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Bioelectrical Circuits: Lecture 2

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BME 205 L02

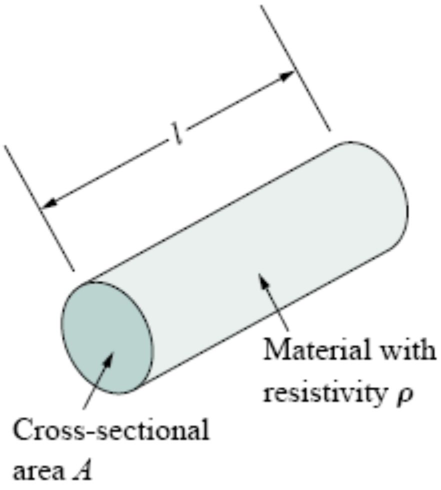
BASIC LAWS

Introduction

- We just learned about current, voltage and power. To actually determine the values of these variables in a given circuit requires we understand some fundamental laws that govern circuits.
- These laws, **Ohm's law** and **Kirchhoff's laws** form the foundation of electric circuit analysis.
- In this lecture we'll cover techniques of circuit analysis such as combining resistors in series and in parallel, voltage division, and current division.
- We'll restrict our focus to resistive circuits for now.

Resistance

- Materials in general have a characteristic behavior of resisting the flow of electric charge. This physical property - the ability to resist current – is known as resistance and is represented by the symbol R.
- The resistance of any material with a uniform cross-sectional area A depends on A and its length l.



- In mathematical form, $$R = \rho \frac{\ell}{A}$$

where ρ is known as the resistivity of the material in ohm-meters.

- The resistance of a material can change with internal or external conditions of the element, e.g., temperature, but our analyses in this course will assume it is always stable.

Resistivity

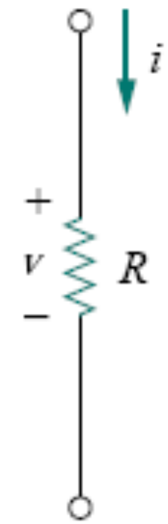
- Good conductors, such as copper and aluminum, have low resistivities, while insulators, such as mica and paper, have high resistivities.
- Table 2.1 presents the values of ρ for some common materials and shows which materials are used for conductors, insulators, and semiconductors.

TABLE 2.1 Resistivities of common materials.

Material	Resistivity ($\Omega \cdot m$)	Usage
Silver	1.64×10^{-8}	Conductor
Copper	1.72×10^{-8}	Conductor
Aluminum	2.8×10^{-8}	Conductor
Gold	2.45×10^{-8}	Conductor
Carbon	4×10^{-5}	Semiconductor
Germanium	47×10^{-2}	Semiconductor
Silicon	6.4×10^{2}	Semiconductor
Paper	10^{10}	Insulator
Mica	5×10^{11}	Insulator
Glass	10^{12}	Insulator
Teflon	3×10^{12}	Insulator

Ohm’s Law

- The circuit element used to model the current-resisting behavior of a material is the resistor.
- For the purpose of constructing circuits, resistors are usually made from metallic alloys and carbon compounds. The circuit symbol for the resistor is shown, where *R* stands for the resistance of the resistor. The resistor is the simplest passive element.



(b)

- the relationship between current and voltage for a resistor is known as *Ohm’s law:*

> **Ohm’s law** states that the voltage *v* across a resistor is directly proportional to the current *i* flowing through the resistor.

Ohm's Law

- That is, $v \propto i$
- The constant of proportionality is the resistance, *R*. Thus we have:

$$\boxed{v = iR}$$

which is the mathematical form of Ohm's law.

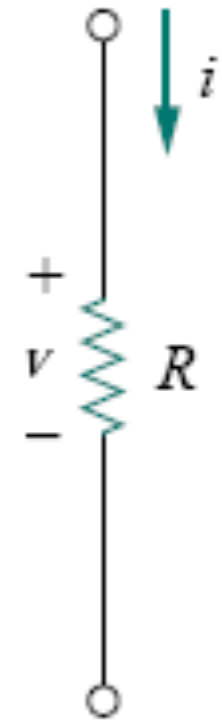
R is measured in the unit of **ohms**, designated Ω.

We then deduce that $$R = \frac{v}{i}$$

so that $1\ \Omega = 1\ \text{V/A}$

Passive sign convention for Ohm's Law

- To apply Ohm's law, we must pay careful attention to the current direction and voltage polarity*.
- Application of the law in the form $v = iR$ assumes that the direction of current i and the polarity of voltage v conform to the passive sign convention (as shown: reference direction for i is from the '+' to the '-' of the voltage reference polarity).
- *This implies that in resistors, current flows from a higher potential to a lower potential.*
- If the reference direction of current i happens to be from the '-' to the '+' end of v instead, then the relationship would be $v = -iR$.



* The term "polarity" refers to the placement of the + and - in the definition of a voltage. It's equivalent to drawing an axis in a mechanical problem to designate one particular direction as "positive displacement" – just like drawing an arrow for a current.

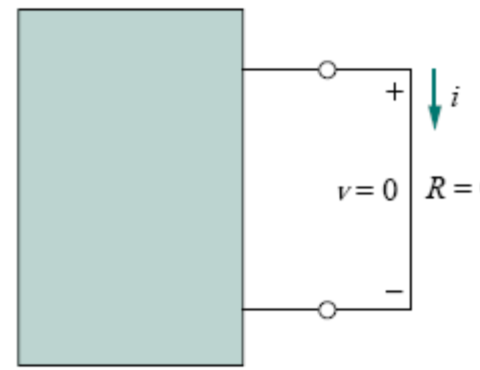
Short circuit

Since the value of *R* can range from zero to infinity, it is important that we consider the two extreme possible values of *R.*

- An element with *R = 0* is called a short circuit, as shown. For a short circuit,

$$v = iR = 0$$

showing that, no matter what the current is, the voltage is zero. In practice, a short circuit is usually a connecting wire assumed to be a perfect conductor.



(a)

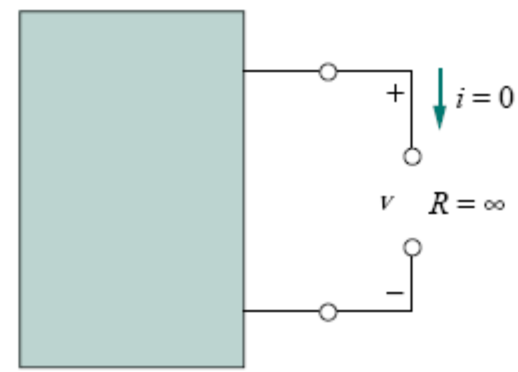
A short circuit is a circuit element with resistance approaching zero.

Open circuit

- Similarly, an element with $R = \infty$ is known as an open circuit, as shown. For an open circuit,

$$i = \lim_{R \to \infty} \frac{v}{R} = 0$$

indicating that no matter what the voltage, the current is zero.



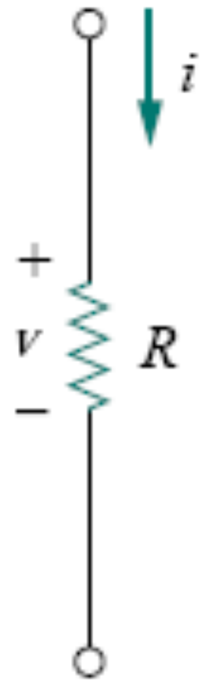
(b)

Thus,

An open circuit is a circuit element with resistance approaching infinity.

