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Terminal Summation: Extending the Concept of Convergence

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Abstract

This paper presents an atypical method for summing divergent series, and provides a sum for the divergent series log(n). We use an idea of T.E. Phipps, called Terminal Summation, which uses asymptotic analysis to assign a value to divergent series. The method associates a series to an appropriate difference equations having boundary conditions at infinity, and solves the difference equations which then provide a value for the original series. We point out connections between Phipps' method, the Euler-MacLaurin sum formula, the Ramanujan sum and other traditional methods for summing divergent series.

1 Introduction

The problem of assigning values to divergent infinite series has engaged the attention of many mathematicians and physicists, dating back to Leibniz, who was supposed to have remarked "all divergent series can be evaluated". Surprisingly, such values proved useful in physical theories like quantum field theory. The problem of evaluating a divergent series falls under the umbrella of summability or summation methods. These fall into two broad classes, semi-continuous and matrix methods, with some methods lying outside of these two classes. The semi-continuous methods are characterized by multiplying each term of a series by a term from a sequence of functions, and then examining the convergence of the new series as its summands approach a limit point of the functions of the sequence. Under the semi-continuous methods fall such methods as the Abel summation method, the Borel summation method, the Mittag-Leffler summation method, the Lindelöf summation method, and the Riesz summation method. The matrix summation methods are characterized by transforming a series either into a sequence or another series, by multiplying the original series by a finite or infinite matrix, and then examining the convergence of the new sequence or series. Some examples of this class include the Voronoi summation method, the Cesàro summation methods, the Euler summation method, and the Hausdorff summation method. Outside of these broad classes are methods that look at various averages of the partial sums like Holder's summation method and a method developed by a physicist named Thomas E. Phipps which uses the asymptotic behavior of a series to assign it a value. In this paper, we will examine Phipps' method, called *terminal summation*, with the hope of spurring further development, or at the very least bringing more attention to it. Phipps created terminal summation with the stated goal of eliminating the concept of "divergence" from discrete infinite processes–infinite series and products, continued fractions and various generalizations of these. In fact, he argues that divergence is a shortcoming of the Cauchy definition of convergence and that a more general definition of convergence is needed to overcome this limitation. In this direction, Chappell Brown has developed a formulation of a generalized limit to eliminate divergence from series of logarithmico-exponential functions [1]. So far, to our knowledge, no summation method or definition has completely eliminated "diveregence" from every infinite series.

The central idea of termimal summation is to formally identify the various discrete infinite processes with corresponding difference equations having boundary conditions at infinity. If the difference equations can be solved, these infinite processes can be assigned a value, or one of several possible values depending upon the order of the difference equation. In this paper, we will focus on terminal summation of non-periodic infinite series, which satisfy a first order difference equation and thus can have at most one value assigned. We will first give a brief overview of the method before going into a more technical exposition.

Formally, any series can be written as a sum of two expressions:

$$V \equiv \sum_{i=1}^{\infty} a_i = S_n + R_{n+1}, \text{ for any } n, \tag{1}$$

where

$$S_n = \sum_{i=1}^{n} a_i, \qquad R_{n+1} = \sum_{i=n+1}^{\infty} a_i.$$

The quantities S_n satisfy the first order difference equation

$$S_n - S_{n-1} = a_n, \qquad n = 2, 3, ...$$

and the expressions $-R_{n+1}$ satisfy the identical equation

$$(-R_{n+1}) - (-R_n) = a_n, \qquad n = 2, 3, ...$$

When the a_n posses a certain property, called by Phipps *broadly asymptotic tractability*, the asymptotic versions of the difference equation are solveable and the quantities S_n and R_{n+1} have asymptotic expansions in a common set of basis functions, the same set as the asymptotic expansions of the a_n. Let σ_n and ρ_{n+1} designate the truncated asymptotic expansions of S_n and R_{n+1}, respectively. In standard notation,

$$S_n \sim \sigma_n, \qquad R_{n+1} \sim \rho_{n+1} \quad \text{as } n \to \infty. \tag{2}$$

