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Series that Probably Converge to One

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Introduction

Infinite series are fascinating objects, but what is often unsatisfactory when working with an infinite series is that at best one can say whether it converges or diverges, but not what it converges to. In a typical calculus sequence, students first covered geometric and telescoping series, whose limits can be easily computed. And once Taylor series are learned, they can be used to calculate the limits of more complicated series if the corresponding functions are known. Using certain simple ideas from probability theory, we will provide a fairly elementary method for constructing infinite series with known sums. We will use the fact that the probabilities of distinct events composing the entire sample space sums up to one. The basic idea is to define a random phenomenon and disjoint events E_i for $i = 1, 2, 3, \ldots$ that exhaust the entire sample space, S, so that

$$1 = P(S) = \sum_{i=1}^{\infty} P(E_i),$$

which gives us an infinite series provided we can calculate $P(E_i)$. As you read through you will find some familiar series as well as some less common series. Of course, even here we are cheating a bit since we are not starting with a convergent series and trying to find its value, but rather we are simply finding some series that converge to one.

Coin Flipping Series

We begin by flipping a fair coin until it lands on a head. Let E_i be the event that the first head appears on the i^{th} flip of the coin. To see that the disjoint events E_i make up the entire sample space, let F_i be the event that we obtain i tails in i tosses of the coin so that $S \setminus \cup_{i=1}^{\infty} E_i = \cap_{i=1}^{\infty} F_i$. Now, $P(F_i) = (1/2)^i$ and so $\lim_{i\to\infty} P(F_i) = 0$, in which case the sample space $S = \cup_{i=1}^{\infty} E_i$. We pause for a moment to note that, in fact, throughout this paper $S \setminus \cup_{i=1}^{\infty} E_i$ may be nonempty, but it will have measure 0 and hence we can say $S = \cup_{i=1}^{\infty} E_i$ for our purposes here. Now, given that $P(E_i) = (1/2)^i$, we obtain our first series, namely

$$1 = P(S) = \sum_{i=1}^{\infty} P(E_i) = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i. \tag{1}$$

We can generalize this to a coin that has probability $0 < p < 1$ or $q = 1 - p$ of landing on heads or tails, respectively. We then get that $P(E_i) = p(1-p)^{i-1} = (1-q)q^{i-1}$ and $\lim_{i\to\infty} P(F_i) = \lim_{i\to\infty} q^i = 0$ so that

$$1 = P(S) = \sum_{i=1}^{\infty} P(E_i) = \sum_{i=1}^{\infty} (1-p)^{i-1} p = \sum_{i=1}^{\infty} (1-q) q^{i-1},$$

which is the geometric series result with $0 < q < 1$. Unfortunately, since probabilities must be positive we cannot obtain by our methods the other half of the domain of the geometric series where $-1 < q < 0$.

Generalizing this approach further, let us fix an integer j and continue to flip a coin with probability $0 < p < 1$ of landing on a head, until j heads appear. Let E_i^j be the event that the j^{th} head appears on the i^{th} flip, and let F_i^k be the event of obtaining exactly k heads in

i tosses of the coin. Now,

$$\begin{aligned}\lim_{i\to\infty} P(F_i^k) &= \lim_{i\to\infty} \binom{i}{k} p^k q^{i-k} \\ &= \lim_{i\to\infty} \frac{i(i-1)...(i-k+1)}{k!}(p/q)^k q^i \\ &\le \lim_{i\to\infty} \frac{i^k}{k!}(p/q)^k q^i = 0,\end{aligned}$$

since an exponential function dominates a power function. Hence, we must eventually obtain j heads and so $S = \cup_{i=j}^{\infty} E_i^j$. Finally, since $P(E_i^j) = \binom{i-1}{j-1} p^j (1-p)^{i-j}$, we use our methods to find that

$$\begin{aligned}1 = P(S) &= \sum_{i=j}^{\infty} P(E_i^j) \\ &= \sum_{i=j}^{\infty} \binom{i-1}{j-1} p^j (1-p)^{i-j} \\ &= \sum_{k=0}^{\infty} \binom{k+j-1}{j-1} p^j (1-p)^k,\end{aligned}$$