Fixed and variable resistors

- A resistor is either fixed or variable.
- Most resistors are of the fixed type, meaning their resistance remains constant.
- The circuit symbol is shown for a fixed resistor, and fig 2.3. shows photos of real fixed resistors.
- You will use the carbon film type in the lab.



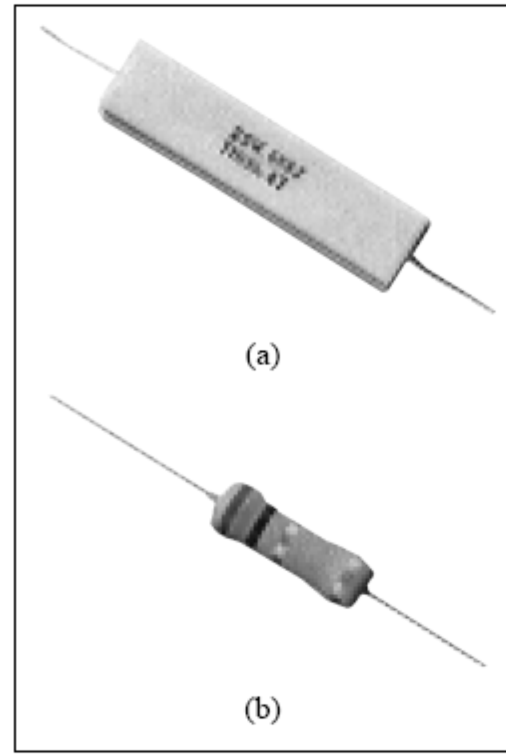


Figure 2.3 Fixed resistors: (a) wire-wound type, (b) carbon film type. *(Courtesy of Tech America.)*

Fixed and variable resistors

- Variable resistors have adjustable resistance. The symbol for a variable resistor is shown in Fig. 2.4(a).
- A common variable resistor is known as a *potentiometer* or *pot* for short, with the symbol shown in Fig. 2.4(b). The pot is a three-terminal element with a sliding contact or wiper. By sliding the wiper, the resistances between the wiper terminal and the fixed terminals vary.
- Real ones are shown in the photo.

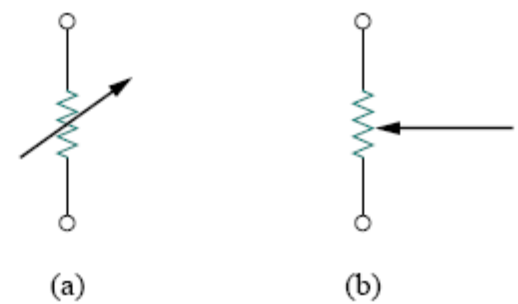


Figure 2.4 Circuit symbol for: (a) a variable resistor in general, (b) a potentiometer.

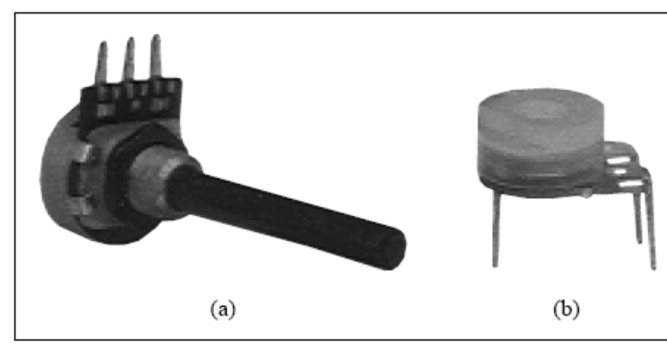
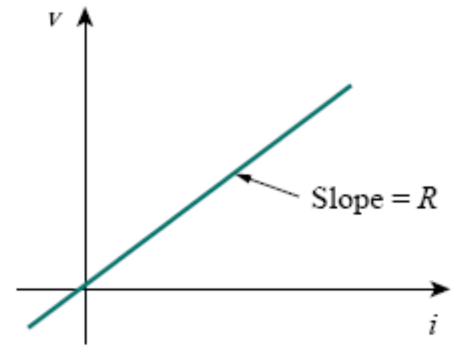


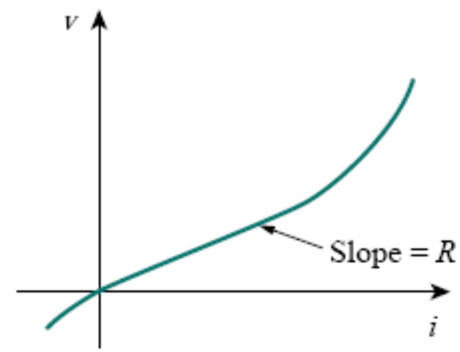
Figure 2.5 Variable resistors: (a) composition type, (b) slider pot. *(Courtesy of Tech America.)*

Linear and nonlinear resistors

- It should be pointed out that not all resistors obey Ohm's law. A resistor that obeys Ohm's law is known as a *linear* resistor. It has a constant resistance and thus its current-voltage characteristic is as illustrated in (a): its *i*-*v* graph is a straight line passing through the origin.
- A nonlinear resistor does not obey Ohm's law. Its resistance varies with current. A typical nonlinear *i*-*v* characteristic is shown in *(b).*
- In this circuit theory course, all of our resistors will be ideal linear ones.



(a)



(b)

Conductance

- A useful quantity in circuit analysis is the reciprocal of resistance *R,* known as *conductance* and denoted by *G:*

$$\boxed{G = \frac{1}{R} = \frac{i}{v}} \qquad \Rightarrow \qquad i = Gv$$

- The conductance is a measure of how well an element will conduct electric current.
- There are two equivalent terms for the unit of conductance:
  - the *mho* (ohm spelled backward) or reciprocal ohm, with inverted omega as its symbol.
  - siemens (S), the SI unit of conductance: $1\ \text{S} = 1\ \mho = 1\ \text{A/V}$

> **Conductance** is the ability of an element to conduct electric current; it is measured in mhos ($\mho$) or siemens (S).

Power dissipated by resistor

- The power dissipated by a resistor can be expressed either in terms of *R*:

$$p = vi = i^2 R = \frac{v^2}{R}$$

  … or in terms of *G*:

$$p = vi = v^2 G = \frac{i^2}{G}$$

note two things:

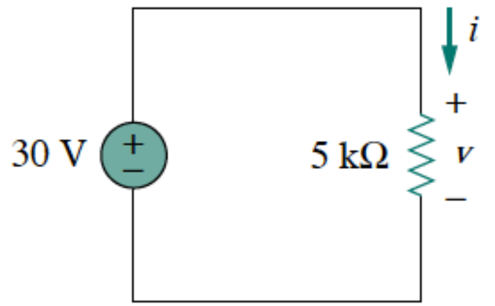
1. The power dissipated in a resistor is a nonlinear function of either current or voltage.
2. Since *R and G* are positive quantities, the power dissipated in a resistor is always positive. Thus, a resistor always absorbs power from the circuit. This confirms the idea that a resistor is a passive element, incapable of generating energy.