Separating out the terms that are independent of n, from σ_n and ρ_{n+1}, the expansion can be expressed as

$$S_n \sim \sigma_n \equiv \overline{\sigma}_n + A, \qquad R_{n+1} \sim \rho_{n+1} \equiv \overline{\rho}_{n+1} + B, \tag{3}$$

where A and B are constants independent of n and the barred expressions denote that additive constants have been removed from the asymptotic expansion so accented. Since S_n and $-R_{n+1}$ satisfy the same difference equation, they differ at most by an additive constant. Consequently, their truncated asymptotic expansions, σ_n and ρ_{n+1} do as well. Since the barred quantities have all constant terms removed, we must have

$$\overline{\sigma}_n = -\overline{\rho}_{n+1}. \tag{4}$$

If the value V of the infinite series $\sum_{i=1}^{\infty} a_i$ exists, then equation (1) must hold for any n value and in particular for arbitrarily large n values. The relations (1), (2) and (3) then yield the asymptotic relation

$$V \sim \overline{\sigma}_n + \overline{\rho}_{n+1} + A + B.$$

Using the relation (4) gives

$$V \sim A + B \text{ as } n \to \infty.$$

If the discrete infinite process can be assigned a value, no difference can exist between it and its asymptotic value. Thus the $\sim$ symbol can be replaced by an equal sign in the above relation to yield,

$$V = A + B.$$

Requiring consistency of this summation process with the Cauchy definition of convergent series put certains constraints on the values of A and B. For Cauchy convergent series, $V = \lim_{n\to\infty} S_n$, and $\lim_{n\to\infty} R_{n+1} = 0$, which implies that $\overline{\rho}_{n+1} \to 0$ as $n \to \infty$ and

$$B = 0.$$

Phipps takes as part of his definition that the equation "$B = 0$" holds even when the series is divergent, and so

$$V = A.$$

To evalute A numerically, use relation (3) to get

$$V = A \sim S_n - \overline{\sigma}_n \text{ as } n \to \infty,$$

or in the conventional limit notation

$$V = \lim_{n\to\infty} [S_n - \overline{\sigma}_n] = \lim_{n\to\infty} \left[S_n + \overline{\rho}_{n+1}\right]. \tag{5}$$

The last expression essentially says that the numerical value of an infinite series can be found by exactly summing a finite number of terms, possibly a large number of terms, then adding on an asymptotic expansion of the remainder from which any additive constant has been removed, while the first expression says that the same value can be obtained by subtracting off an asymptotic expansion of the "divergence". Subtracting away "infinities", so called *renormalization* techniques, have been used by physicists to handle divergences in field theories to obtain numbers that match those obtained by experiments to a high degree of precision. See [4].

Since this process uses at each limiting stage an asymptotic approximation to a *terminus* or remainder or "part at infinity" of the infinite process, Phipps called it "terminal summation," so as not to fall foul of established terminology. In his view this method is not a summation method, since no modification is made to the summand or the series as a whole, but is more of a generalization, a covering theory, of Cauchy convergence.

A natural question to ask is whether the terminal summation process give a unique value, since at each step of "going to the limit" we can have a potentially different remainder. The condition to avoid such a situation is inherent in the difference equation "equivalent" to the infinite process. The quantities σ_n or ρ_{n+1} are evaluated directly from the asymptotic forms of the associated linear difference equations, (2), namely

$$\sigma_n - \sigma_{n-1} \sim a_n \text{ as } n \to \infty \tag{6}$$

or

$$\rho_n - \rho_{n+1} \sim a_n \text{ as } n \to \infty, \tag{7}$$

and are unique up to an additive constant. But since this additive constant is deleted to get $\overline{\sigma}_n$ or $\overline{\rho}_{n+1}$, these functions are unique and hence the teminal sum (5) must be unique.

In order to actually carry out calculation, a common asymptotic expansion basis for σ_n, ρ_{n+1}, and a_n must be used. If equation (6) is used, we express σ_n as a trial series of these basis functions with undetermined coefficients. This trial series will also give a series for σ_{n-1}, with the same coefficients in different combinations as coefficients of various members of the same basis set. The a_n is a known function of n and so its asymptotic expansion will have numerically known coefficients. Putting these three asymptotic expansions into (6) and equating the coefficients of the same basis functions yield a sequence of equations for the unknown coefficients in the original trial series. Solving these equations successively provides as many expansion terms as required for σ_n and $\overline{\sigma}_n$. Guessing which trial series to use requires some ingenuity and solving the recurrence equations for the undetermined coefficients can be tedious. Fortunately, modern computer algebra software can do most of the tedious calculations, once an appropriate trail series is determined.