which is, in fact, the sum of the mass function of a negative binomial random variable. This series may be familiar as

$$\frac{1}{(1-q)^{j+1}} = \sum_{k=0}^{\infty} = \sum_{k=0}^{\infty} \binom{k+j}{k} q^k,$$

which can be obtained by taking j derivatives with respect to q of

$$\frac{1}{1-q} = \sum_{k=0}^{\infty} q^k.$$

We will now consider flipping a coin until we a run of 2 consecutive heads is achieved. Let E_i be the event of obtaining the first instance of 2 heads in a row concluding with the i^{th} toss and let F_i be the event of no run of two heads or more in i tosses of the coin. To calculate $P(E_i)$ we will use the general idea from Berresford's paper [1]. First note that, $P(E_1) = 0$, $P(E_2) = p^2$, $P(E_3) = qp^2$, and $P(E_i) = P(F_{i-3})qp^2$ for $i > 3$, since to have a run of two heads appear for the first time on the ith toss the last three tosses must be THH and no run of two heads can occur before that. We now must calculate $P(F_i)$. First, $P(F_1) = 1$ and $P(F_2) = 1 - p^2$. The i^{th} toss must be either T or H, and in the latter case the previous toss was T, so $P(F_i) = qP(F_{i-1}) + pqP(F_{i-2})$. Solving this recurrence relation for $p = 0.5$ we obtain

$$P(F_i) = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{4}\right)^i + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{4}\right)^i$$

Note that the terms raised to the power of i are each less than 1, so $\lim_{i\to\infty} P(F_i) = 0$. This is what we need to conclude that $S = \cup_{i=1}^{\infty} E_i$. Therefore,

$$1 = \frac{1}{4} + \frac{1}{8} + \frac{1}{8}\sum_{i=1}^{\infty} \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{4}\right)^i + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{4}\right)^i$$

In general we find that

$$\begin{aligned} P(F_i) = \frac{1}{\sqrt{q^2+4pq}} &\left[\left(\frac{q+\sqrt{q^2+4pq}}{2}\right)^i - \left(\frac{q-\sqrt{q^2+4pq}}{2}\right)^i\right] \\ &+ \frac{pq}{\sqrt{q^2+4pq}}\left[\left(\frac{q+\sqrt{q^2+4pq}}{2}\right)^{i-1} - \left(\frac{q-\sqrt{q^2+4pq}}{2}\right)^{i-1}\right]. \end{aligned}$$

Factorials and Products from Marbles in Bags

Start with a bag containing one blue and one red marble. We will continue to remove marbles from the bag under the rule that if we select a red marble from the bag we will put back the red marble along with an additional red marble. The game ends once a blue marble is selected. Some games may take a long time to end, but thankfully we are using a magical bag that can expand to accommodate added marbles. Let E_i be the event that the blue marble is selected on the i^{th} draw from the bag, which is a variation of Polya's Urn Scheme.

Clearly, $P(E_1) = 1/2$, and

$$\begin{aligned} P(E_i) &= (1 - P(E_1))(1 - P(E_2)) \ldots (1 - P(E_{i-1}))(1/(i+1)) \\ &= \frac{1}{2}\frac{2}{3}\frac{3}{4} \cdots \frac{i-1}{i}\frac{1}{i+1} \\ &= \frac{1}{i(i+1)}. \end{aligned}$$

Again we need to show that the events E_i complete the sample space. Let F_i be the event that i red marbles are selected and note that $P(F_i) = 1/(i+1)$ so that $\lim_{i\to\infty} P(F_i) = 0$. Now since $S = \cup_{i=1}^{\infty} E_i$, we derive that

$$1 = P(S) = \sum_{i=1}^{\infty} P(E_i) = \sum_{i=1}^{\infty} \frac{1}{i(i+1)}. \tag{2}$$