Examples

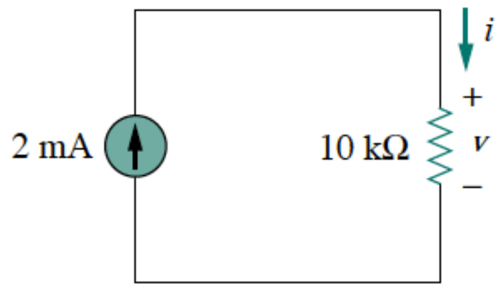
1. An electric iron draws 2A at 120V. Find its resistance.

2. The essential component of a toaster is an electrical element (a resistor) that converts electrical energy to heat energy. How much current is drawn by a toaster with resistance 10Ω at 110V?

3. In a simple circuit composed of an independent voltage source of 30V connected to a 5kΩ resistor, calculate the current i drawn from the voltage source, the conductance G, and the power absorbed p.



4. In a simple circuit with an independent current source of 2mA connected to a 10kΩ resistor, calculate the voltage v across the resistor, the conductance G, and the power absorbed p.



Nodes, Branches and Loops

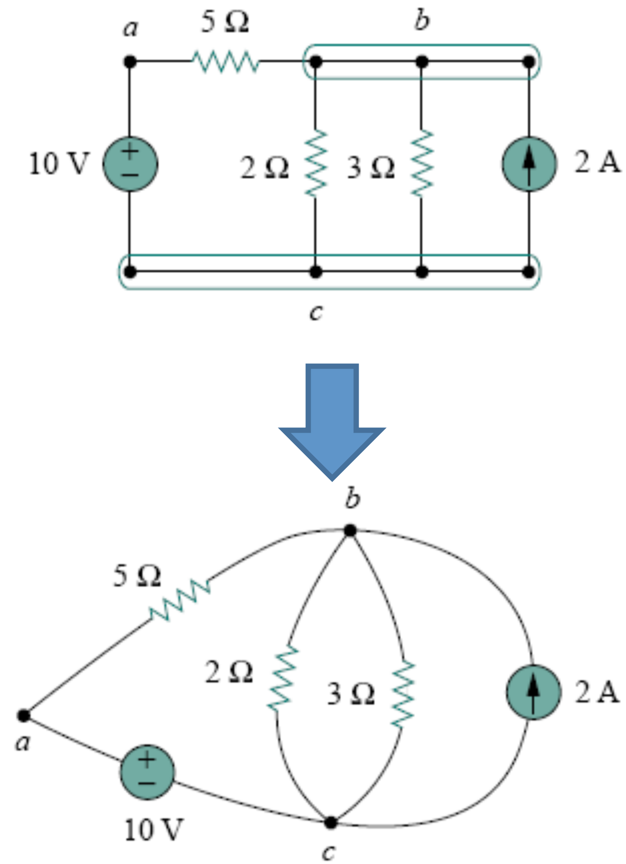
- Now for some "network topology" – the study of the placement of elements in a network (i.e., circuit).

> A **branch** represents a single element such as a voltage source or a resistor.

- A branch represents any two-terminal element. The circuit shown has five branches - the 10-V voltage source, the 2-A current source, and the 3 resistors.

> A **node** is the point of connection between two or more branches.

- A node is usually indicated by a dot in a circuit. If a short circuit (a connecting wire) connects two nodes, the two nodes constitute a single node. The circuit shown has three nodes *a, b, and c.* Notice that the three points that form node *b* are connected by perfectly conducting wires and therefore constitute a single point. Same goes for four points of node *c*.



A **loop** is any closed path in a circuit.

- A loop is a closed path formed by starting at a node, passing through a set of nodes, and returning to the starting node without passing through any node more than once. A loop is said to be **independent** if it contains at least one branch which is not part of any other independent loop. Independent loops or paths result in independent sets of equations.
- A network with *b* branches, *n* nodes, *and* l independent loops will satisfy the fundamental theorem of network topology:

$$b = l + n - 1$$

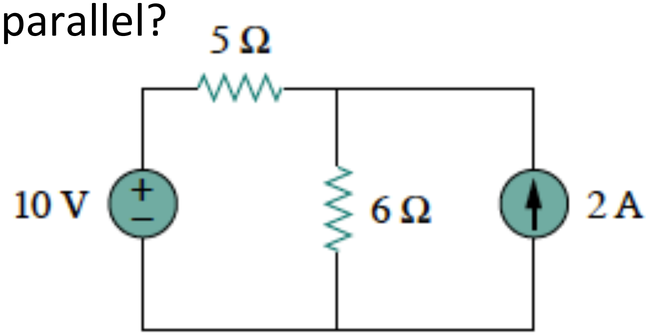
Definition:

Two or more elements are in **series** if they are cascaded or connected sequentially and consequently carry the same current.
Two or more elements are in **parallel** if they are connected to the same two nodes and consequently have the same voltage across them.

- Elements are in **series** when they are chain-connected or connected sequentially, end to end. E.g. two elements are in series if they share one common node and *no other element* is connected to that common node.
- Elements in **parallel** are connected to the same pair of terminals. E.g. on last slide the 2-Ω and 3-Ω resistors are in parallel. The 2-Ω and 5-Ω resistors are **neither in series nor in parallel**.

Example

Determine the branches and nodes in the circuit below. Which elements are in series? Which elements are in parallel?



Kirchhoff's laws

- Ohm's Law is not enough by itself to analyze circuits.
- Kirchhoff's laws were first introduced in 1847 by the German physicist Gustav Robert Kirchhoff (1824–1887). These laws are formally known as Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL).
- Kirchhoff's first law is based on the law of conservation of charge, which requires that the algebraic sum of charges within a system cannot change.

> **Kirchhoff's current law (KCL)** states that the algebraic sum of currents entering a node (or a closed boundary) is zero.

- Mathematically:

$$\sum_{n=1}^{N} i_n = 0$$

where N is the number of branches connected to the node and i_n is the n^{th} current entering (or leaving) the node.

Currents entering a node may be regarded as positive, while currents leaving the node may be taken as negative or vice versa.

KCL proof

To prove KCL, assume a set of currents $i_k(t)$, $k = 1, 2, . . . ,$ flow into a node. The algebraic sum of currents at the node is

$i_T(t) = i_1(t) + i_2(t) + i_3(t) + \cdots$

Integrating both sides of the Eq. gives

$q_T(t) = q_1(t) + q_2(t) + q_3(t) + \cdots$

where $q_k(t) = \int i_k(t)\,dt$ and $q_T(t) = \int i_T(t)\,dt$.

But the law of conservation of electric charge requires that the algebraic sum of electric charges at the node must not change; that is, the node stores no net charge.

Thus $q_T(t) = 0 \rightarrow i_T(t) = 0$, confirming the validity of KCL.

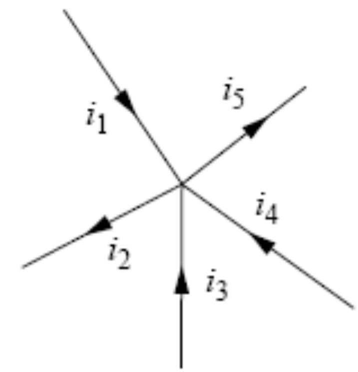
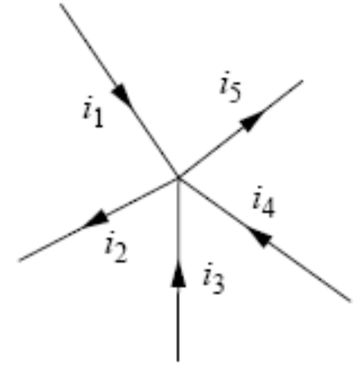


Figure 2.16 Currents at a node illustrating KCL.