Using his techniques Phipps was able to sum Brown's series,

$$1 + x + 2x^2 + 6x^3 + \cdots + n!x^n + \cdots, \tag{8}$$

divergent everywhere except at $x = 0$, at enough values to show that it is a continuous function at zero! [9]. According to Phipps, nearly all other summation methods fail to give it a value for $x \neq 0$. Terminal summation can also be used to speed up convergence for convergent series since it typically includes a nonzero remainder at each step. See [8] for some explanations and examples on using terminal summation to speed up convergence. For a complete exposition of terminal summation dealing with periodic series, and higher order discrete infinite process like continued fractions, we refer to the Naval Ordnance Technical Report [7]. In the next section we give the technical definitions and sufficient conditions for an infinite series to have a termial sum.

2 Terminal Summation

We now provide some definitions and a formal exposition of Phipps' concepts.

Definition 1. An infinite sequence of functions $\phi_i(n), i = 1, 2, ...$ is called a *standard asymptotic sequence*, or *basis* if it has the following five properties:

1. Asymptotic sequence property:
   $\phi_{i+1}(n) = o(\phi_i(n))$ as $n \to \infty$, i = 1,2,....
2. Multiplicative group property:
   For all i, j, $\phi_i(n)\phi_j(n) = \phi_k(n)$ for some k.
3. Group unit:
   $\phi_i(n) = 1$ for some i.
4. Reflexive property:
   $\phi_i(n + v) \sim \mathcal{S}(\phi_i(n))$ as $n \to \infty$, $i = 1, 2, ...$
   where '$\mathcal{S}(\phi_i(n))$' denotes a standard asymptotic expansion in the $\phi_i(n)$, not necessarily the same from one occurrence to another.
5. Stability property:
   $\phi_i(n + v) \simeq k_v\phi_i(n)$ as $n \to \infty$, $i = 1, 2, ...$
   where '$\simeq$' denotes equality of asymptotically dominat terms as $n \to \infty$, $v = \pm1, \pm2, ...$, and k_v is a constant independent of n and i.

Property (1) enables us to get an estimate on the error term of an asymptotic expansion, while properties (2) and (3) ensure that any linear combination of products of functions $\sim \mathcal{S}(\phi(n))$ is also $\sim \mathcal{S}(\phi(n))$ as $n \to \infty$. Properties (4) and (5) ensure that translates of the asymptotic basis functions have asymptotic expansion in the basis functions themselves. An asymptotic sequence of functions is also called an *asymptotic scale*, since they give a sense of how fast a function represented by an asymptotic series is growing or decaying. In Phipps' papers, the prototypical asymptotic squence $\phi_i(n) = n^{-i}$, $i = 0, 1, 2, ...$, is used.

Definition 2 (Broad asymptotic expansion)**.** A function $f(n)$, $n = 1, 2, ...$ will be said to posses a *broad asymptotic expansion* if either one of the following is true:

$$f(n) \sim \theta(n)\mathcal{S}(\phi(n)) \text{ as } n \to \infty, \tag{9}$$

where

$$\theta(n + v) \sim \theta(n)\mathcal{S}(\phi(n)) \text{ as } n \to \infty, v = \pm1, \pm2, ... \tag{10}$$

or

$$f(n) \sim c\psi(n) + \mathcal{S}(\phi(n)) \text{ as } n \to \infty, \tag{11}$$

where

$$\psi(n + v) \sim \psi(n) + \mathcal{S}(\phi(n)) \text{ as } n \to \infty, v = \pm1, \pm2, ... \tag{12}$$

A *standard asymptotic expansion* is just a special case with $c = 0$ and $\theta(n) = 1$. It is useful to have the above two cases combined into one:

$$f(n) \sim c\psi(n) + \theta(n)\mathcal{S}(\phi(n)) \text{ as } n \to \infty, \tag{13}$$

where $\theta(n)$ and $\psi(n)$ are as above. With this formulation, the functions that have expansions of the form (13) for a fixed asymptotic scale will be elements of a linear vector space.

