### TOPIC 3: MECHANICAL SYSTEMS

#### 3-1 1-DOF MECHANICAL SYSTEMS (1st-ORDER, INERTIA-DAMPER)

<table>
<thead>
<tr>
<th></th>
<th>Translational System</th>
<th>Rotational System</th>
</tr>
</thead>
<tbody>
<tr>
<td>inertia-damper system</td>
<td><img src="translational_system.png" alt="Free Body Diagram" /></td>
<td><img src="rotational_system.png" alt="Free Body Diagram" /></td>
</tr>
<tr>
<td>free-body diagram</td>
<td><img src="translational_system.png" alt="Free Body Diagram" /></td>
<td><img src="rotational_system.png" alt="Free Body Diagram" /></td>
</tr>
<tr>
<td><strong>Newton’s 2nd law</strong></td>
<td>( ma = \sum f \rightarrow m\ddot{v}(t) = f(t) - f_b(t) )</td>
<td>( J\alpha = \sum T \rightarrow J\ddot{\omega}(t) = T(t) - T_b(t) )</td>
</tr>
<tr>
<td><strong>damping force</strong></td>
<td>( f_b(t) = bv(t) )</td>
<td>( T_b(t) = b_r\omega(t) )</td>
</tr>
<tr>
<td><strong>spring force</strong></td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td><strong>differential equation</strong></td>
<td>( m\ddot{v}(t) + bv(t) = f(t) )</td>
<td>( J\ddot{\omega}(t) + b_r\omega(t) = T(t) )</td>
</tr>
<tr>
<td><strong>initial condition</strong></td>
<td>( v(t)</td>
<td>_{t=0} = v(0) = v_0 )</td>
</tr>
<tr>
<td><strong>time constant</strong></td>
<td>( \tau = \frac{m}{b} )</td>
<td>( \tau = \frac{J}{b_r} )</td>
</tr>
</tbody>
</table>
### 3-1 1-DOF Mechanical Systems (1st-Order, Inertia-Damper) (continued)

<table>
<thead>
<tr>
<th>Laplace transform</th>
<th>Translational System</th>
<th>Rotational System</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m[sV(s) - v_0] + bV(s) = F(s)$</td>
<td>$J[s\Omega(s) - \omega_0] + b,\Omega(s) = T(s)$</td>
<td></td>
</tr>
<tr>
<td>$V(s) = \frac{mv_0 + F(s)}{ms + b} = \frac{r\nu_0}{s^2 + \tau s + 1} + \frac{F(s)}{\tau s + 1}$</td>
<td>$\Omega(s) = \frac{J\omega_0 + T(s)}{Js + b_r} = \frac{r\omega_0}{s^2 + \tau s + 1} + \frac{T(s)}{s^2 + \tau s + 1}$</td>
<td></td>
</tr>
</tbody>
</table>

#### characteristic equation

$ms + b = rs + 1 = 0$  
$Js + b_r = rs + 1 = 0$

#### characteristic root

$s = -\frac{b}{m} = -\frac{1}{\tau}$  
$s = -\frac{b_r}{J} = -\frac{1}{\tau}$

#### free (or natural or initial) response

$v_{free}(t) = v_0 e^{-bt/m} = v_0 e^{-t/\tau}$  
$\omega_{free}(t) = \omega_0 e^{-bt/j} = \omega_0 e^{-t/\tau}$

#### energy balance

(or thermodynamics 1st law)  
(or Lagrangian dynamics)

$\Delta E = W_{input} + Q_{input} \rightarrow \Delta(K + U) = \int P_{input} dt - \int D_{output} dt \rightarrow \dot{K} + \dot{U} = P_{input} - D_{output}$

where  
$\Delta E = \text{change in internal energy}$  
$\Delta K = \text{change in kinetic energy}$  
$\Delta U = \text{change in potential energy}$  
$W_{input} = \text{work input}$  
$Q_{input} = \text{heat input}$  
$P_{input} = \text{power input}$  
$D_{output} = \text{dissipated power}$

#### kinetic energy

$K = \frac{1}{2} m[v(t)]^2 \rightarrow$  
$\dot{K} = \frac{dK}{dt} = \frac{dK}{dv} \dot{v} = m\dot{v}(t)\dot{v}(t)$  
$K = \frac{1}{2} J[\omega(t)]^2 \rightarrow$  
$\dot{K} = \frac{dK}{dt} = \frac{dK}{d\omega} d\omega = \frac{1}{2} J\dot{\omega}(t)\dot{\omega}(t)$

#### potential energy

N/A  
N/A

#### power input

$P_{input} = \frac{dW_{input}}{dt} = f(t)v(t)$  
$P_{input} = \frac{dW_{input}}{dt} = T(t)\omega(t)$

#### dissipated power

$D_{output} = f_b v(t) = b[v(t)]^2$  
$D_{output} = T_b \omega(t) = b_r[\omega(t)]^2$