Consider the node shown. Applying KCL gives $i_1 + (-i_2) + i_3 + i_4 + (-i_5) = 0$

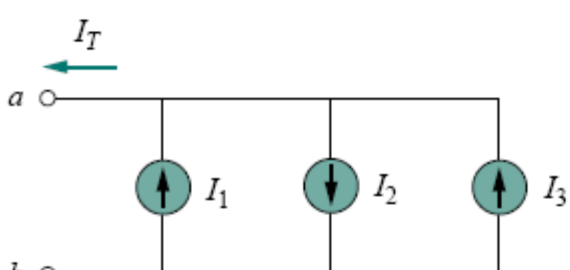
since currents i_1, i_3, and i_4 are entering the node, while currents i_2 and i_5 are leaving it. By rearranging the terms, we get $i_1 + i_3 + i_4 = i_2 + i_5$

This last equation is an alternative form of KCL:



> The sum of the currents entering a node is equal to the sum of the currents leaving the node.

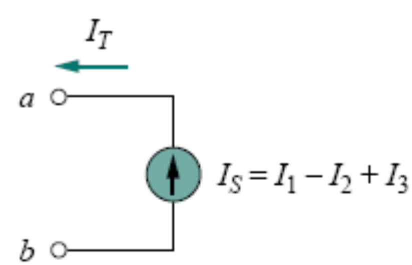
- A simple application of KCL is combining current sources in parallel.
- The combined current is the algebraic sum of the current supplied by the individual sources. For example, the current sources shown in (a) can be combined as in (b). The combined or equivalent current source can be found by applying KCL to node *a*.



(a)

$$I_T + I_2 = I_1 + I_3$$

$$I_T = I_1 - I_2 + I_3$$



(b)

Kirchhoff's Voltage Law

- Kirchhoff's second law is based on the principle of conservation of energy:

Kirchhoff's voltage law (KVL) states that the algebraic sum of all voltages around a closed path (or loop) is zero.

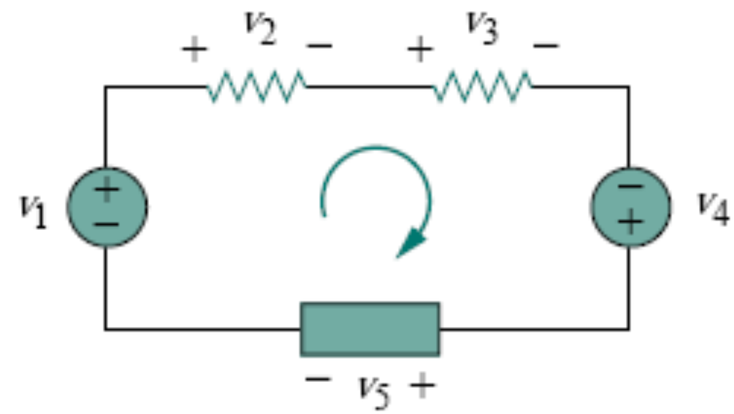
- Expressed mathematically, KVL states that

$$\sum_{m=1}^{M} v_m = 0$$

where M is the number of voltages in the loop (or the number of branches in the loop) and v_m is the m^{th} voltage.

Kirchhoff's Voltage Law

- To illustrate KVL, consider the circuit shown.



- The sign on each voltage is the polarity of the terminal encountered first as we travel around the loop. We can start with any branch and go around the loop either clockwise or counterclockwise. Suppose we start with the voltage source on the left and go clockwise around the loop as shown; then voltages would be *−v1,+v2,+v3,−v4, and +v5,* in that order. For example, as we reach branch 3, the positive terminal is met first; hence we have +*v3.* For branch 4, we reach the negative terminal first; hence, −*v4.* Thus, KVL yields

$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0$$

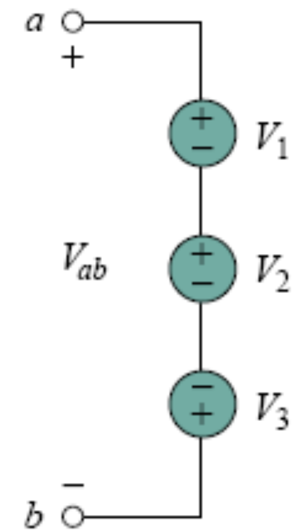
- Rearranging: $v_2 + v_3 + v_5 = v_1 + v_4$

Which may be interpreted as

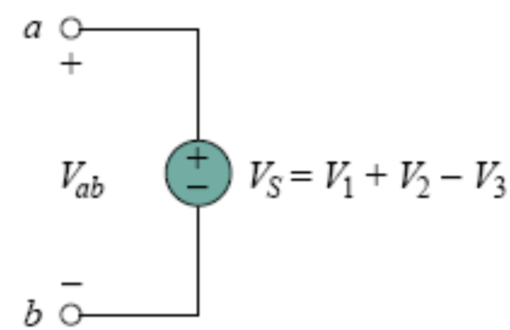
Sum of voltage drops = Sum of voltage rises

- When voltage sources are connected in series, KVL can be applied to obtain the total voltage. The combined voltage is the algebraic sum of the voltages of the individual sources. For example, for the voltage sources shown in (a), the combined or equivalent voltage source in (b) is obtained by applying KVL.

$$-V_{ab} + V_1 + V_2 - V_3 = 0$$



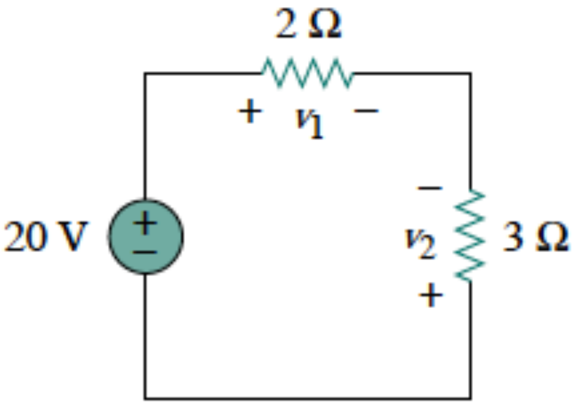
(a)

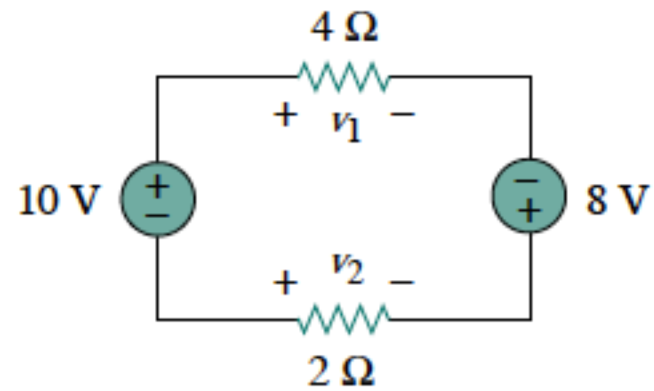


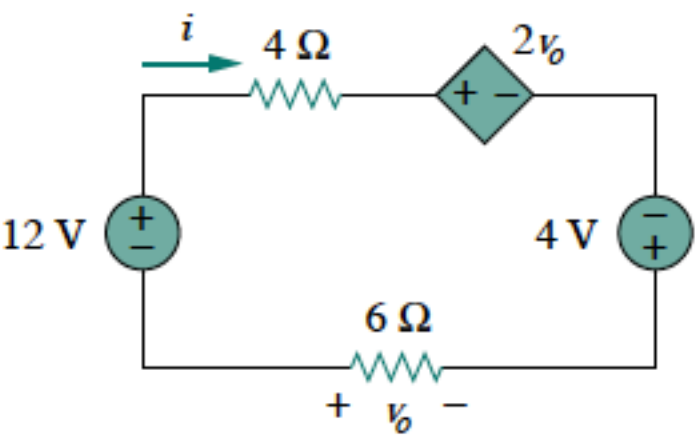
(b)