Definition 3 (Asymptotically tractable series)**.** An infinite series is *asymptotically tractable* if for all sufficiently large n a remainder term R_{n+1} exists and possesses a broad asymptotic expansion, $\rho_{n+1} \sim R_{n+1}$ as $n \to \infty$.

A theorem proved by Phipps, specialized to non-periodic series in this paper, provides sufficient conditions for a series to be broadly asymptotically tractable.

Theorem 1. *The series $\sum_{n=1}^{\infty} a_n$ is broadly asymptotically tractable, except possibly at isolated singularities in the ranges of parameters contained in a_n, if each a_n possesses a broad asymptotic expansion in the sense of* (9) *in a common set of basis functions $\phi_i(n)$, with a common multiplier $\theta(n)$ conforming to* (10).

The formal definition of a terminal sum for an asymptotically tractable series is then

$$V = \lim_{n\to\infty} [S_n - \overline{\sigma}_n] = \lim_{n\to\infty} \left[S_n + \overline{\rho}_{n+1}\right]. \tag{14}$$

A pecularity of this method are points of ambiguity where the method can give different values to seemingly the same series. For instance, for the series $\sum_{i=1}^{\infty} 1$ it gives the value of zero, while for the series $\zeta(0) = \sum_{n=1}^{\infty} n^{-0}$ it gives the value of $-\frac{1}{2}$. See [7] for more details on why this can occur. Other summation methods suffer from a similar defect; for instance the Euler method assigns the value of $\frac{2}{3}$ to the series $1+0-1+1+0-1+1+0\cdots$ but the value of $\frac{1}{2}$ to the series $1-1+1-1+\cdots$. See pg 14 of [5].

Calculation of Terminal Sum

The following refers to the generic infinite series

$$\sum_{i=1}^{\infty} a_i$$

.

Step 1. **Preliminary verification of asymptotic tractability**:

By trial and error, discover a function $\theta(n)$ that is a factor of the dominant term of the a_n as $n \to \infty$ and discover a set of expansion basis functions $\phi_i(n)$, $i = 1, 2, 3, ...$, such that

$$\theta(n \pm 1) \sim \theta(n)\mathcal{S}(\phi(n)), \text{ as } n \to \infty$$

and

$$a_n \sim \theta(n)\mathcal{S}(\phi(n)), \text{ as } n \to \infty$$

where $\mathcal{S}$ denotes any standard asymptotic expansion. If this step is not achieved or is impossible, the method fails.

Step 2. **Introduction of trial series**:

Two cases are possible.

Case A. The function $\theta(n)$ found in Step 1 is not a constant. In this case the trial series for σ_n takes the form

$$\sigma_n \sim \theta(n)\mathcal{S}(\phi(n)) \text{ as } n \to \infty.$$

Case B. $\theta(n)$ is a constant. In this case a trial series of the form

$$\sigma_n \sim c\psi(n) + \mathcal{S}(\phi(n)) \text{ as } n \to \infty$$

may be used where c is an undetermined constant and where ψ is a function satisfying

$$\psi(n \pm 1) \sim \psi(n) + \mathcal{S}(\phi(n)) \text{ as } n \to \infty.$$

According to Phipps, it should be possible to take $c = 0$ and omit the ψ-term when the indefinite integral with respect to n of the dominant term in the asymptotic expansion of a_n lies in the set of basis functions $\phi_i(n)$.

Step 3. **Evaluation of the coefficients**:

Insert the trial series for σ_n and a_n into the asymptotic form of Equation (6). Cancel the multiplier $\theta(n)$, if present, and equate coefficients of successive members of the basis set $\phi_i(n)$, for as many i as required to meet the computational needs of the problem. In essence, this step determines the unique asymptotic expansion of σ_n and completes the verification of asymptotic tractability of the series. Phipps gave the following proviso: another step is needed if $\lim_{n\to\infty} |\theta(n)\phi_i(n)| = \infty$ for all i. In this situation, the additional step is to check the absolute Cauchy converegence of the asymptotic expansion of $\sigma_n/\theta(n)$, or the weaker condition of having a sufficiently rapid convergent beginning. If the $\theta(n)$ grow slowly enough so that there exists an M sufficiently large and independent of n, to make the error $\theta(n)\phi_M(n)$, the first omitted term, approach zero as $n \to \infty$, then $\mathcal{S}(\phi(n))$ can just be an asymptotic series and not a convergent one.

