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Bioelectrical Circuits: Lecture 3

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BME 205 L03

DC Circuits – Methods of Analysis

Introduction

- We now use Ohm's Law and Kirchhoff's Laws to develop powerful techniques for circuit analysis:
- Nodal Analysis
  - based on application of Kirchhoff's Current Law
- Mesh Analysis
  - based on application of Kirchhoff's Voltage Law
- For almost any circuit we can use these techniques to obtain a set of simultaneous equations and solve for all voltages & currents.
- We'll use Cramer's Rule to solve the simultaneous equations; this is easy…

Nodal Analysis

- *Nodal analysis* (also known as the *node-voltage method)* provides a general procedure for analyzing circuits using node voltages as the circuit variables.
- Choosing node voltages instead of element voltages as circuit variables is convenient and reduces the number of equations one must solve simultaneously.
- To simplify matters, we'll assume for now that circuits do not contain voltage sources. Circuits that contain voltage sources will be analyzed later on.
- In this case, the steps to determine node voltages are...

Steps to determine Node Voltages

- Given a circuit with *n* nodes without voltage sources:

1. Select a node as the reference node. Assign voltages v_1, $v_2, \ldots, v_{n-1}$ to the remaining $n - 1$ nodes. The voltages are referenced with respect to the reference node.
2. Apply KCL to each of the $n - 1$ nonreference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

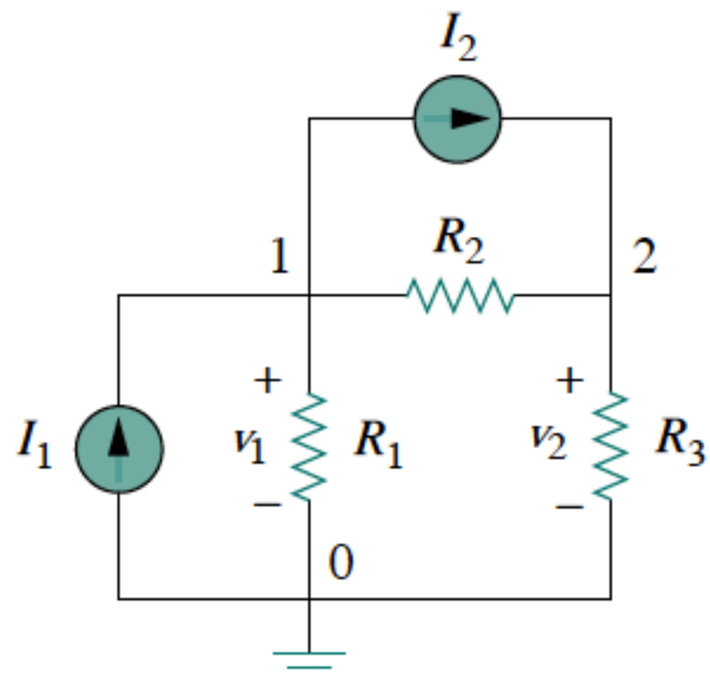
Reference / Ground

- The first step in nodal analysis is selecting a node as the *reference* or *datum node.* The reference node is commonly called the *ground* since it is assumed to have zero potential. A reference node is indicated by any of the three symbols below. The type of ground in (b) is called a *chassis ground* and is used in devices where the case, enclosure, or chassis acts as a reference point for all circuits. When the potential of the earth is used as reference, we use the *earth ground* in (a) or (c). We shall always use the symbol in (b).

(a) (b) (c)

Assign nonreference nodes

- Once we have selected a reference node, we assign voltage variable labels to nonreference nodes. Consider, for example, the circuit shown. Node 0 is the reference node *(v = 0),* while nodes 1 and 2 are assigned voltages v_1 and v_2, respectively.
- Keep in mind that the node voltages are defined with respect to the reference node.



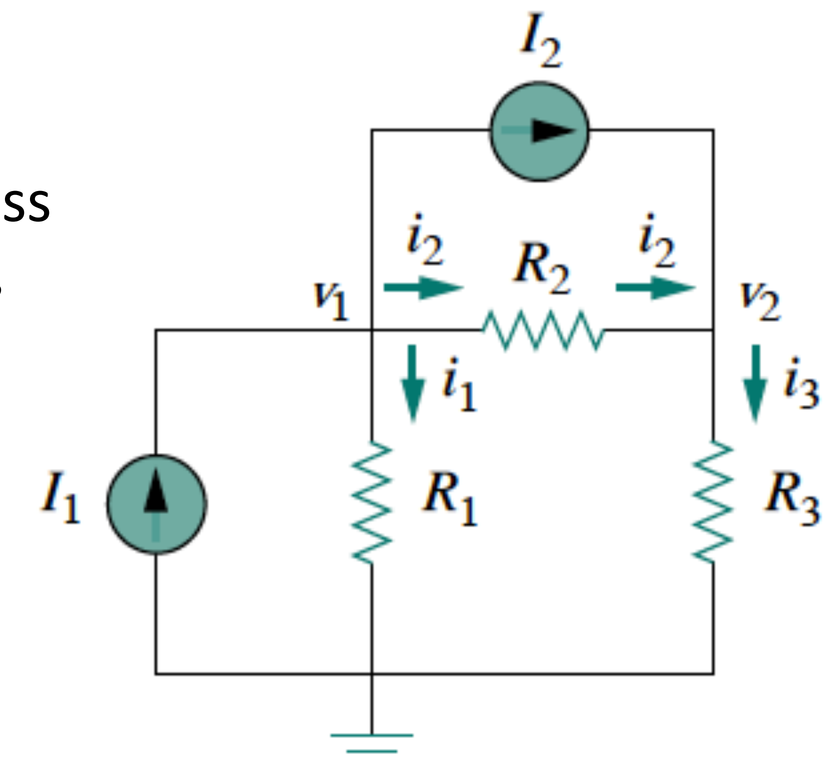
- Each node voltage is the voltage rise from the reference node to the corresponding nonreference node or simply the voltage of that node with respect to the reference node.

Apply KCL

- As the second step, we apply KCL to each nonreference node in the circuit. Label i_1, i_2, and i_3 as the currents through resistors R_1,R_2, and R_3, respectively. At node 1, applying KCL gives $I_1 = I_2 + i_1 + i_2$

- At node 2: $I_2 + i_2 = i_3$
- We now apply Ohm's law to express the unknown currents i_1, i_2, *and* i_3 in terms of node voltages.

- Keep in mind the passive sign convention: in the equation *v=iR*, the "*i*" refers to current flowing from the + to the – of v.

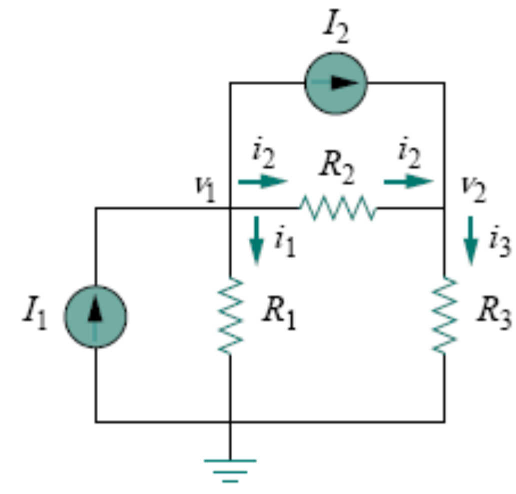


- This gives us

$$i_1 = \frac{v_1 - 0}{R_1} \qquad \text{or} \qquad i_1 = G_1 v_1$$

$$i_2 = \frac{v_1 - v_2}{R_2} \qquad \text{or} \qquad i_2 = G_2(v_1 - v_2)$$

$$i_3 = \frac{v_2 - 0}{R_3} \qquad \text{or} \qquad i_3 = G_3 v_2$$



(b)

- We substitute these into our KCL equations:

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2} \qquad\qquad I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3}$$

... or in terms of conductances:

$$I_1 = I_2 + G_1 v_1 + G_2(v_1 - v_2)$$
$$I_2 + G_2(v_1 - v_2) = G_3 v_2$$

Solve for Node Voltages

- The third step in nodal analysis is to solve for the node voltages. If we apply KCL to *n−1* nonreference nodes, we obtain n−1 simultaneous equations. For the current example circuit we could use any standard method, such as substitution, elimination, Cramer's rule, or matrix inversion.
- To use either of the last two methods, one must cast the simultaneous equations in matrix form. For example, our conductance equations can be cast in matrix form as

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

which can be solved to get v_1 *and* v_2.