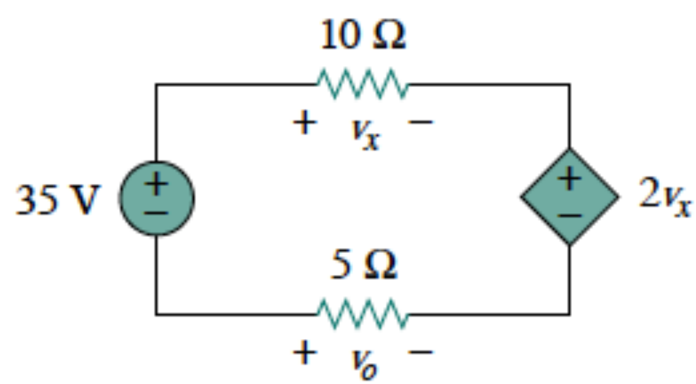
- To avoid violating KVL, a circuit cannot contain two different voltages *V1* and *V2* in parallel unless *V1 = V2.*

Examples

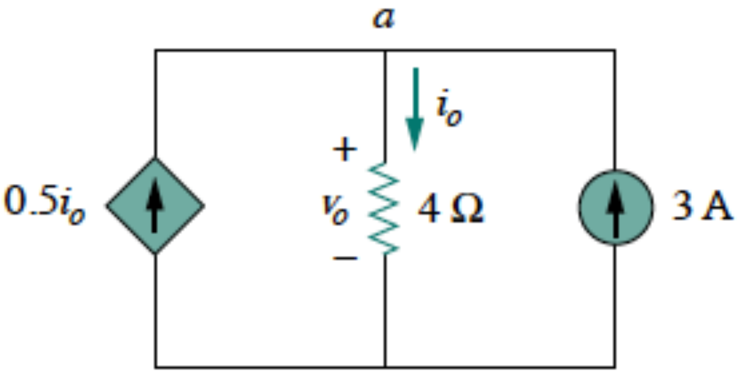

2 Ω
+ v_1 −
20 V
− v_2 +
3 Ω


4 Ω
+ v_1 −
10 V
8 V
+ v_2 −
2 Ω

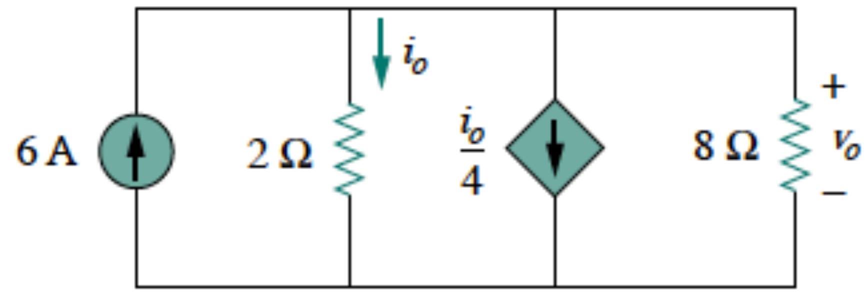

i
4 Ω
$2v_o$
12 V
4 V
6 Ω
+ v_o −


10 Ω
+ v_x −
35 V
$2v_x$
5 Ω
+ v_o −

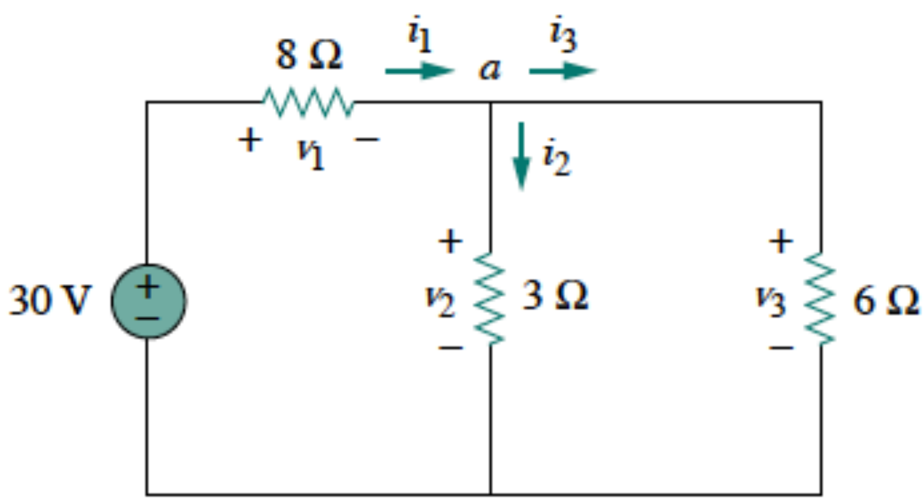
Examples



a
i_o
+
v_o
4 Ω
−
$0.5i_o$
3 A



i_o
6 A
2 Ω
$\frac{i_o}{4}$
8 Ω
+
v_o
−



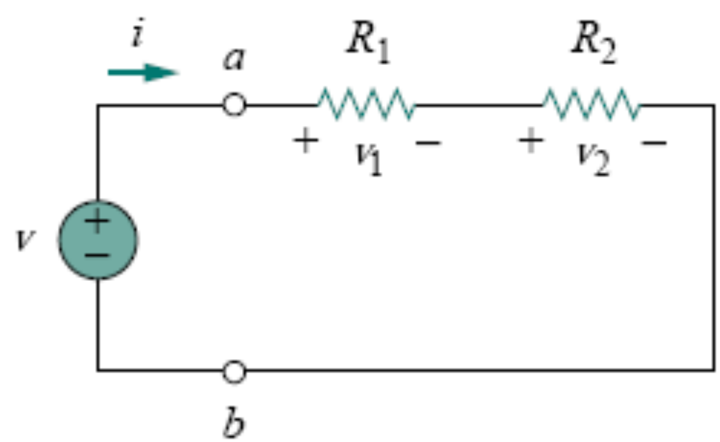
8 Ω
i_1
a
i_3
+
v_1
−
i_2
30 V
+
v_2
−
3 Ω
+
v_3
−
6 Ω

Series Resistors and Voltage Division

- We often have to combine resistors in series and in parallel.
- Consider the single-loop circuit shown. The two resistors are in series, since the same current *i* flows in both of them. Applying Ohm's law to each of the resistors, we obtain

$$v_1 = i R_1, \qquad v_2 = i R_2$$

- If we apply KVL to the loop (moving in the clockwise direction), we have $-v + v_1 + v_2 = 0$