Step 4. **Evaluation of infinite series**:

Compute for increasing n the values of V_n, where

$$V_n = \sum_{i=1}^{n} a_i - \overline{\sigma}_n.$$

Here $\overline{\sigma}_n$ is the series σ_n obtained in step 3, from which the n-independent term has been removed. If $V_n \to V$ for increasing n, V is the terminal sum $\sum_{i=1}^{\infty} a_i$. If not, repeat steps 3 and 4 with more terms in the expansion and larger values of n. According to Phipps,"If $|\theta(n)|$, the multiplier in $|\overline{\sigma}_n|$, is a strongly increasing function of n, a very delicate balance must be maintained between the number of expansion terms and the value of n for minimum absolute error." In this situation, using the first omitted or last included term in the expansion of $\overline{\sigma}_n$ as an error estimate can be helpful. If the series $\sum_{i=1}^{\infty} a_i$ can be terminal summed to a number V, we write $V = \sum_{i=1}^{\infty} a_i \ (T)$.

3 Terminal summation examples

We will illustrate this method first on the divergent series $\sum_{n=1}^{\infty} \log(n)$, using σ_n and its corresponding difference equation (6). Let

$$a_n = \log(n), \tag{15}$$

and the asymptotic expansion basis be integral powers of n, that is $\phi_i(n) = n^{-i}$. After trying several different possibilities for $\theta(n)$, the one that gives us a solvable equation is $\theta(n) = 1$ and an appropriate $\psi(n)$. We are in the case B scenario of Step 2 in which Phipps suggests using the dominant term of $\int \log(n)dn$, $n\log(n)$, for $\psi(n)$. Trying several different expansions for σ_n, the one that gives a solvable equation includes the term $n\log(n)$ and several others, and is given below:

$$\sigma_n \sim c_0 n\log(n) + c_1 n + c_2\log(n) + b_0 + \frac{b_1}{n} + \frac{b_2}{n^2} + \frac{b_3}{n^3} + \cdots \tag{16}$$

where $c_0, c_1, c_2, b_0, b_1, b_2 ...$ are to be determined. To get the expansion series for σ_{n-1}, we substitute $n-1$ for n into the above relation and express the terms as functions of n:

$$\sigma_{n-1} \sim c_0 n\log(n) + c_0 n\log(1-\frac{1}{n}) - c_0\log(n) - c_0\log(1-\frac{1}{n}) + c_1 n - c_1 + c_2\log(n) + \\ c_2\log(1-\frac{1}{n}) + b_0 + \frac{b_1}{n}\left(1-\frac{1}{n}\right)^{-1} + \frac{b_2}{n^2}\left(1-\frac{1}{n}\right)^{-2} + \cdots \tag{17}$$

Taking the difference of (16) and (17), and substituting the result and equation (15) into (6) yields

$$\log(n) \sim c_0\log(n) - c_0 n\log(1-\frac{1}{n}) + (c_0 - c_2)\log(1-\frac{1}{n}) + c_1 + \\ \frac{b_1}{n}\left[1-\left(1-\frac{1}{n}\right)^{-1}\right] + \frac{b_2}{n^2}\left[1-\left(1-\frac{1}{n}\right)^{-2}\right] + \cdots$$

For n sufficiently large, we can use the Taylor series expansion for all the above terms, excluding the term $\log(n)$, and collect the coefficients of the same basis function to get

$$\log(n) \sim c_0\log(n) + (c_0 + c_1) + (2c_2 - c_0)\frac{1}{2n} + (\frac{c_2}{2} - \frac{c_0}{6} - b_1)\frac{1}{n^2} + \cdots .$$

Equating the coefficients of the same basis function on both side of the relation yield the recurrence equations needed to solve for the constants c_i's and b_n's:

$$\log(n) = c_0\log(n), \qquad c_0 + c_1 = 0, \qquad \frac{c_2 - c_0/2}{n} = 0, \qquad \frac{c_2/2 - c_0/6 - b_1}{n^2} = 0$$
$$\frac{5c_2/6 - c_0/3 - b_1 - b_2}{n^3} = 0, \qquad \frac{-c_0/20 + c_2/4 - b_1 - 2b_2 - 3b_3}{n^4} = 0 \qquad ....$$