Example

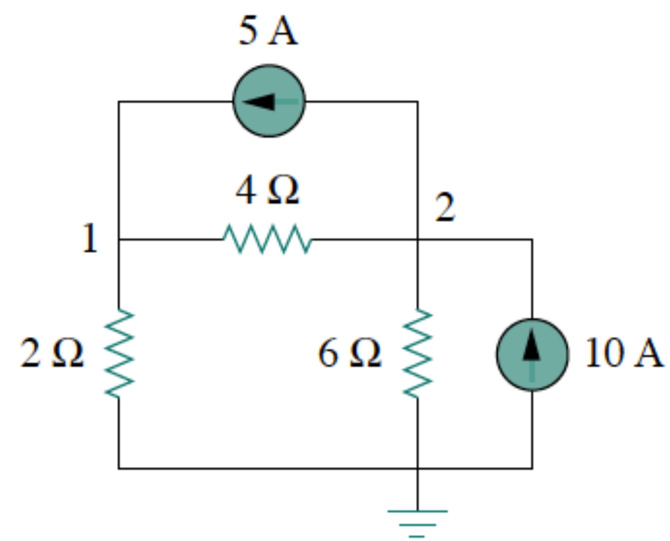
Calculate the node voltages in this circuit.

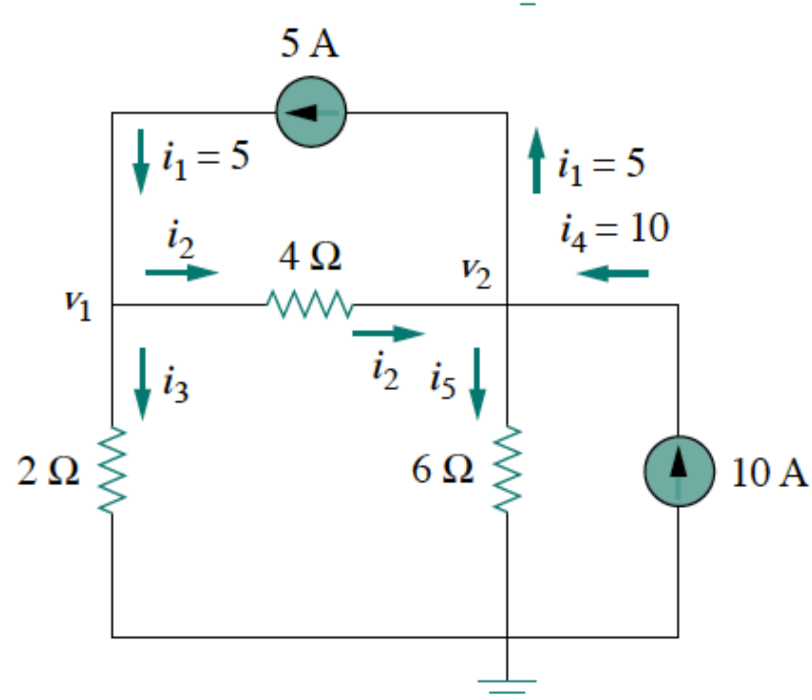
Solution:

First prepare the circuit for nodal analysis, labeling the node voltages and currents as shown below.

Notice how the currents are selected for the application of KCL - Except for the branches with current sources, the labeling is arbitrary but consistent, i.e. the currents entering a resistor are in the same direction when they go out the other end.

The reference node is selected, and the node voltages v_1 and v_2 are now to be determined.



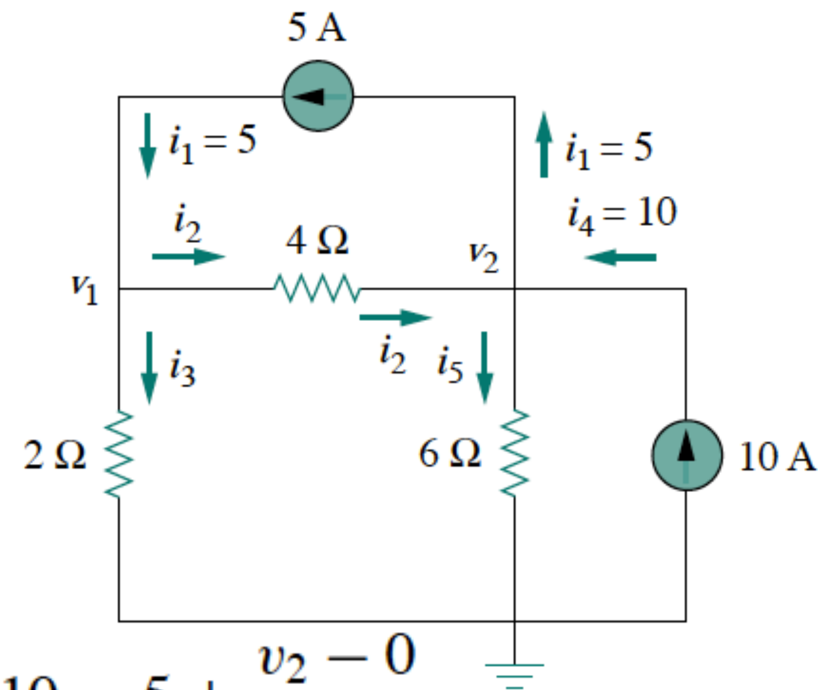


- At node 1, applying KCL and Ohm's law gives

$$i_1 = i_2 + i_3 \Rightarrow 5 = \frac{v_1 - v_2}{4} + \frac{v_1 - 0}{2}$$

- Multiplying each term in the last equation by 4, we obtain

$$20 = v_1 - v_2 + 2v_1 \quad \text{or} \quad \underline{3v_1 - v_2 = 20}$$



At node 2, we do the same thing and get

$$i_2 + i_4 = i_1 + i_5 \qquad \Longrightarrow \qquad \frac{v_1 - v_2}{4} + 10 = 5 + \frac{v_2 - 0}{6}$$

Multiplying each term by 12 results in

$$3v_1 - 3v_2 + 120 = 60 + 2v_2$$

$$\underline{-3v_1 + 5v_2 = 60}$$

We could use elimination: add underlined equations & substitute

$$4v_2 = 80 \qquad \Longrightarrow \qquad v_2 = 20 \text{ V}$$

$$3v_1 - 20 = 20 \qquad \Longrightarrow \qquad v_1 = \frac{40}{3} = 13.33 \text{ V}$$

- Alternatively, we could use Cramer's rule:

$$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix}$$

The determinant of the matrix is

$$\Delta = \begin{vmatrix} 3 & -1 \\ -3 & 5 \end{vmatrix} = 15 - 3 = 12$$

We now obtain v_1 and v_2 as

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} 20 & -1 \\ 60 & 5 \end{vmatrix}}{\Delta} = \frac{100 + 60}{12} = 13.33 \text{ V}$$

$$v_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 3 & 20 \\ -3 & 60 \end{vmatrix}}{\Delta} = \frac{180 + 60}{12} = 20 \text{ V}$$

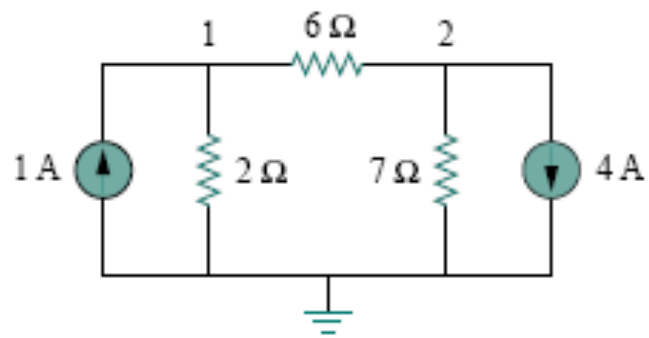
which is as we got before.

- We can then get the currents easily if required.

