\phi\|_{2,\Omega} = \Big(\sum_{T\in\mathcal{T}_h} \|T\phi\|_{2,T}^2\Big)^{1/2} \le C\|\phi\|_{-1,\Omega} \le C\|\phi\|_{0,\Omega}.$$

Using the regularity result (**??**) and inequality (**??**) with $s = 3$, we get

$$\Big(\sum_{T\in\mathcal{T}_h} \|T\phi - T_h\phi\|_{2,T}^2\Big)^{1/2} \le Ch\|T\phi\|_{3,\Omega} \le C\|\phi\|_{-1,\Omega} \le C\|\phi\|_{0,\Omega}.$$

Using the definition of T_h and $\tilde{T}_h$, Corollary **??** with $u = g$, we have

$$\Big(\sum_{T\in\mathcal{T}_h} \|T_h\phi - \tilde{T}_h\phi\|_{2,T,\Omega}^2\Big)^{1/2} \le Ch\|\phi\|_{0,\Omega}.$$

Combining the above inequalities gives the desired result. □

Proposition 5.7.

$$|a(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j}) - a_h(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j})| \le Ch^2|\phi_{h,i}|_{0,\Omega}\ |\phi_{h,j}|_{0,\Omega}.$$

Proof. Now $|a(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j}) - a_h(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j})| \le \sum_{T\in\mathcal{T}_h} |E_T(\tilde{S}_h\phi_{h,i}\tilde{S}_h\phi_{h,i})|$. By Lemma **??** we have,

$$|a(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j}) - a_h(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j})| \le Ch^{2k} \sum_{T\in\mathcal{T}_h} |\tilde{S}_h\phi_{h,i}|_{k,T}|\tilde{S}_h\phi_{h,j}|_{k,T}.$$

Using the inverse inequality (**??**) with $m = k$ and $l = 1$, we obtain

$$|a(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j}) - a_h(\tilde{S}_h\phi_{h,i}, \tilde{S}_h\phi_{h,j})| \le Ch^2 \sum_{T\in\mathcal{T}_h} |\tilde{S}_h\phi_{h,i}|_{1,T}|\tilde{S}_h\phi_{h,j}|_{1,T} \le Ch^2|\tilde{S}_h\phi_{h,i}|_{1,\Omega}\ |\tilde{S}_h\phi_{h,j}|_{1,\Omega}.$$

By Proposition **??**, we have $|\tilde{S}_h\phi_{h,i}|_{1,\Omega} \le C|\phi_{h,i}|_{0,\Omega}$, which when combined with the last inequality implies the desired result. □

Proposition 5.8.

$$|\langle\phi_{h,i}, \tilde{T}_h\phi_{h,j}\rangle - \langle\phi_{h,i}, \tilde{T}_h\phi_{h,j}\rangle_h| \le Ch^2|\phi_{h,j}|_{0,\Omega}\ |\phi_{h,i}|_{0,\Omega}.$$

Proof. Since $|\langle\phi_{h,i},\tilde{T}_h\phi_{h,j}\rangle-\langle\phi_{h,i},\tilde{T}_h\phi_{h,j}\rangle_h|\le\sum_{T\in\mathcal{T}_h}|E_T(\phi_{h,i}\tilde{T}_h\phi_{h,j})|$, by Lemma **??**, we have

$$|\langle\phi_{h,i},\tilde{T}_h\phi_{h,j}\rangle-\langle\phi_{h,i},\tilde{T}_h\phi_{h,j}\rangle_h|\le Ch^k\sum_{T\in\mathcal{T}_h}|\phi_{h,i}|_{0,T}|\tilde{T}_h\phi_{h,j}|_{k,T}.$$

Using the inverse inequality (**??**) on the last term with $m=k$ and $l=2$, we get

$$|\langle\phi_{h,i},\tilde{T}_h\phi_{h,j}\rangle-\langle\phi_{h,i},\tilde{T}_h\phi_{h,j}\rangle_h|\le Ch^2\sum_{T\in\mathcal{T}_h}|\phi_{h,i}|_{0,T}|\tilde{T}_h\phi_{h,j}|_{2,T}\le Ch^2(\sum_{T\in\mathcal{T}_h}|\phi_{h,i}|^2_{0,T})^{1/2}(\sum_{T\in\mathcal{T}_h}|\tilde{T}_h\phi_{h,j}|^2_{2,T})^{1/2}.$$

Proposition **??** with the above inequality gives the desired result. □

Now we are ready to prove one of our main results, the next theorem.