Solving these recurrence equations in succession gives for the first few c_n the values below:

$$c_0 = 1, \quad c_1 = -1, \qquad c_2 = \frac{1}{2}, \quad b_1 = \frac{1}{12}, \qquad b_2 = 0, \quad b_3 = -\frac{1}{360}.$$

n	V_n Approximation	Approximate Error
10	0.9189357633	3 $\times 10^{-6}$
30	0.9189384303	1 $\times 10^{-7}$
50	0.9189385106	2 $\times 10^{-8}$
70	0.9189385251	8 $\times 10^{-9}$
90	0.9189385294	4 $\times 10^{-9}$
120	0.9189385311	2 $\times 10^{-9}$

Table 1: Approximation to $C(1) = log(2\pi)/2 \approx 0.9189385332$

The only constant not determined is b_0, but it is deleted from σ_n since it has no dependence upon n. Thus

$$\overline{\sigma}_n \sim (n + \frac{1}{2})\log(n) - n + \frac{1}{12n} - \frac{1}{360n^3} + \cdots$$

This expression is really just the first few terms of the Euler-Maclaurin sum formula for the natural log function, with the constant deleted:

$$\sum_{i=1}^{n} \log(i) \sim \int_1^n \log(x)dx + C(1) + \frac{1}{2}\log(n) + \sum_{r=1}^{\infty}(-1)^{r-1}\frac{B_{2r}}{(2r)!n^{2r-1}},$$

where the number B_{2r} are the even Bernoulli numbers and $C(1) = \frac{1}{2}\log(2\pi)$ [5]. This series is a divergent asymptotic series, so we can not use the entire series to get a value. But we can use the first few terms as an approximation and estimate the approximation error by looking at the order of the first omitted term. Thus if we use the approximation

$$\overline{\sigma}_n \approx (n + \frac{1}{2})\log(n) - n + \frac{1}{12n},$$

the error in approximating the terminal sum of $\sum_{n=1}^{\infty} \log(n)$, $C(1)$, by

$$V_n \equiv \sum_{i=1}^{n} \log(i) - (n + .5)\log(n) + n - \frac{1}{12n}$$

is of order $1/(360n^3)$, the first omitted term in the asymptotic expansion of $\overline{\sigma}_n$. Using Mathematica to do 40 digit precision calculations, we get a sequence of approximations to $C(1)$ that is summarized in Table (1). For $n = 10$, we get an approximation of around 0.918936 for $C(1)$ which agrees with its actual value to five decimal places.

Next we terminal sum the series $S(z) = 1 + 2z + 3z^2 + 4z^3 + \cdots nz^{n-1} + \cdots$, which converges absolutely for $|z| < 1$ and diverges for $|z| \geq 1$. First, we find a closed expression for the n^{th} partial sum $S_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$.

$$\begin{aligned} S_n(z) &= \frac{d}{dz}(1 + z + z^2 + z^3 + \cdots + z^n) \\ &= \frac{d}{dz}\left(\frac{1 - z^{n+1}}{1 - z}\right) \text{ for } z \neq 1 \\ &= \frac{-(n+1)z^n}{1 - z} + \frac{1 - z^{n+1}}{(1 - z)^2} \text{ for } z \neq 1. \end{aligned}$$

Let $\theta(n) = a_n \equiv nz^{n-1}$, then $\theta(n-1) = \theta(n)\frac{n-1}{nz}$. We use the following trial expansion for σ:

$$\sigma_n \sim \theta(n)(c_0 n + c_1 + \frac{c_2}{n} + \frac{c_3}{n^2} + ...)$$

which gives the expansion:

$$\sigma_{n-1} \sim \theta(n)(\frac{c_0(n-1)^2}{zn} + \frac{c_1(n-1)}{nz} + \frac{c_2}{nz} + \frac{c_3}{n(n-1)z} + ...).$$

The equation $\sigma_n - \sigma_{n-1} \sim a_n$ in this situation becomes

$$\theta(n) \sim \theta(n)(nc_0(1-1/z) + c_1 - (c_1 + 2c_0)/z + \frac{c_2 + (c_1 + c_0 - c_2)/z}{n} + \frac{c_3(1-1/z)}{n^2} + \cdots)$$