Series Resistors and Voltage Division

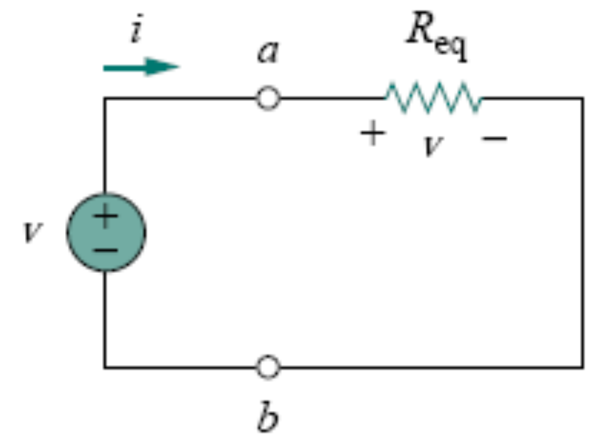
- Combining, we get $v = v_1 + v_2 = i(R_1 + R_2)$

or

$$i = \frac{v}{R_1 + R_2}$$

which can be written $v = i\,R_{eq}$

where $R_{eq} = R_1 + R_2$



- Thus, the previous circuit can be replaced by this equivalent circuit. The two circuits are equivalent because they exhibit the same voltage-current relationships at the terminals *a-b.*

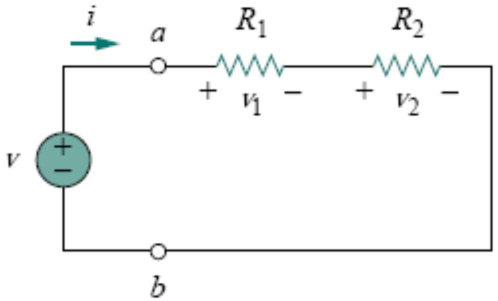
The equivalent resistance of any number of resistors connected in series is the sum of the individual resistances.

- For *N* resistors in series:

$$R_{eq} = R_1 + R_2 + \cdots + R_N = \sum_{n=1}^{N} R_n$$

Series Resistors and Voltage Division

- To determine the voltage across each resistor in the original circuit, we substitute $i = \frac{v}{R_1 + R_2}$ into $v_1 = iR_1$, & $v_2 = iR_2$ and obtain



$$v_1 = \frac{R_1}{R_1 + R_2} v, \qquad v_2 = \frac{R_2}{R_1 + R_2} v$$

- Notice that the source voltage *v* is divided among the resistors in direct proportion to their resistances; the larger the resistance, the larger the voltage drop. This is called the *principle of* ***voltage division****,* and the above circuit is called a *voltage divider.* In general, if a voltage divider has N resistors ($R_1, R_2, \ldots, R_N$) in series with the source voltage v, the n^{th} resistor (R_n) will have a voltage drop of

$$v_n = \frac{R_n}{R_1 + R_2 + \cdots + R_N} v$$

Parallel Resistors and Current Division

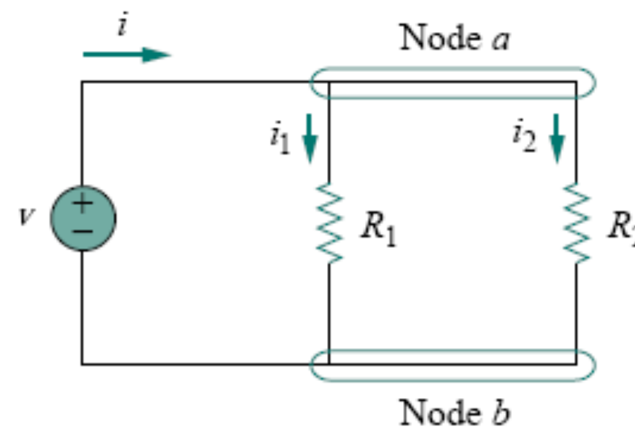
- Consider this circuit, where two resistors are connected in parallel and therefore have the same voltage across them. From Ohm's law,

$$v = i_1 R_1 = i_2 R_2 \quad \text{or} \quad i_1 = \frac{v}{R_1}, \qquad i_2 = \frac{v}{R_2}$$

Applying KCL at node *a* gives the total current *i* as $i = i_1 + i_2$

Substituting, we get

$$i = \frac{v}{R_1} + \frac{v}{R_2} = v\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{v}{R_{eq}}$$



where R_{eq} is the equivalent resistance of the resistors in parallel:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \Longrightarrow \frac{1}{R_{eq}} = \frac{R_1 + R_2}{R_1 R_2} \Longrightarrow \boxed{R_{eq} = \frac{R_1 R_2}{R_1 + R_2}}$$ *

The equivalent resistance of two parallel resistors is equal to the product of their resistances divided by their sum.

- It must be emphasized that this applies only to two resistors in parallel. From *, if $R_1 = R_2$, *then* $R_{eq} = R_1/2$.
- We can extend the result to the general case of a circuit with *N* resistors in parallel. The equivalent resistance is given by

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}$$

- Note that R_{eq} is always smaller than the resistance of the smallest resistor in the parallel combination. If $R_1 = R_2 = \cdots = R_N = R$, *then*

$$R_{\text{eq}} = \frac{R}{N}$$

In terms of conductance: $$G_{\text{eq}} = G_1 + G_2 + G_3 + \cdots + G_N$$

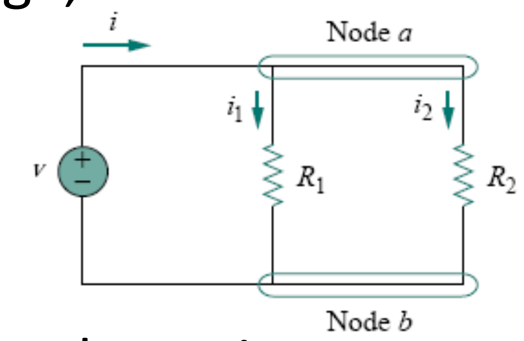
The equivalent conductance of resistors connected in parallel is the sum of their individual conductances.

- Notice that the equivalent conductance of parallel resistors is obtained in the same way as the equivalent resistance of series resistors – by adding. In the same manner, the equivalent conductance of resistors in series is obtained just the same way as the resistance of resistors in parallel. Thus the equivalent conductance G_{eq} *of N* resistors *in series* is

$$\frac{1}{G_{\text{eq}}} = \frac{1}{G_1} + \frac{1}{G_2} + \frac{1}{G_3} + \cdots + \frac{1}{G_N}$$

- Given the total current *i* entering node *a* in our original parallel circuit, how do we obtain current i_1 and i_2? We know that the equivalent resistor has the same voltage, or

$$v = i R_{\text{eq}} = \frac{i R_1 R_2}{R_1 + R_2}$$



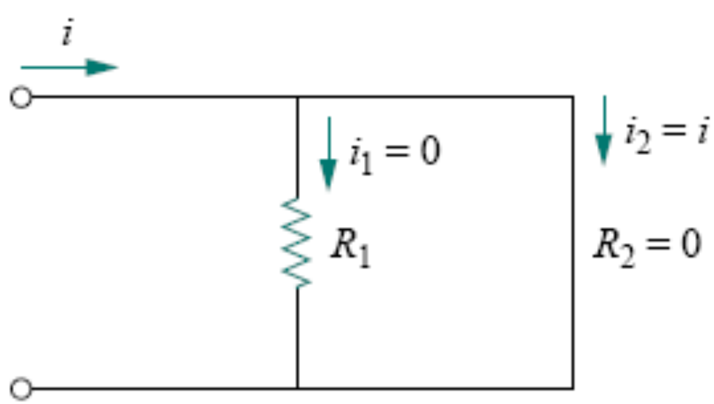
- Combining gives

$$i_1 = \frac{R_2 i}{R_1 + R_2}, \qquad i_2 = \frac{R_1 i}{R_1 + R_2}$$

- which shows that the total current *i* is shared by the resistors in inverse proportion to their resistances → **current division**.