The next step here is to use the same setup but if a red marble is chosen we put back the original red along with two more red marbles. Again $P(E_1) = 1/2$, but

$$\begin{aligned} P(E_i) &= (1 - P(E_1))(1 - P(E_2)) \ldots (1 - P(E_{i-1}))(1/(2i)) \\ &= \frac{(2i-3)!!}{(2i-2)!!}\frac{1}{2i} \\ &= \frac{(2i-3)!!}{(2i)!!}, \end{aligned}$$

where $n!! = n(n-2)(n-4)\ldots 2($ or $1)$. With F_i defined as above we can show that

$$\lim_{i\to\infty} P(F_i) = \lim_{i\to\infty} \frac{(2i-1)!!}{(2i)!!} = \lim_{i\to\infty} \frac{(2i)!}{(2^i i!)^2} = 0,$$

by using Stirling's formula $n! \sim n^n e^{-n}\sqrt{2\pi n}$ to prove that $(2i)!/(2^i i!)^2 \sim \sqrt{4\pi i}/2\pi i$. Hence, we conclude that

$$1 = P(S) = \sum_{i=1}^{\infty} P(E_i) = \sum_{i=1}^{\infty} \frac{(2i-3)!!}{(2i)!!}. \tag{3}$$

Generalizing this idea, we will start with a bag that has $b > 0$ blue marbles and $r > 0$ red marbles. We will select marbles from the bag until a blue marble is chosen under the rule that when a red marble is chosen we will add l blue marbles and $d+1$ red marbles. Again

E_i is the event that a blue marble is selected on the ith draw from the bag. In this scenario we see that $P(E_1) = b/(b+r)$, and $P(E_i) = (1 - P(E_1))(1 - P(E_2)) \ldots (1 - P(E_{i-1}))(b + (i-1)l)/(b + (i-1)l + r + (i-1)d)$. To show that the whole sample space is covered notice that

$$P(F_i) = \prod_{j=0}^{i-1} \frac{r + jd}{b + r + j(l+d)}$$

and so

$$\lim_{i \to \infty} P(F_i) = \prod_{j=0}^{\infty} \frac{r + jd}{b + r + j(l+d)}.$$

Taking the log of the product we consider

$$\sum_{j=1}^{\infty} \ln \left(\frac{r + jd}{b + r + j(l+d)} \right)$$

and it is easy to see that

$$\int_1^{\infty} \ln \left(\frac{r + xd}{b + r + x(l+d)} \right) dx = -\infty.$$

We can now conclude that

$$\lim_{i \to \infty} P(F_i) = \prod_{j=1}^{\infty} \frac{r + jd}{b + r + j(l+d)} = e^{-\infty} = 0.$$

Intuitively, for j sufficiently large, $\frac{r+jd}{b+r+j(l+d)} \approx \frac{d}{l+d}$, which is less than one when $l \neq 0$, thus the limit of the product is zero.

Calculating $P(E_i)$ for $i = 1, 2, 3, 4$ gives

$$\begin{aligned}
P(E_1) &= \frac{b}{b+r} \\
P(E_2) &= \frac{r(b+l)}{(b+r)(b+r+l+d)} \\
P(E_3) &= \frac{r(r+d)(b+2l)}{(b+r)(b+r+l+d)(b+r+2l+2d)} \\
P(E_4) &= \frac{r(r+d)(r+2d)(b+3l)}{(b+r)(b+r+l+d)(b+r+2l+2d)(b+r+3l+3d)},
\end{aligned}$$

and we see that

$$P(E_i) = \frac{b+(i-1)l}{r+(i-1)d}\prod_{j=0}^{i-1}\frac{r+jd}{b+r+(j)(l+d)}. \tag{4}$$