Examples

Obtain the node voltages

1)



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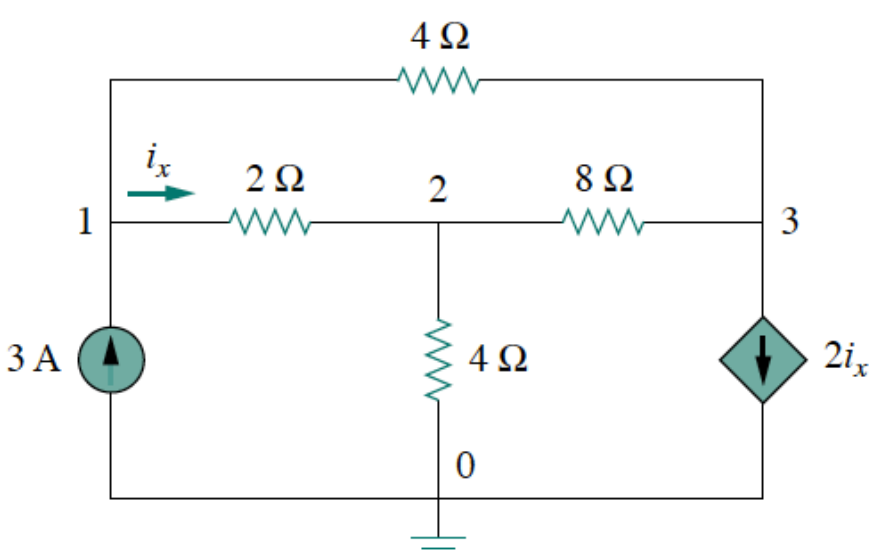

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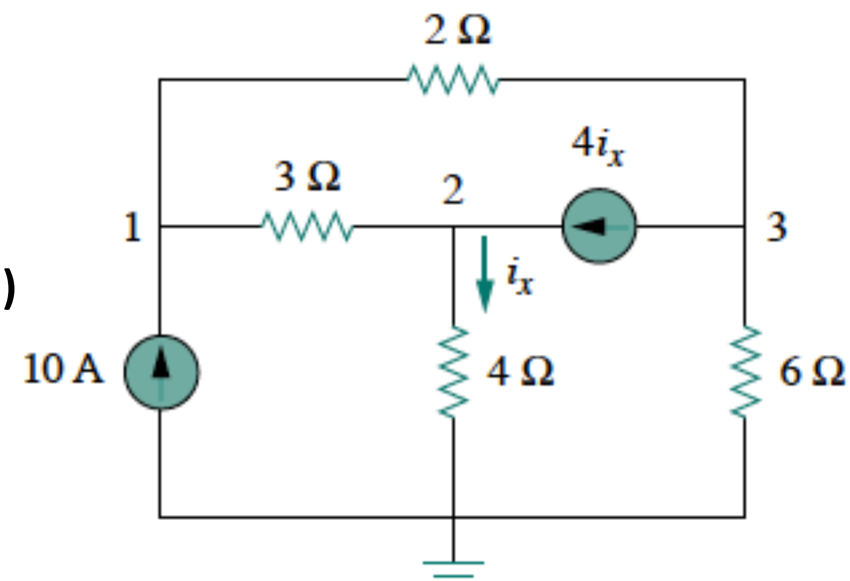
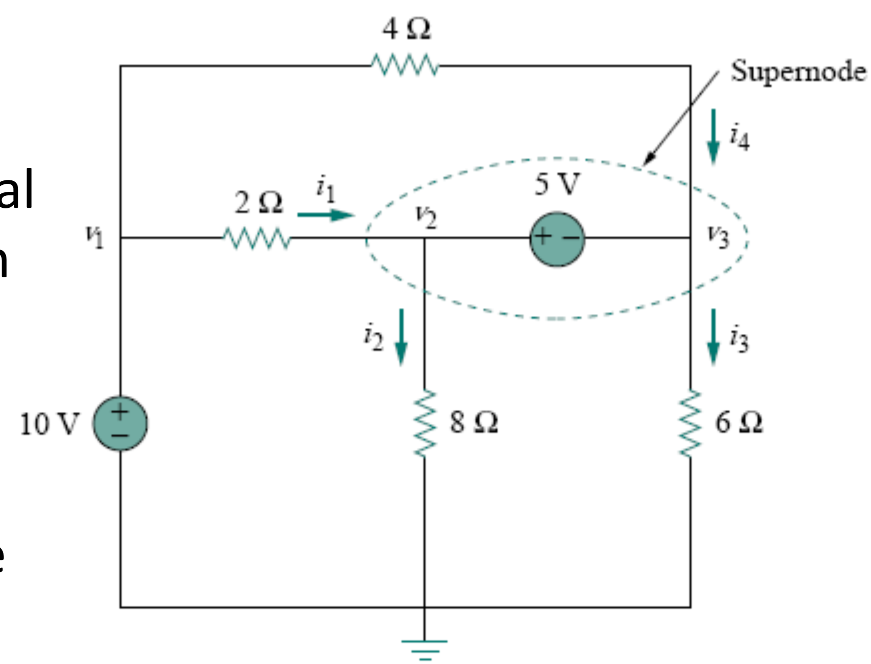


Nodal Analysis with Voltage Sources

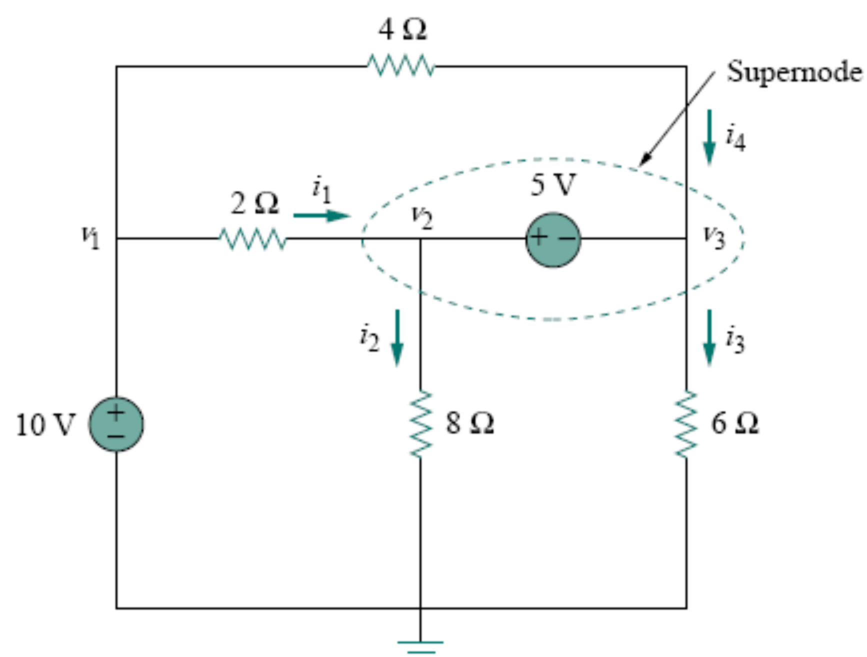
- CASE I. If a voltage source is connected between the reference node and a nonreference node, we simply set the voltage at the nonreference node equal to the voltage of the voltage source. In this example, v_1 = 10 V.
- CASE II. If the voltage source (dependent or independent) is connected between two nonreference nodes, the two nonreference nodes form a *generalized node* or *supernode*; we apply both KCL and KVL to determine the node voltages.



A supernode is formed by enclosing a(dependent or independent) voltage source connected between two nonreference nodes and any elements connected in parallel with it.