Cancelling the multiplier $\theta(n)$ and equating the coefficients of n^i on both side of the relation yield the equations needed to solve for the undetermined c_n:

$$nc_0(1-1/z) = 0 \text{ which implies for } z \neq 1 \text{ that } c_0 = 0$$

and

$$c_1(1-1/z) - 2c_0 = 1 \text{ with } c_0 = 0 \text{ implies for } z \neq 1 \text{ that } c_1 = \frac{-z}{1-z}.$$

and

$$c_2(1-1/z) - c_1/z = 0 \text{ with } c_1 \text{ as above implies for } z \neq 1 \text{ that } c_2 = \frac{-z}{(1-z)^2}.$$

The other equations starting with

$$c_3(1-1/z) = 0$$

all implies that $c_n = 0$ for $n = 3, 4, ...$ In this case then σ_n and $\overline{\sigma}_n$ have closed forms that are the same for $z \neq 1$:

$$\sigma_n = \overline{\sigma}_n = -\frac{nz^n}{1-z} - \frac{z^n}{(1-z)^2}.$$

Thus for $z \neq 1$, we get a sequence of approximations,

$$V_n \equiv S_n - \overline{\sigma}_n = \frac{1}{(1-z)^2}$$

to the terminal sum of the series independent of n. For $z \neq 1$, taking limits as $n \to \infty$, yield the terminal sum of the series:

$$S(z) \equiv 1 + 2z + 3z^2 + 4z^3 + \cdots + nz^{n-1} + \cdots = \frac{1}{(1-z)^2} \quad (T)$$

In particular for $z = -1$, $S(-1) = \frac{1}{4} \quad (T)$.

The case when $z = 1$ needs to be handled separately, since the above process can not be completed and is left as an exercise for the interested reader.

4 Properties of Terminal Summation

We will now examine if terminal summation satisfies some desirable properties which many summation methods share. In what follows, we let S denote any summation method.

Definition (Regularity)**.** A summation method is regular if it yields the same value for a convergent series as that given by the Cauchy definition.

Definition (Linearity)**.** A summation method S is linear if it satisfies both of the following:
(a) Scalability: For β a constant, $S(\sum(\beta a_i)) = \beta S(\sum a_i)$.
(b) Additivity : $S(\sum(a_i + b_i)) = S(\sum a_i) + S(\sum b_i)$.

Definition (Stability / Translativity)**.** Removing a finite number of terms from the series and adding it back to the summation of the remaining tail gives the same value as the original series.
$S(\sum_{i=1}^{\infty} a_i) = \sum_{i=1}^{n} a_i + S(\sum_{i=n+1}^{\infty} a_i)$ for any finite n.

Stability is really a property of the series and not of any particluar summation method. A prime example of an unstable series is $\sum_{i=1}^{\infty} 1$. If the series is stable then any linear summation method which assigns it a finite value will yield the inconsistent equation: $0 = 1$.

Terminal summation is constructed to be regular, and is linear for stable series that have asymptotic expansions in the same asymptotic scale. For two series that have asymptotic expansions in different asymptotic scales the additivity of the two series is not so clear.

It would be interesting to see what conditions on a series will ensure that its terminal sum will satisfy the following four useful properties:

Cauchy product: $S\left(\sum a_i \sum b_i\right) = S\left(\sum c_i\right)$ where $c_i = \sum_{k=1}^{i} a_k b_{i-k+1}$

Continuity:

$$\lim_{x \to c} S\left(\sum a_i(x)\right) = S\left(\sum \lim_{x \to c} a_i(x)\right).$$

Commutativity with differentiation:

$$\frac{d}{dx} S\left(\sum a_i(x)\right) = S\left(\sum \frac{d}{dx} a_i(x)\right).$$

Commutativity with integration:

$$\int_b^y S\left(\sum a_i(x)\right) dx = S\left(\sum \int_b^y a_i(x) dx\right)$$

A summation method which satisfies these properties on some class of series would give one the freedom to manipulate a series term by term and get valid results. Establishing these desirable properties using Phipps' definition and formulation is analytically difficult. Of these four, perhaps the easiest to establish may be commutativity with integration, since a theorem from asymptotics theory tells us that the sum of the integrals of an asymptotic series is also an asymtotic series for the integral of the sum. Phipps used this property on the Brown series to show that it can be written as a convergent series [9]. There are some theorems which can be used in this analysis when the asymptotic expansions are restricted to standard asymptotic expansions where $\psi(n) = 0$ and $\theta(n) = 1$ in (13) for some fixed asymptotic scale. With standard asymptotic expansions and the sequence n^{-i}, $i = 0, 1, 2, 3...$ as the asymptotic scale, the above properties do hold given certain conditions on the summand functions; see [2] and [6].