The two cases above, (2) and (3), arose from $(b=1, r=1, l=0, d=1)$ and $(1,1,0,2)$. The case $(1,1,0,0)$, and in fact whenever $b=r$ and $l=d$, has (4) reducing to $1/2^i$ and the sum we derive is given in (1), since these cases are the same as flipping a fair coin. Some other cases that can be worked out nicely are

$$\sum_{i=1}^{\infty}\frac{i}{(i+1)!} = 1 \qquad (1,1,1,0),$$

$$\sum_{i=1}^{\infty}\frac{2i-1}{2^i(i!)} = 1 \qquad (1,1,2,0),$$

$$\sum_{i=1}^{\infty}\frac{k(i-1)!k!}{(k+i)!} = 1 \qquad (k,1,0,1) \text{ where } k>1,$$

$$\sum_{i=1}^{\infty}\frac{b}{r}\left(\frac{r}{b+r}\right)^i = 1 \qquad (b,r,b,r).$$

As an exercise one can show that the first three series are telescoping series and the last one is a special case of the geometric series.

In a similar manner we will again start with a bag containing $b>0$ blue marbles and $r>0$ red marbles. In this case, if a red marble is chosen at the i^{th} turn we will add $i\times l$ blue marbles and $i\times d+1$ red marbles. Again we have $P(E_1)=b/(b+r)$ and conditioning on the previous draw and using $\sum_{k=1}^{n} k = n(n+1)/2$ we discover that

$$P(E_i) = \frac{2b+i(i-1)l}{2r+i(i-1)d}\prod_{j=1}^{i}\frac{2r+j(j-1)d}{2b+2r+j(j-1)(l+d)}.$$

As usual we let F_i be the event that i consecutive red marbles are selected. The event F_i is calculated in essentially the same manner as E_i except that the last draw is now a red instead of a blue. Hence,

$$P(F_i) = \prod_{j=1}^{i} \frac{2r + j(j-1)d}{2b + 2r + j(j-1)(l+d)}.$$

In this case, $\lim_{i\to\infty} P(F_i)$ does not always equal 0. For instance, when $l = 0$ and $d = r = b$ we discover, according to *Mathematica* and [3], that

$$\lim_{i\to\infty} P(F_i) = \prod_{j=1}^{\infty} \frac{2 + j(j-1)}{4 + j(j-1)} = \cosh[\sqrt{7}\pi/2]\text{sech}[\sqrt{15}\pi/2] \approx 0.1455.$$

Intuitively this makes sense since we are continually adding a lot of red marbles and no blue marbles thus making it possible to never draw a blue marble. But now $S = F \cup (\cup_{i=1}^{\infty} E_i)$, where F is the event that a blue marble is never drawn. Hence,

$$1 = P(S) = P(F) + \sum_{i=1}^{\infty} P(E_i) = \cosh[\sqrt{7}\pi/2]\text{sech}[\sqrt{15}\pi/2] + \sum_{i=1}^{\infty} \left[\frac{2}{2 + i(i-1)} \prod_{j=1}^{i} \frac{2 + j(j-1)}{4 + j(j-1)} \right].$$

On the other hand, if we let $d = 0$ and $r = b = l$, adding more blue marbles than red, then

$$\lim_{i\to\infty} P(F_i) = \lim_{i\to\infty} \prod_{j=1}^{i} \frac{2}{4 + j(j-1)} \leq \lim_{i\to\infty} \prod_{j=1}^{i} \frac{2}{4} \leq \lim_{i\to\infty} \left(\frac{2}{4}\right)^i = 0,$$

and so

$$1 = P(S) = \sum_{i=1}^{\infty} P(E_i) = \sum_{i=1}^{\infty} \frac{2^{i-1}[2 + i(i-1)]}{\prod_{j=1}^{i}[4 + j(j-1)]}. \tag{5}$$

For a slight variation we can continue to select marbles from the bag until a blue is chosen under the rule that if a red is chosen at the i^{th} turn add $i!l$ blue marbles and $i!d+1$ red marbles, starting with $b > 0$ blue marbles in the bag and $r > 0$ red marbles. In this case, calculating in the same manner, we get that $P(E_1) = b/(b+r)$, $P(E_2) = r(b+l)/(b+r)(b+r+l+d)$, and for $i > 2$

$$P(E_i) = \frac{r[b + l\sum_{k=1}^{i-1} k!]}{(b+r)[r + d\sum_{k=1}^{i-1} k!]} \prod_{j=1}^{i-1} \frac{r + d\sum_{k=1}^{j} k!}{b + r + (l+d)\sum_{k=1}^{j} k!},$$

and

$$P(F_i) = \frac{r}{b+r} \prod_{j=1}^{i-1} \frac{r + d\sum_{k=1}^{j} k!}{b + r + (l+d)\sum_{k=1}^{j} k!}.$$