- So in this example, nodes 2 and 3 form a supernode. (We could have more than two). We analyze a circuit with supernodes using the same three steps mentioned in the previous section except that the supernodes are treated differently.
- Why? Because an essential component of nodal analysis is applying KCL, which requires knowing the current through each element. There is no way of knowing the current through a voltage source in advance. However, KCL must be satisfied at a supernode like any other node. Hence, at the

supernode in this example,

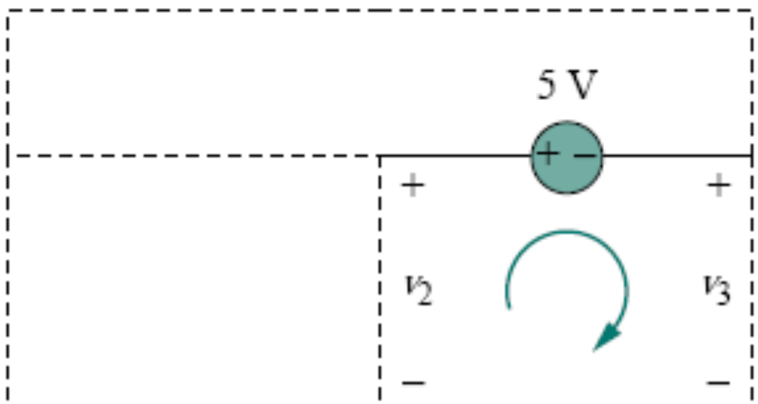


$$i_1 + i_4 = i_2 + i_3$$

or

$$\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_2 - 0}{8} + \frac{v_3 - 0}{6}$$

- To apply Kirchhoff's voltage law to the supernode, we redraw the circuit as shown. Going around the loop in the clockwise direction gives $-v_2 + 5 + v_3 = 0 \implies v_2 - v_3 = 5$
- So now we have 3 equations in v_1, v_2, v_3.



- Note the following properties of a supernode:

1. The voltage source inside the supernode provides a constraint equation needed to solve for the node voltages.
2. A supernode has no voltage of its own.
3. A supernode requires the application of both KCL and KVL.

Example

For the circuit shown in Fig. 3.9, find the node voltages.

Solution:

The supernode contains the 2-V source, nodes 1 and 2, and the 10-Ω resistor. Applying KCL to the supernode as shown in Fig. 3.10(a) gives

$$2 = i_1 + i_2 + 7$$

Expressing i_1 and i_2 in terms of the node voltages

$$2 = \frac{v_1 - 0}{2} + \frac{v_2 - 0}{4} + 7 \quad \Longrightarrow \quad 8 = 2v_1 + v_2 + 28$$

or

$$v_2 = -20 - 2v_1 \tag{3.3.1}$$

To get the relationship between v_1 and v_2, we apply KVL to the circuit in Fig. 3.10(b). Going around the loop, we obtain

$$-v_1 - 2 + v_2 = 0 \quad \Longrightarrow \quad v_2 = v_1 + 2 \tag{3.3.2}$$

From Eqs. (3.3.1) and (3.3.2), we write

$$v_2 = v_1 + 2 = -20 - 2v_1$$

or

$$3v_1 = -22 \quad \Longrightarrow \quad v_1 = -7.333 \text{ V}$$

and $v_2 = v_1 + 2 = -5.333$ V. Note that the 10-Ω resistor does not make any difference because it is connected across the supernode.

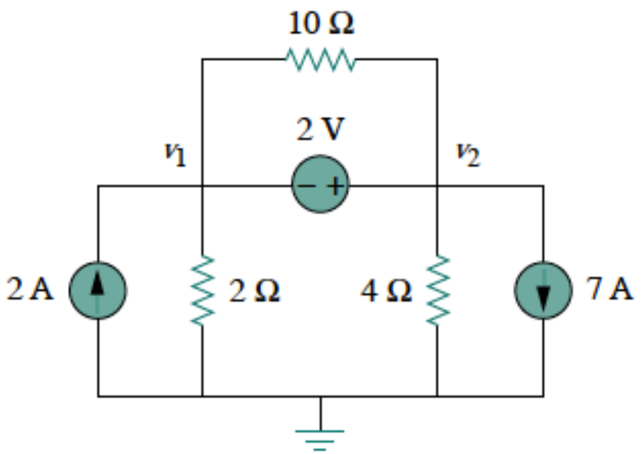


Figure 3.9 For Example 3.3.

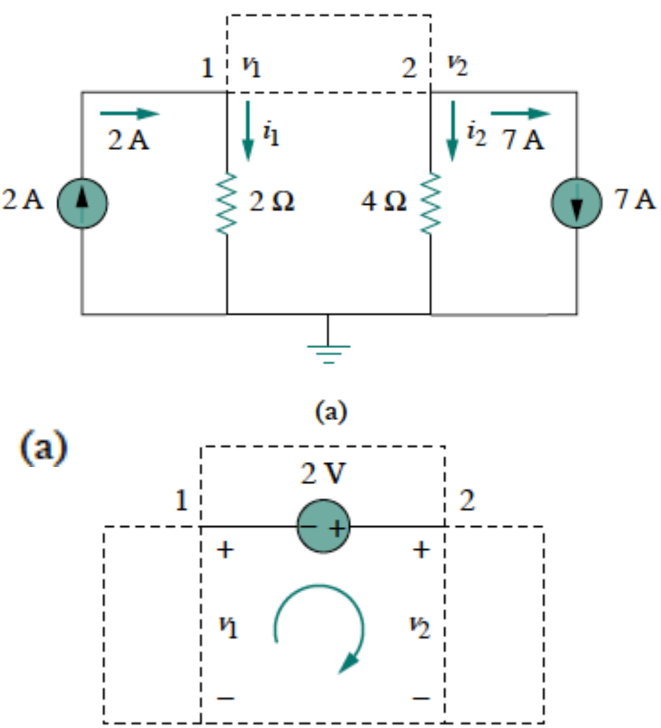
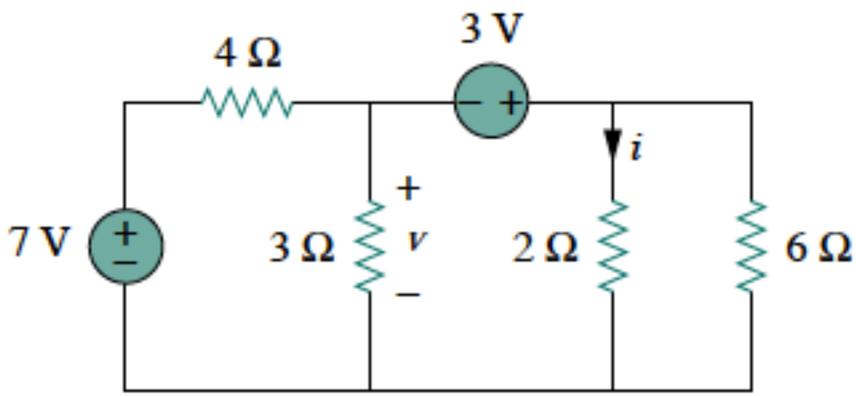