5 Conclusion and Connections

When calculating the terminal sum of the series $\sum \log(n)$ using the method outlined by Phipps, we are basically calculating the terms of the Euler-Maclaurin Sum formula for the natural log function. This indicates that there is a connection between the two, which we will examine next. It is unlcear if Phipps was aware of this connection, since it was never explicitly mentioned in his works. The Euler-Maclaurin sum formula relates a finite sum of a function to a finite integral of the same function plus a remainder term and has many uses in mathematics. Using the convention followed by Hardy and others, the Euler-Maclaurin formula for a function f of class C^{2k} is as follows:

$$\sum_{i=a}^{n} f(i) = \int_a^n f(x)dx + \frac{f(n)}{2} + C_k(a) + \sum_{r=1}^{k} (-1)^{r-1} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(n) + R_k(n),$$

where B_{2r} are the Bernoulli numbers, $C_k(a)$ a constant, and $R_k(n)$ is the remainder term, whose exact forms are given below:

$$C_k(a) = \frac{f(a)}{2} + \sum_{r=1}^{k} (-1)^r \frac{B_{2r}}{(2r)!} f^{(2r-1)}(a),$$

and

$$R_k(n) = -\frac{1}{(2k)!} \int_a^b B_{2k}(x - \lfloor x \rfloor) f^{(2k)}(x) dx$$

with $B_{2r}(x)$ a Bernoulli polynomial. The number a is usual taken to be zero or one, depending upon the integrability of f down to those numbers.

In his book on divergent series, Hardy shows that if the following two hypotheses are true for all k from a certain K, for f that is infinitely differentiable,

$$\int^{\infty} |f^{(2k+2)}(x)|dx < \infty \tag{18}$$

and

$$f^{(2k+1)}(x) \to 0 \text{ as } x \to \infty, \tag{19}$$

then the constant

$$C(a) = \frac{f(a)}{2} + \sum_{r=1}^{\infty} (-1)^r \frac{B_{2r}}{(2r)!} f^{(2r-1)}(a), \tag{20}$$

depends only upon f and a. This constant is called by him the Euler-Maclaurin constant of f and the $(\mathcal{R}, a)$ sum of the series $\sum f(n)$. In Hardy's words: "The $\mathcal{R}$ stands for Ramanujan, whose work with divergent series was mainly based on this definition. The definition is implicit in much of Euler's work. The sum which it attritubutes to a series depends on the value chosen for a. We shall, however, find that there is usually one value of a which is natural to choose in any special case." See specifically pages 326-327 of [5].

The two conditions (18) and (19) imply that the Euler-Maclariun sum formula series is asymptotically tractable when the infinitely differentible function f eventually has higher order derivatives that form an asymptotic scale. Thus they give different sufficient conditions for a series to be asymptotically tractable, and even yield a formula for the terminal sum in terms of a convergent or semiconvergent series. In practice, checking if the conditions (18) and (19) are met or even using the formula (20) may be difficult when the function f is complicated like the gamma function, as it is in the case of the Brown series given by (8). A natural question that comes to mind is "Does the Phipps condition of broadly asymptotic tractability imply the above two conditions, (18) and (19)?" This may be an open question and is certainly not clear to the authors.

Eric Delabaere has developed a summation method based on the concepts of Ramanujan, using the Borel transform in a suitable space of analytic functions to solve a difference equation similar to Phipps' method but with a different boundary condition. Unfortunately, Delabaere's method does not agree with the sum for convergent series, so can not be said to be a generalization of convergence as Phipps seek to do with his method of terminal summation. See [3] for a summary of Delabaere's method. One advantage of Phipps method over the Delabaere method and the Euler-Maclariun sum formula is its straight forward computation, using only the machinery of calculus.

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