- Accordingly, circuits with parallel resistances are often called *current dividers.* Notice that the larger current flows through the smaller resistance.
- As an extreme case, suppose one of the resistors in parallel is zero, say $R_2 = 0$; that is, R_2 is a short circuit, as shown. From current division, $R_2 = 0$ implies that $i_1 = 0, i_2 = i$. This means that the entire current i bypasses R_1 and flows through the short circuit $R_2 = 0$, the path of least resistance. Thus when a circuit is short circuited, two things should be kept in mind:

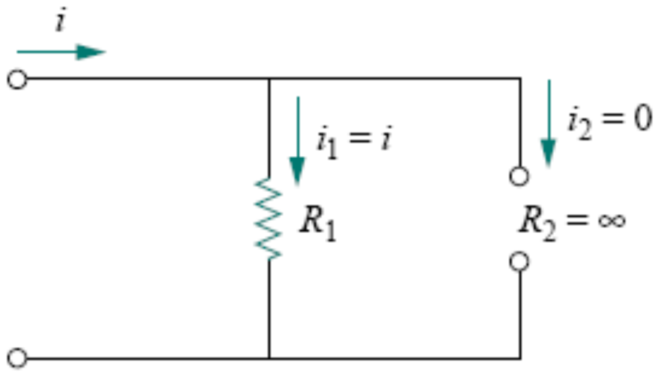
1. The equivalent resistance $R_{eq} = 0$.
2. The entire current flows through the short circuit.



- As another extreme case, suppose $R_2 = \infty$, that is, R_2 is an open circuit, as shown below. The current still flows through the path of least resistance, R_1. By taking the limit of

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

as $R_2 \rightarrow \infty$, we obtain $R_{eq} = R_1$ in this case.



- How does current division work in terms of conductance?
- If we divide both the numerator and denominator by R_1R_2,

$$i_1 = \frac{R_2\, i}{R_1 + R_2}, \qquad i_2 = \frac{R_1\, i}{R_1 + R_2}$$

becomes

$$i_1 = \frac{G_1}{G_1 + G_2} i \qquad i_2 = \frac{G_2}{G_1 + G_2} i$$

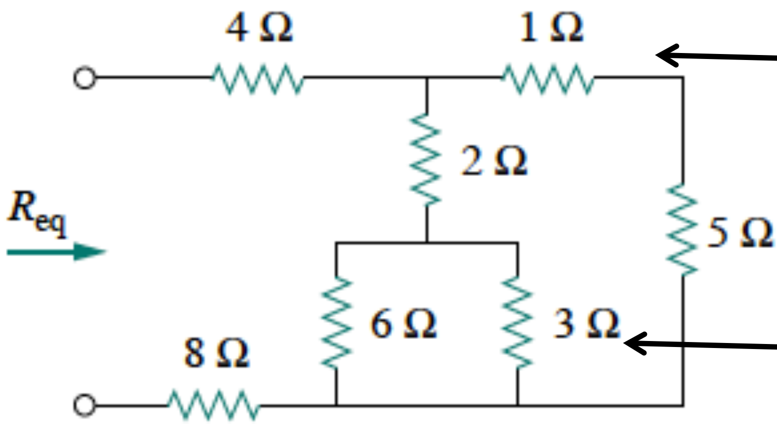
- Thus, in general, if a current divider has *N* conductors ($G_1, G_2, \ldots, G_N$) in parallel with the source current *i,* the n^{th} conductor (G_n) will have current

$$i_n = \frac{G_n}{G_1 + G_2 + \cdots + G_N} i$$

- In general, it is often convenient and possible to combine resistors in series and parallel and reduce a resistive network to a single equivalent resistance R_{eq}. Such an equivalent resistance is the resistance between the designated terminals of the network and must exhibit the same *i-v* characteristics as the original network at the terminals.