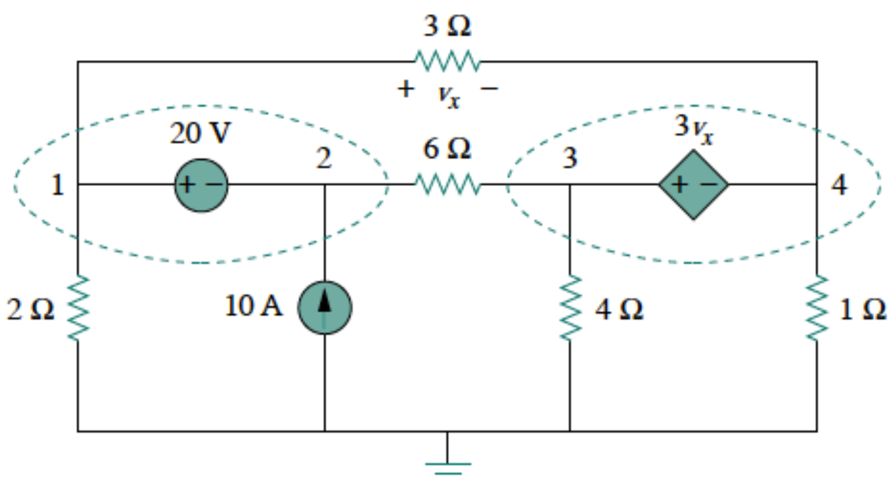
Figure 3.10

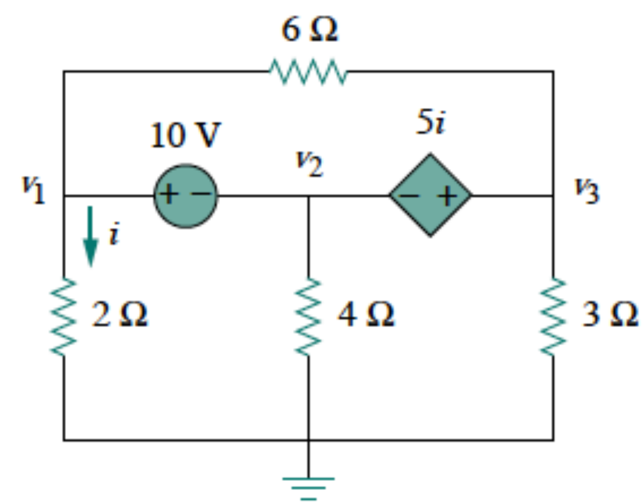
Examples

Find *v and i*



Find node voltages

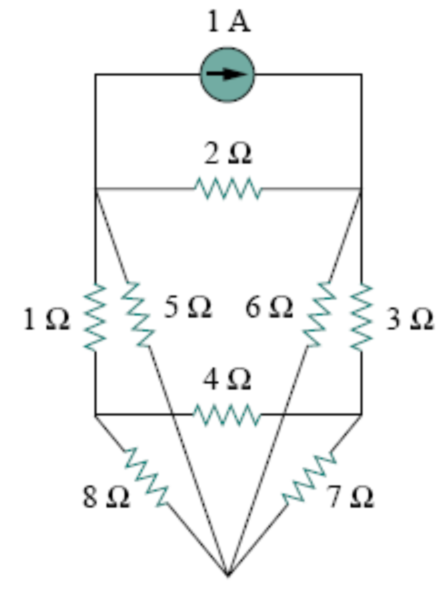




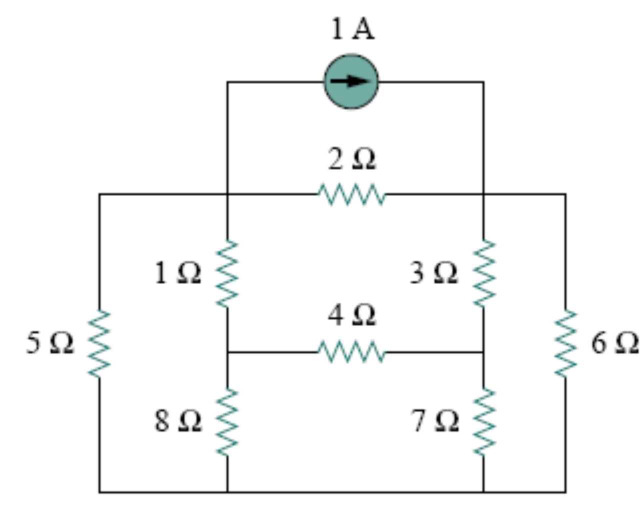
Mesh Analysis

- Mesh analysis provides another general procedure for analyzing circuits, using mesh currents as the circuit variables.
- Using mesh currents instead of element currents as circuit variables is convenient and reduces the number of equations that must be solved simultaneously.
- Recall that a loop is a closed path with no node passed more than once. A **mesh** is a loop that does not contain any other loop within it.
- Nodal analysis applies KCL to find unknown voltages in a given circuit, while mesh analysis applies KVL to find unknown currents. Mesh analysis is not quite as general as nodal analysis because it is only applicable to a circuit that is *planar.*
- *A **planar circuit** is one that can be* drawn in a plane with no branches crossing one another; otherwise it is *nonplanar.* A circuit may have crossing branches and still be planar if it can be redrawn such that it has no crossing branches.

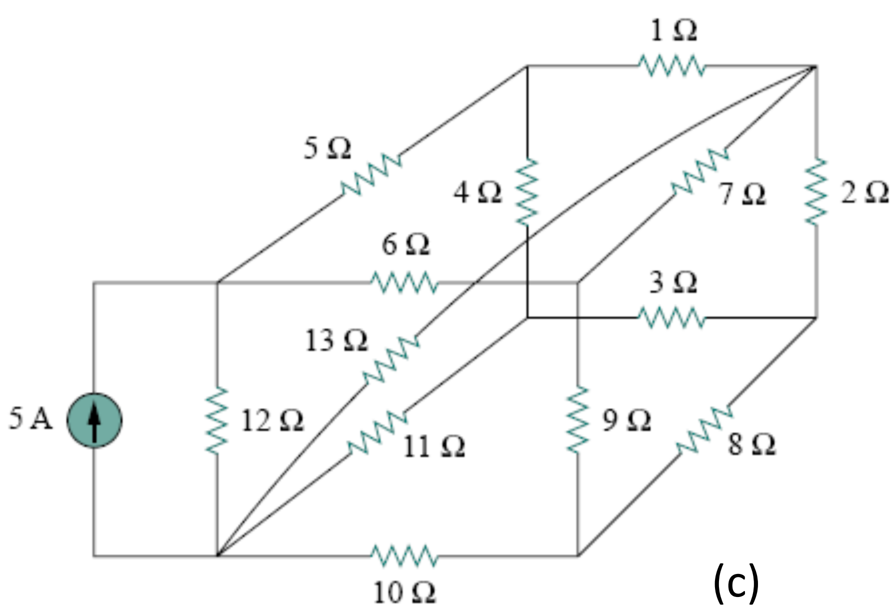
- For Example, (a) has two crossing branches, but it can be redrawn as in (b), so it is planar.
- However, circuit (c) is nonplanar, because there is no way to redraw it and avoid the branches crossing. Nonplanar circuits can be handled using nodal analysis, but they will not be considered in this course.



(a)



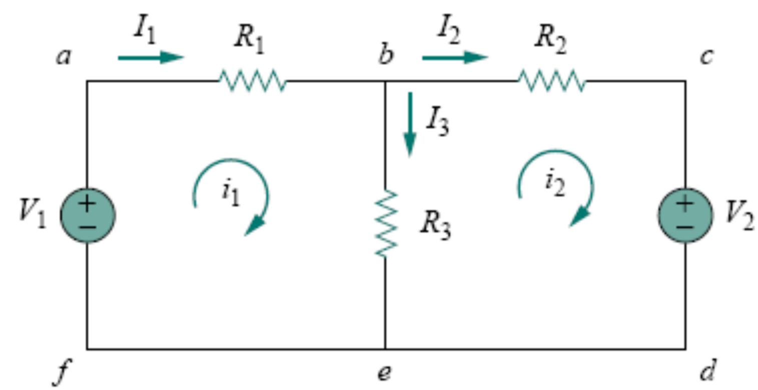
(b)



(c)

A mesh is a loop which does not contain any other loops within it.

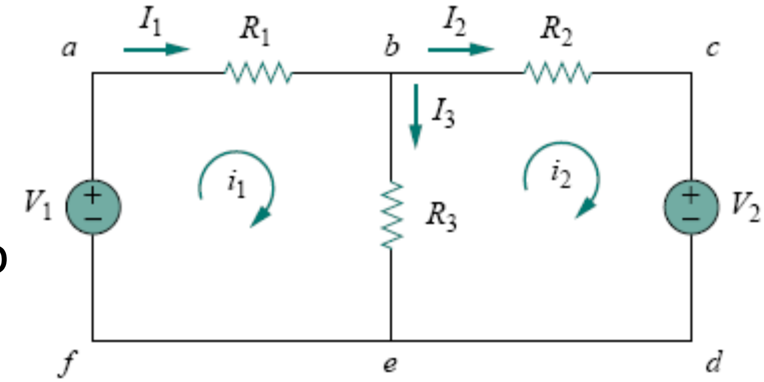
- For example, paths *abefa* and *bcdeb* are meshes, but path *abcdefa* is not a mesh. The current through a mesh is known as mesh current.
- In mesh analysis, we are interested in applying KVL to find the mesh currents in a given circuit.



In the mesh analysis of a circuit with *n* meshes and no current sources, we take the following three steps:

1. Assign mesh currents $i_1, i_2, \ldots, i_n$ to the *n* meshes.
2. Apply KVL to each of the *n* meshes. Use Ohm's law to express the voltages in terms of the mesh currents.
3. Solve the resulting *n* simultaneous equations to get the mesh currents.