For $d = 0$ and $r = b = l$ we have

$$P(F_i) = \frac{1}{2} \prod_{j=1}^{i-1} \frac{1}{1 + \sum_{k=0}^{j} k!},$$

and clearly $\lim_{i\to\infty} P(F_i) = 0$ and so

$$1 = \frac{1}{2} + \frac{1}{3} + \sum_{i=3}^{\infty} \frac{\sum_{k=0}^{i-1} k!}{2 \prod_{j=1}^{i-1} [1 + \sum_{k=0}^{j} k!]}. \tag{6}$$

On the other hand, for $l = 0$ and $r = d = b$, we find that

$$P(F_i) = \frac{1}{2} \prod_{j=1}^{i-1} \frac{\sum_{k=0}^{j} k!}{1 + \sum_{k=0}^{j} k!}. \tag{7}$$

In this case, we don't necessarily expect the limit to be 0 since we are adding a lot of red marbles but no blue marbles. In fact, *Mathematica* suggests that the limit is around 0.23. We leave it as a challenge to the reader to calculate the limit or at least show that it is positive or what until the next section for a reason it is positive.

In yet another variation, we start with $b > 0$ blue marbles and $r > 0$ red marbles in the bag, but this time if a red is chosen at the i^{th} turn add l^i blue marbles and $d^i + 1$ red marbles. Here $P(E_1) = b/(b + r)$, $P(E_2) = r(b + l)/(r + b)(r + d + b + l)$,

$$P(E_i) = \frac{r[b + \sum_{k=1}^{i} l^k]}{(r + b)[r + \sum_{k=1}^{i} d^k]} \prod_{j=1}^{i-1} \frac{r + \sum_{k=1}^{j} d^k}{r + b + \sum_{k=1}^{j} d^k + \sum_{k=1}^{j} l^k},$$

and

$$P(F_i) = \frac{r}{r + b} \prod_{j=1}^{i-1} \frac{r + \sum_{k=1}^{j} d^k}{r + b + \sum_{k=1}^{j} d^k + \sum_{k=1}^{j} l^k}.$$

When $d = 0$ and $b = r = l > 1$, we have

$$\lim_{i\to\infty} P(F_i) = \frac{1}{2 \prod_{j=1}^{i-1} [2 + (1 - r^{j+1})/(1 - r)]} = 0,$$

$P(E_1) = 1/2$, $P(E_2) = 1/3$, and for $i > 2$

$$P(E_i) = \frac{(r^{i-1} + r - 2)(r - 1)^{i-2}}{2 \prod_{j=1}^{i-1} [r^j + 2r - 3]},$$

which will yield, in the specific case when $r = 2$, the series

$$1 = 1/2 + 1/3 + \sum_{i=3}^{\infty} \frac{2^{i-2}}{\prod_{j=1}^{i-1}[2^j + 1]}. \tag{8}$$

Of course, when $r = b$ and $l = d$, we again come up with the sum given in (1) since the bag always has equal amounts of red and blue marbles. We leave it to the reader to check this if they wish. For the case $l = 0$ and $b = r = d$ we don't expect $P(F_i)$ to converge to 0 so we leave that case alone.

Certainly there are other avenues to pursue since there are many possibilities of what to put back in the bag and we can also stop when we have chosen k blue marbles in a row along with adding marbles to the bag after each draw. Also, we could work scenarios with more than two color marbles in the bag, say we have red, blue, and green marbles and we will stop if a red is chosen on an even draw or a blue is chosen on an odd draw, while adding marbles after each draw.