Examples

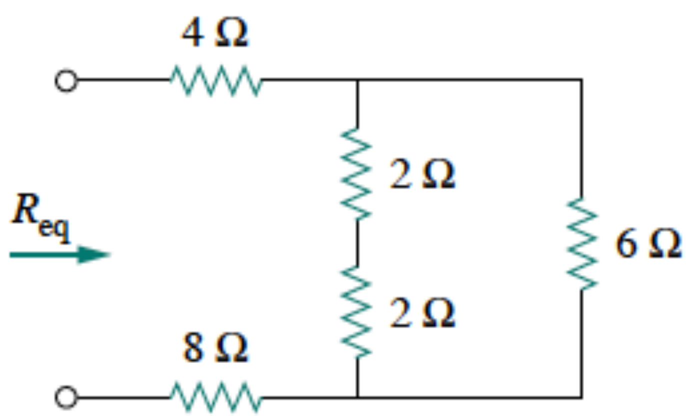
1. Find Req



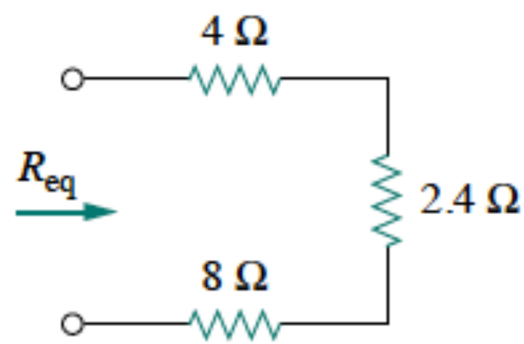
$$1\ \Omega + 5\ \Omega = 6\ \Omega$$

$$6\ \Omega \parallel 3\ \Omega = \frac{6 \times 3}{6+3} = 2\ \Omega$$

Combine step by step…

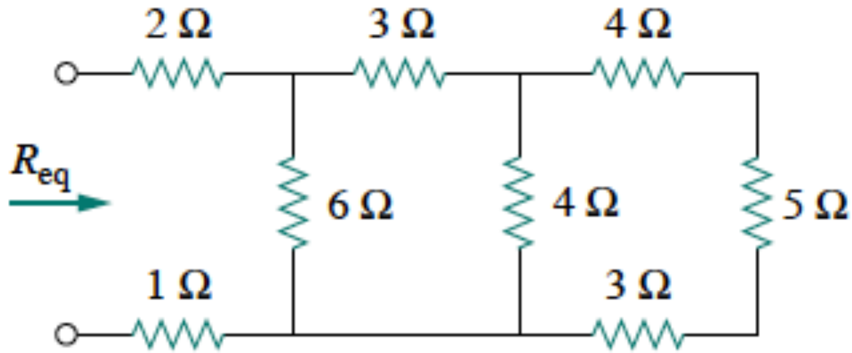


…

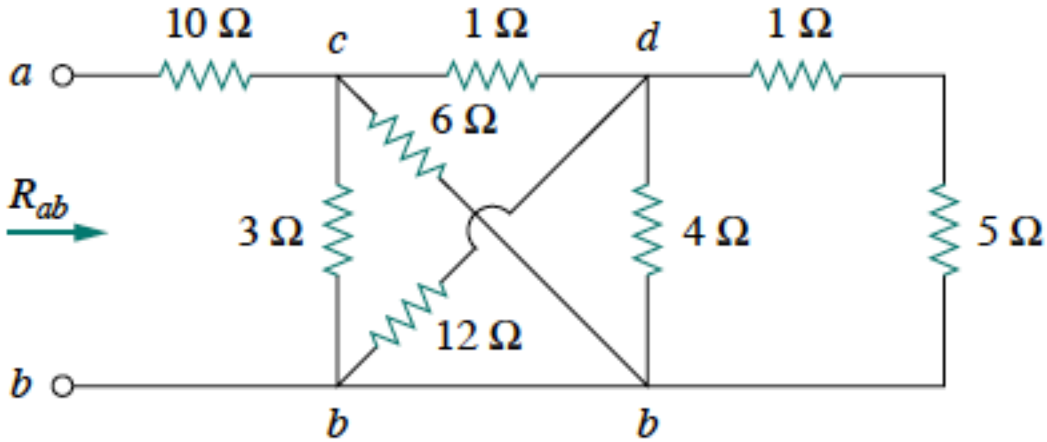


$$R_{eq} = 4\ \Omega + 2.4\ \Omega + 8\ \Omega = 14.4\ \Omega$$

2



3 Calculate the equivalent resistance R_{ab}



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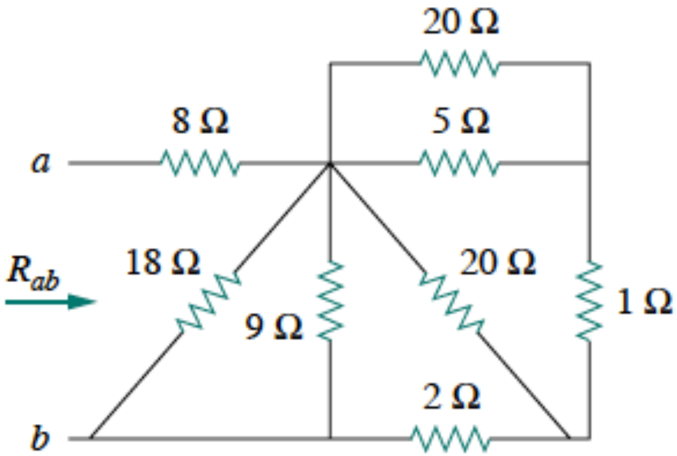

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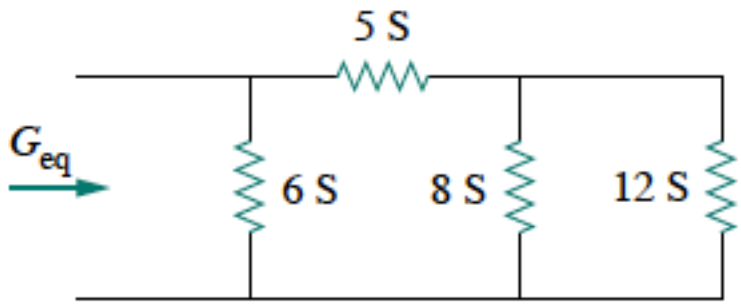
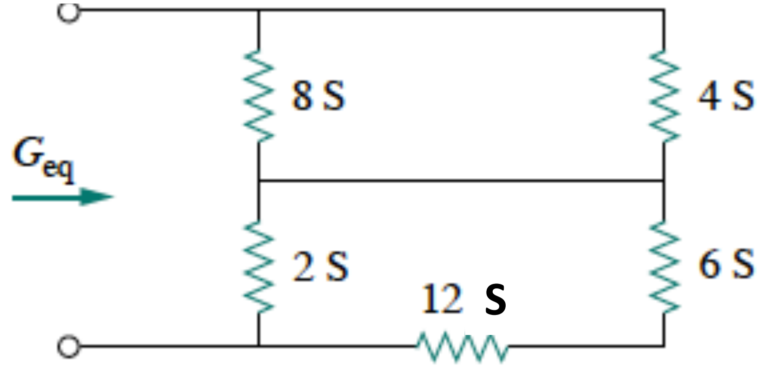
Find the equivalent conductance G_{eq}



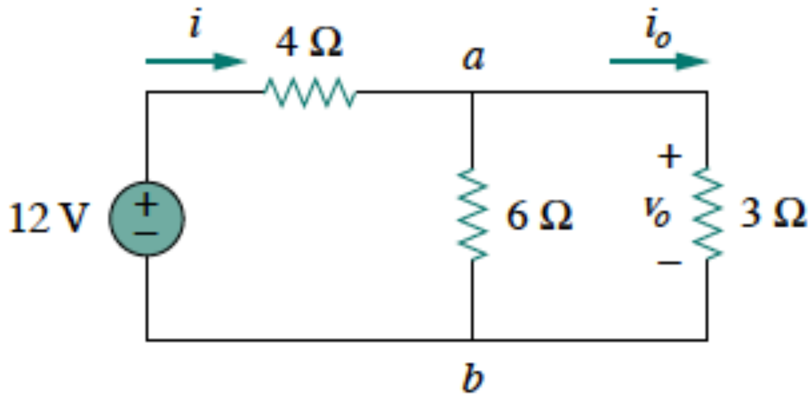
6

Find the equivalent conductance G_{eq}



Find i_o and v_o Calculate the power dissipated in the 3-Ω resistor.

7



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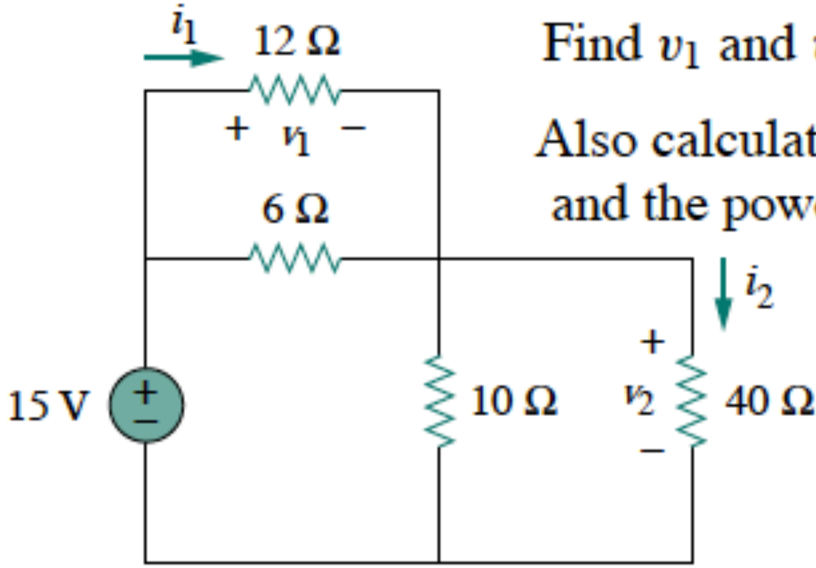

Find v_1 and v_2

Also calculate i_1 and i_2 and the power dissipated in the 12-Ω and 40-Ω resistors.

9 determine: (a) the voltage v_o, (b) the power supplied by the current source, (c) the power absorbed by each resistor.