- For this example, the first step requires that mesh currents i_1 and i_2 are assigned to meshes 1 and 2. Although a mesh current may be assigned to each mesh in an arbitrary direction, it is conventional to assume that each mesh current flows clockwise.



As the second step, we apply KVL to each mesh. Applying KVL to mesh 1, we obtain

$$-V_1 + R_1 i_1 + R_3(i_1 - i_2) = 0 \quad \text{or} \quad (R_1 + R_3)i_1 - R_3 i_2 = V_1$$

For mesh 2, applying KVL gives

$$R_2 i_2 + V_2 + R_3(i_2 - i_1) = 0 \quad \text{or} \quad -R_3 i_1 + (R_2 + R_3)i_2 = -V_2$$

Note that in the 1st equation the coefficient of i_1 is the sum of the resistances in the first mesh, while the coefficient of i_2 is the negative of the resistance common to meshes 1 and 2. The same is true for the 2nd equation. The third step is to solve for the mesh currents. In matrix form:

$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}$$

- If a circuit has n nodes, b branches, and l independent loops or meshes, then $l = b - n + 1$. Hence, l independent simultaneous equations are required to solve the circuit using mesh analysis.
- Notice that the branch currents are different from the mesh currents unless the mesh is isolated. To distinguish between the two types of currents, we use i for a mesh current and I for a branch current. The current elements I_1, I_2, and I_3 are algebraic sums of the mesh currents.
- In the last example it is evident that

$$I_1 = i_1, \qquad I_2 = i_2, \qquad I_3 = i_1 - i_2$$

Example

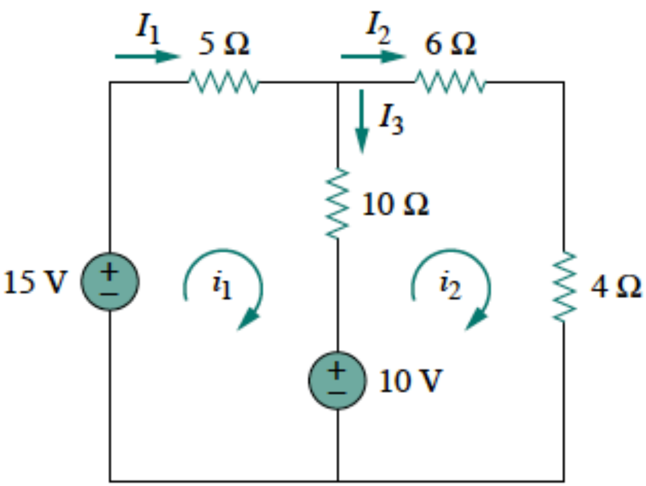


Figure 3.18 For Example 3.5.

For the circuit in Fig. 3.18, find the branch currents I_1, I_2, and I_3 using mesh analysis.

Solution:

We first obtain the mesh currents using KVL. For mesh 1,

$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0$$

or

$$3i_1 - 2i_2 = 1 \tag{3.5.1}$$

For mesh 2,

$$6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0$$

or

$$-i_1 + 2i_2 = 1$$

We can use Cramer's Rule: $\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

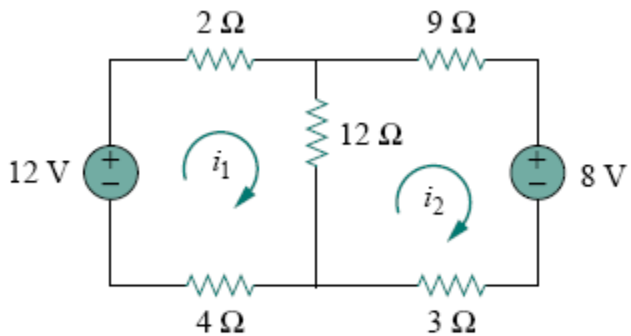
We obtain the determinants $\Delta = \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} = 6 - 2 = 4$

$$\Delta_1 = \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} = 2 + 2 = 4, \qquad \Delta_2 = \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} = 3 + 1 = 4$$

$$\Longrightarrow \quad i_1 = \frac{\Delta_1}{\Delta} = 1 \text{ A}, \qquad i_2 = \frac{\Delta_2}{\Delta} = 1 \text{ A}$$

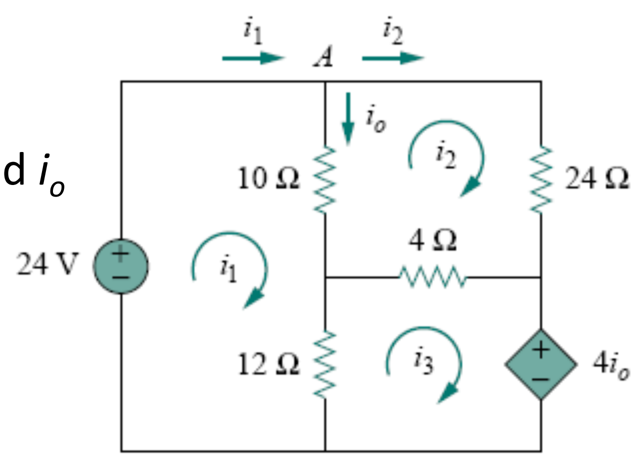
More examples

1)



Find i_o

2)



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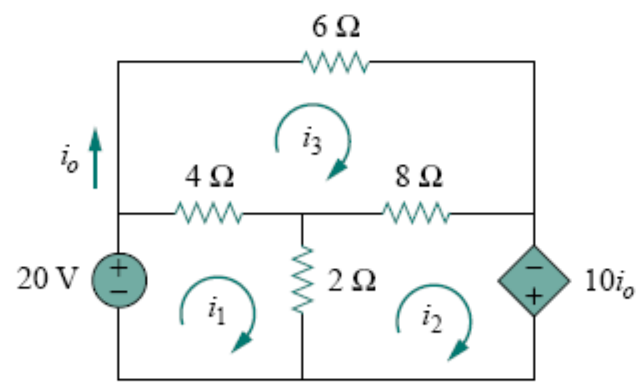
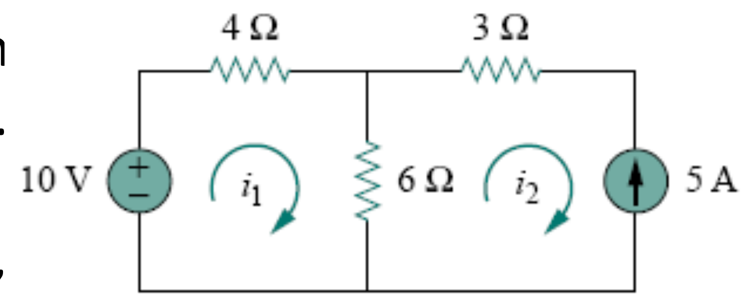


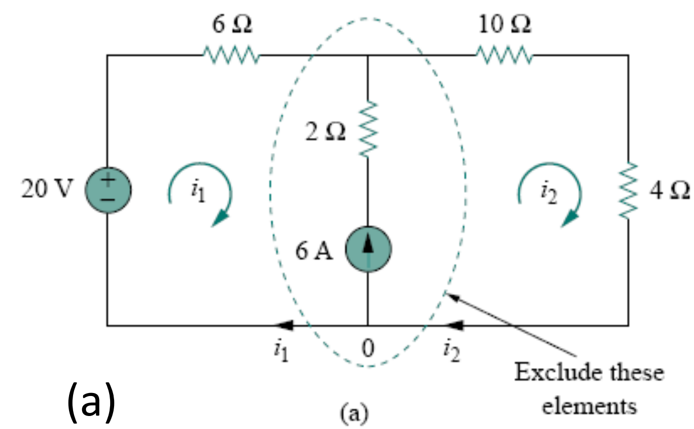
Mesh Analysis with Current Sources

- CASE I. When a current source exists only in one mesh. Consider top circuit for example. We set $i_2 = -5\ A$ and write a mesh equation for the other mesh in the usual way, that is,

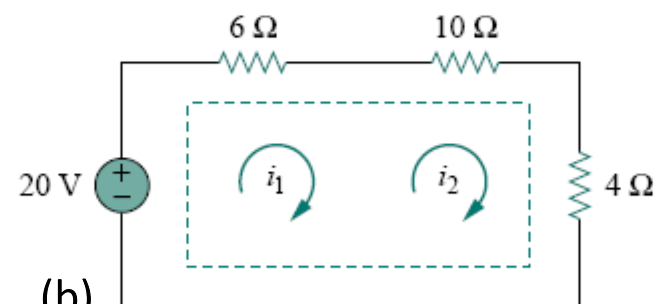


$$-10 + 4i_1 + 6(i_1 - i_2) = 0 \qquad \Longrightarrow \qquad i_1 = -2\ \text{A}$$

- CASE II. When a current source exists between two meshes. Consider circuit (a), for example. We create a *supermesh* by excluding the current source and any elements connected in series with it, as shown in (b). Thus,



(a)



(b)

A supermesh results when two meshes have a (dependent or independent) current source in common.