Connections and Conclusions

Theorem 3.20 in Real Infinite Series [2] states that $\sum_{n=1}^{\infty} a_n$ and $\prod_{n=1}^{\infty}(1+a_n)$ both converge or diverge. Obviously by taking the log of the product we can include $\sum_{n=1}^{\infty} \log(1 + a_n)$ to the list. As a corollary to this we can show that for $\{a_n\}$ positive, $\prod_{n=1}^{\infty} \frac{a_n}{1+a_n}$ is positive if and only if $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges, since

$$\log\left(\prod_{n=1}^{\infty} \frac{a_n}{1 + a_n}\right) = \sum_{n=1}^{\infty} -\log\left(1 + \frac{1}{a_n}\right)$$

which only converges to a nonzero value if and only if $\sum_{n=1}^{\infty} 1/a_n$ converges. This shows that the limit in (7) must be positive since taking $a_n = \sum_{k=0}^{n} k!$ we clearly have $\sum_{n=1}^{\infty} 1/a_n$ converging.

To prove another corollary we will need the fact that for $\{a_n\}$ positive

$$\sum_{i=1}^{n} \frac{a_i}{(1 + a_1)(1 + a_2)\ldots(1 + a_i)} = 1 - \frac{1}{(1 + a_1)(1 + a_2)\ldots(1 + a_n)},$$

which can be proven by setting $a_i = 1 + a_i - 1$ to make the sum telescope. Hence, for a_n positive, we obtain that

$$\sum_{i=1}^{\infty} \frac{a_i}{(1+a_1)(1+a_2)\dots(1+a_i)} = 1 \quad \text{if} \quad \sum_{i=1}^{\infty} a_i \quad \text{diverges}, \tag{9}$$

and

$$\sum_{i=1}^{\infty} \frac{a_i}{(1+a_1)(1+a_2)\dots(1+a_i)} = 1 - \frac{1}{\prod_{i=1}^{\infty}(1+a_i)} \quad \text{if} \quad \sum_{i=1}^{\infty} a_i \quad \text{converges}, \tag{10}$$

both of which follow from the relationship between $\sum_{i=1}^{\infty} a_i$ and $\prod_{n=1}^{\infty}(1+a_n)$.

Many of our results in can be derived from these facts. For example take $a_i = 1 + \binom{i}{2}$ to derive (5), $a_i = \sum_{k=1}^{i}(k-1)!$ to obtain (6) and $a_i = 2^i$ to get (8). Of course, our methods provide a physical interpretation for these series and it is natural to wonder if the methods here can be used to prove (9) and (10). Indeed, they can as the following will show.

Given a sequence of positive a_n define the following game where on the i^{th} turn we throw a dart at a square board of size one. The game ends if the dart lands in the region of the square under the height of $a_i/(1+a_i)$ and continues if it lands above that height. Let E_i be the event that the game ends at the i^{th} turn and F_i be the event that the game continues to the next turn. Now, calculating some probabilities shows that $P(F_1) = 1/(1+a_1)$ and so

$$P(F_i) = P(F_{i-1})\frac{1}{1+a_i} = \prod_{n=1}^{i} \frac{1}{1+a_n}$$

and

$$P(E_i) = P(F_{i-1})P(\text{ hitting target area on the } i^{\text{th}} \text{ throw }) = \prod_{n=1}^{i} \frac{a_i}{1+a_n}.$$

Finally, with $F = \cap_{i=1}^{\infty} F_i$ and $S = \cup_{i=1}^{\infty} E_i \cup F$ we see that (9) and (10) are equivalent to the fact that $P(S) = 1$, and that $\sum_{i=1}^{\infty} a_i$ converges if and only if $\prod_{i=1}^{\infty}(1+a_i)$ converges. Of course, there are series that so far have not been derived by these methods such $\sum_{n=1}^{\infty} 6/(\pi n)^2 = 1$ and $\sum_{n=0}^{\infty} 1/e(n!) = 1$. More generally, if we have a series of positive terms is there always a corresponding probabilistic scenario.

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