Theorem 5.9. *If the numerical integration scheme used to obtained the approximate eigenvalues* $\{\tilde{\lambda}_{h,i}\}_{i=1}^m$ *has precision of degree* $2k-1$*, then*

$$|\frac{1}{m}\sum_{i=1}^m\lambda_{h,i}-\frac{1}{m}\sum_{i=1}^m\tilde{\lambda}_{h,i}|\le Ch^2.$$

Proof. Theorem **??** gives the estimates

$$\begin{aligned}|\frac{1}{m}\sum_{i=1}^m\lambda_{h,i}-\frac{1}{m}\sum_{i=1}^m\tilde{\lambda}_{h,i}|\le C\{&\|(S_h-\tilde{S}_h)|_{M_h}\|^2_{H^0(\Omega),H^0(\Omega)}\\&+\|(S_h-\tilde{S}_h)|_{M_h}\|_{H^0(\Omega),H^1(\Omega)}\|(T_h-\tilde{T}_h)|_{M_h}\|_{H^0(\Omega),H^1(\Omega)}\\&+\|(T_h-\tilde{T}_h)|_{M_h}\|^2_{H^0(\Omega),H^0(\Omega)}\\&+\sum_{i,j=1}^m|a(\tilde{S}_h\phi_{h,i},\tilde{S}_h\phi_{h,j})-a_h(\tilde{S}_h\phi_{h,i},\tilde{S}_h\phi_{h,j})|\\&+\sum_{i,j=1}^m|\langle\phi_{h,i},\tilde{T}_h\phi_{h,j}\rangle_G-\langle\phi_{h,i},\tilde{T}_h\phi_{h,j}\rangle_h|\}.\end{aligned}$$

Using Propositions **??**, **??**, **??** and the fact that $\|\phi_{h,i}\|_{0,\Omega}=1$ in the above inequality we get the desired result. □

For completeness we state the next theorem, which is obtained by using Theorem **??**, Theorem **??** and the triangle inequality.

Theorem 5.10. *If the numerical integration scheme used to obtained the approximate eigenvalues* $\{\tilde{\lambda}_{h,i}\}_{i=1}^m$ *has precision of degree* $2k-1$*, then*

$$|\lambda-\frac{1}{m}\sum_{i=1}^m\tilde{\lambda}_{h,i}|\le Ch^2.$$

Remark 9. We note that one gets the same order of convergence, $O(h^2)$, for the approximate eigenvalues obtained by using numerical integration as the approximation obtained by using exact integration, (see Theorem **??**).

Remark 10. We obtained the order of h^2 convergence, under the condition that the eigenvector of λ lies in $H^3(\Omega)$, (cf. the regularity result (**??**)). But our methods can be used to show that the order of convergence is h^{2k-2} if the eigenvector lies in $H^{k+1}(\Omega)$ where k is the maximum degree of the polynomials, used in defining V_h.

6 Conclusion

We would like to extend our theorems, Theorem **??** and Theorem **??**, to the case where the domain is a connected and bounded domain, with a curved boundary, and where the finite element spaces are constructed by using isoparametric elements. From now on, by Ω we mean a connected and bounded domain, with a curved boundary. We will discuss some of the problems that arises and must be overcome when trying to extend our theorems in these directions.

Given Ω, there are two ways to approximate it. The first is to use a polygonal domain Ω_h lying complete inside Ω. For fourth order source problems, the approximation may not converge to the actual solution. An example of this is construction by Babuška in the book of Nečas [**?**]. The other way to approximate Ω is to use an 'external domain' Ω_h, which does not lie completely in Ω. Usually curved elements are used to construct Ω_h, so that the boundaries of Ω and Ω_h are very 'close' to each other. Generally the closer the boundaries are to each other the better the approximations.

There are a lot of issues that must be confronted when using an external approximation. The solution to the source problem, u, is only defined on Ω, and the solution to the approximate source problem is only defined on Ω_h. In order to compare the two solutions, we need to consider a set $\tilde{\Omega}$ which contains both Ω and Ω_h. After choosing the set $\tilde{\Omega}$, we need to consider extending the solution and approximate solution operators to the bigger set. Then we need to consider over which set to estimate the error of approximations. Most of the time, the error is approximated over the approximating set Ω_h.

In the author view, it seems very difficult to extend the theorems and methods used herein to a mixed formulation over domain with curved boundaries, the major difficulties lie in analyzing the source problems associated with the eigenvalue problems. This of course does not preclude that there may be an easier approach than the one used herein to analyze the problem. We end with a quote of Ciarlet and Raviart, from their paper in which they studied the source problems using mixed finite element methods:

> Because of the numerical complexity involved with the standard conforming finite elements for solving fourth-order problems, even in the case of polygonal boundaries, it seems unrealistic to handle curved boundaries with the associated curved isoparametric finite elements.
>
> With some numerical evidence to support this statement, it is suggested here that one proper way to handle fourth-order problems on curved domains is to use the method analogous to that described here, with corresponding isoparametric finite elements.

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