- We create a supermesh as the periphery of the two meshes and treat it differently. (If a circuit has two or more supermeshes that intersect, they should be combined to form a larger supermesh.)
- Why do we need supermeshes that "go over the head" of a current source? Because mesh analysis applies KVL—which requires that we know the voltage across each branch—and we do not know the voltage across a current source in advance.
- However, a supermesh must satisfy KVL like any other mesh or loop. Therefore, applying KVL to the supermesh in (b) gives

$$-20 + 6i_1 + 10i_2 + 4i_2 = 0$$

or

$$6i_1 + 14i_2 = 20$$

- We apply KCL to a node in the branch where the two meshes intersect. Applying KCL to node 0 gives

$$i_2 = i_1 + 6$$

- Solving the two equations in i_1 and i_2 we get

$$i_1 = -3.2 \text{ A}, \qquad i_2 = 2.8 \text{ A}$$

- Note the following properties of a supermesh:

1. The current source in the supermesh is not completely ignored; it provides the constraint equation necessary to solve for the mesh currents.
2. A supermesh has no current of its own.
3. A supermesh requires the application of both KVL and KCL.

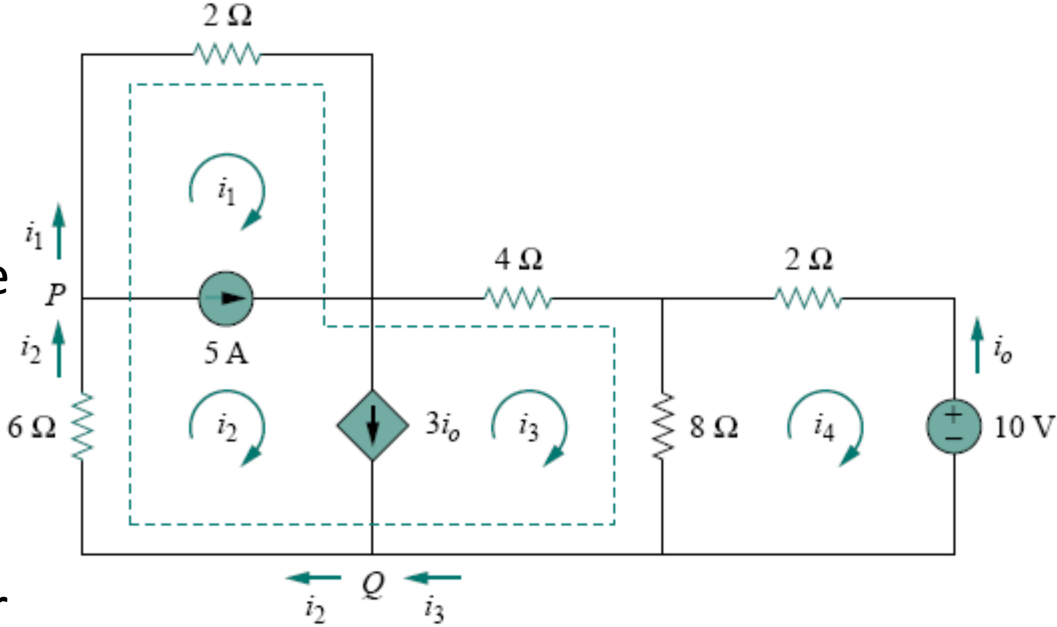
Example

Find i_1 to i_4 using mesh analysis.

Solution:

Note that meshes 1 and 2 form a supermesh since they have an independent current source in common. Also, meshes 2 and 3 form another supermesh because they have a dependent current source in common.

The two supermeshes intersect and form a larger supermesh as shown. Applying KVL to the larger supermesh,



$$2i_1 + 4i_3 + 8(i_3 - i_4) + 6i_2 = 0$$

or

$$i_1 + 3i_2 + 6i_3 - 4i_4 = 0$$

For the independent current source, we apply KCL to node P:

$$i_2 = i_1 + 5$$

For the dependent current source, we apply KCL to node Q:

$$i_2 = i_3 + 3i_o$$

But $i_o = -i_4$, hence,

$$i_2 = i_3 - 3i_4$$

Applying KVL in mesh 4,

$$2i_4 + 8(i_4 - i_3) + 10 = 0$$

or

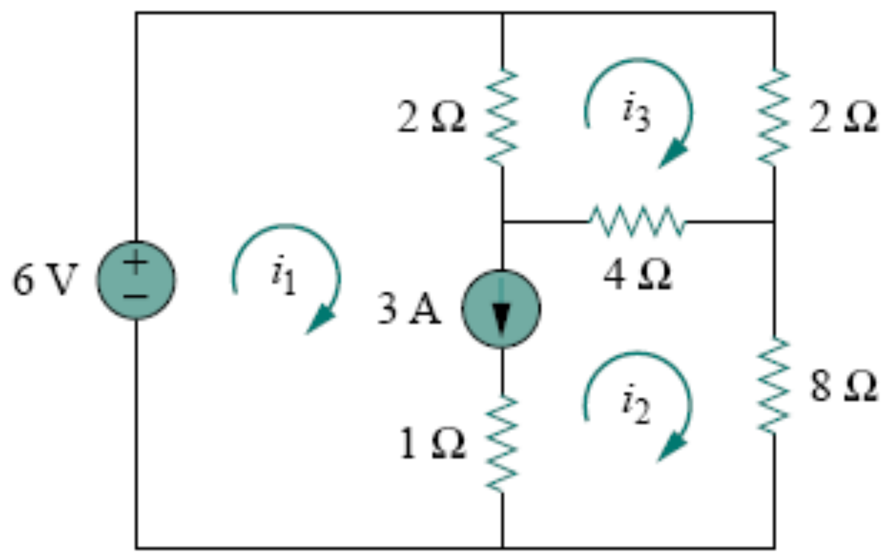
$$5i_4 - 4i_3 = -5$$

Solve to get:

$i_1 = -7.5$ A, $\quad i_2 = -2.5$ A, $\quad i_3 = 3.93$ A, $\quad i_4 = 2.143$ A

Practice Problem

- Use mesh analysis to determine i_1, i_2, and i_3

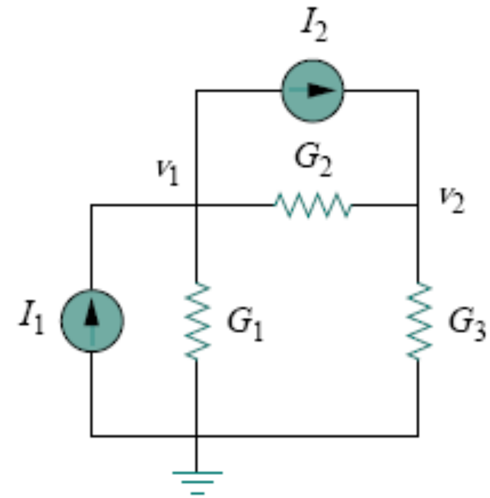


Nodal and Mesh Analysis by Inspection

- When all sources in a circuit are independent current sources, we do not need to apply KCL to each node to obtain the node-voltage equations as we did in last section. We can obtain the equations by mere inspection of the circuit. As an example, let us reexamine the circuit shown. The circuit has two nonreference nodes and the node equations were derived before as

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

Observe that each of the diagonal terms is the sum of the conductances connected directly to node 1 or 2, while the off-diagonal terms are the negatives of the conductances connected between the nodes.



(a)

- Also, each term on the right-hand side is the algebraic sum of the current sources entering the node.
- In general, if a circuit with independent current sources has N nonreference nodes, the node-voltage equations can be written in terms of the conductances as

$$\begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1N} \\ G_{21} & G_{22} & \cdots & G_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ G_{N1} & G_{N2} & \cdots & G_{NN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix}$$

or simply

$$\mathbf{Gv} = \mathbf{i}$$

where G_{kk} = sum of the conductances connected to node k

$G_{kj} = G_{jk}$ = Negative of the sum of the conductances directly connecting nodes k and j, $k \neq j$

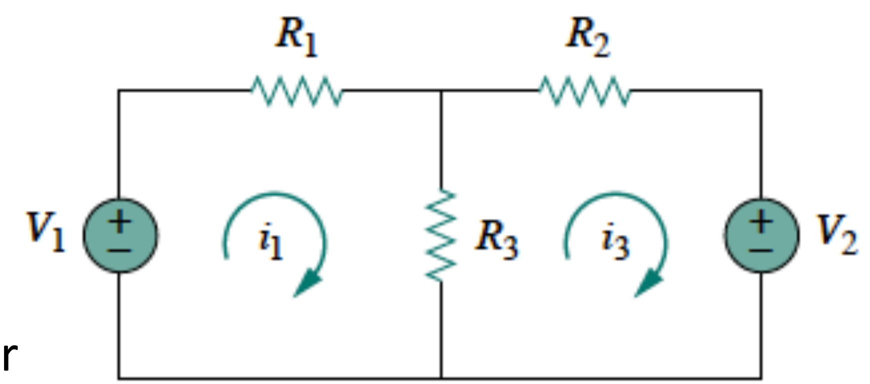
v_k = Unknown voltage at node *k;*

i_k = Sum of all independent current sources directly connected to node *k*, with currents entering the node treated as positive.

G is called the *conductance matrix*, **v** is the output vector; and **i** is the input vector.

- This equation can be solved to obtain the unknown node voltages. Keep in mind that this is valid for circuits with only independent current sources and linear resistors.
- Similarly, we can obtain mesh-current equations by inspection when a linear resistive circuit has only independent voltage sources.
- Consider again the circuit shown. The circuit has two nonreference nodes and the node equations were derived before as

$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}$$



- We notice that each of the diagonal ter the related mesh, while each of the off-diagonal terms is the negative of the resistance common to meshes 1 and 2. Each term on the right-hand side is the algebraic sum taken clockwise of all independent voltage sources in the related mesh.

- In general, if the circuit has N meshes, the mesh-current equations can be expressed in terms of the resistances as

$$\begin{bmatrix} R_{11} & R_{12} & \dots & R_{1N} \\ R_{21} & R_{22} & \dots & R_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ R_{N1} & R_{N2} & \dots & R_{NN} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}$$

or simply

$$\mathbf{Ri} = \mathbf{v}$$

where R_{kk} = Sum of the resistances in mesh k

$R_{kj} = R_{jk}$ = Negative of the sum of the resistances in common with meshes k and j, $k \neq j$

i_k = Unknown mesh current for mesh k in the clockwise direction

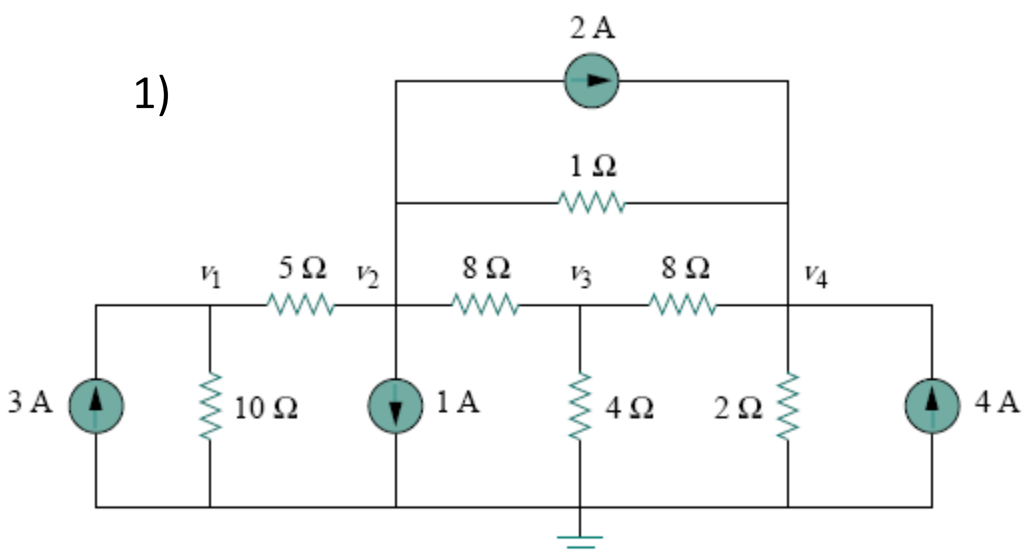
v_k = Sum taken clockwise of all independent voltage sources in mesh k, with voltage rise treated as positive

R is called the *resistance matrix*, **i** is the output vector; and **v** is the input vector. We can solve this matrix equation to obtain the unknown mesh currents.

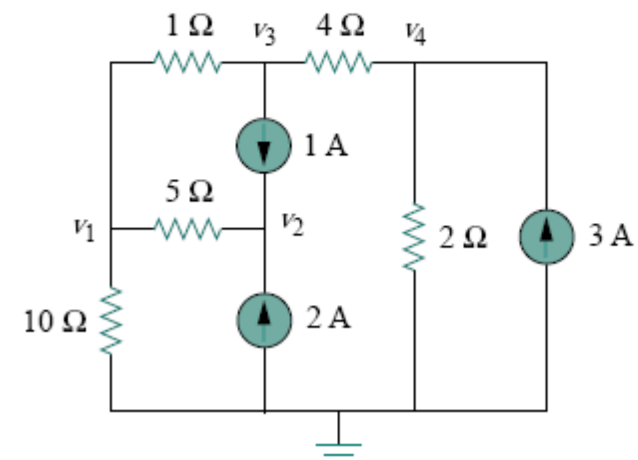
Examples

Write the node-voltage matrix equations by inspection.

1)



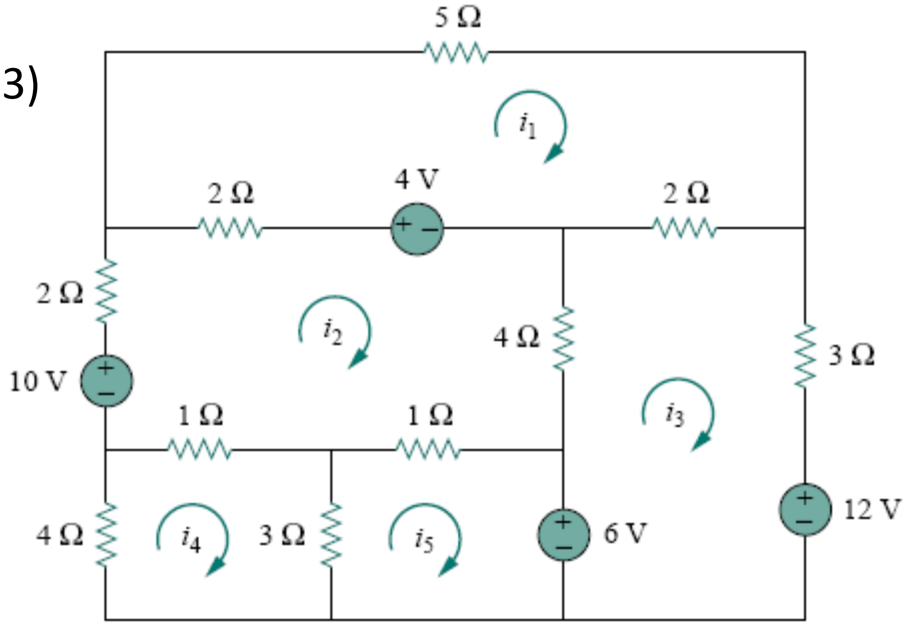
2)



Example

write the mesh-current equations by inspection

3)



4)