#### governing equation

$mv(t)\dot{v}(t) = f(t)v(t) - b[v(t)]^2$  
$\rightarrow m\ddot{v}(t) + bv(t) = f(t)$  
$J\omega(t)\dot{\omega}(t) = T(t)\omega(t) - b_r[\omega(t)]^2$  
$\rightarrow J\ddot{\omega}(t) + b_\omega(t) = T(t)$
### 3-2 1-DOF MECHANICAL SYSTEMS (2\textsuperscript{ND}-ORDER, SPRING-INERTIA-DAMPER)

<table>
<thead>
<tr>
<th>spring-inertia-damper system</th>
<th>Translational System</th>
<th>Rotational System</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="https://example.com/freebody_diagram.png" alt="Free-body diagram" /></td>
<td><img src="https://example.com/translational_system.png" alt="Translational system" /></td>
<td><img src="https://example.com/rotational_system.png" alt="Rotational system" /></td>
</tr>
</tbody>
</table>

| Newton’s 2\textsuperscript{nd} law | $ma = \sum f \rightarrow m\ddot{x}(t) = f(t) - f_{\text{b}}(t) - f_{\text{k}}(t)$ | $J\alpha = \sum T \rightarrow J\ddot{\theta}(t) = T(t) - T_{\text{b}}(t) - T_{\text{k}}(t)$ |
| damped force (or torque) | $f_{\text{b}}(t) = b\dot{x}(t) = b\ddot{x}(t)$ | $T_{\text{b}}(t) = b_\text{b}\omega(t) = b_\text{b}\dot{\theta}(t)$ |
| Spring force (or torque) | $f_{\text{k}}(t) = kx(t)$ | $T_{\text{k}}(t) = k_\text{r}\theta(t)$ |
| Differential equation | $m\dddot{x}(t) + b\ddot{x}(t) + kx(t) = f(t)$ | $J\dddot{\theta}(t) + b_\theta\dot{\theta}(t) + k_\theta\theta(t) = T(t)$ |
| Initial conditions | $x(t)|_{t=0} = x(0) = x_0$ & $\dot{x}(t)|_{t=0} = \dot{x}(0) = v_0$ | $\theta(t)|_{t=0} = \theta(0) = \theta_0$ & $\dot{\theta}(t)|_{t=0} = \dot{\theta}(0) = \omega_0$ |
## 3-2 1-DOF MECHANICAL SYSTEMS (2\textsuperscript{nd}-ORDER, SPRING-INERTIA-DAMPER) (continued)

<table>
<thead>
<tr>
<th></th>
<th>Translational System</th>
<th>Rotational System</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>(undamped) natural frequency</strong></td>
<td>$\omega_n = \sqrt{\frac{k}{m}}$</td>
<td>$\omega_n = \sqrt{\frac{k_r}{J}}$</td>
</tr>
<tr>
<td><strong>damping ratio</strong></td>
<td>$\zeta = \frac{b}{2\sqrt{mk}}$</td>
<td>$\zeta = \frac{b_r}{2\sqrt{Jk_r}}$</td>
</tr>
<tr>
<td><strong>damped natural frequency</strong></td>
<td>$\omega_d = \omega_n\sqrt{1-\zeta^2} = \frac{\sqrt{4mk-b^2}}{2m}$</td>
<td>$\omega_d = \omega_n\sqrt{1-\zeta^2} = \frac{\sqrt{4Jk_r-b_r^2}}{2J}$</td>
</tr>
<tr>
<td><strong>Laplace transform</strong></td>
<td>$m[s^2X(s)-sx_0-x_0'] + b[sX(s)-x_0] + kX(s) = F(s)$</td>
<td>$J[s^2\Theta(s)-s\theta_0-\omega_0] + b_r[s\Theta(s)-\theta_0] + k_r\Theta(s) = T(s)$</td>
</tr>
<tr>
<td></td>
<td>$\rightarrow X(s) = \left(\frac{ms+b}{ms^2+bs+k}\right)x_0 + \frac{mv_0}{ms^2+bs+k} + \frac{F(s)}{ms^2+bs+k}$</td>
<td>$\rightarrow \Theta(s) = \frac{(Js+b_r)\theta_0 + J\omega_0 + T(s)}{Js^2+b_r s + k_r}$</td>
</tr>
<tr>
<td><strong>characteristic equation</strong></td>
<td>$ms^2+bs+k = m(s^2+2\zeta\omega_n s + \omega_n^2) = 0$</td>
<td>$Js^2 + b_r s + k_r = J(s^2+2\zeta\omega_n s + \omega_n^2) = 0$</td>
</tr>
<tr>
<td><strong>characteristic roots</strong></td>
<td>$s_{1,2} = \frac{-b \pm \sqrt{b^2-4mk}}{2m} = \omega_n \left(-\zeta \pm \sqrt{\zeta^2-1}\right)$</td>
<td>$s_{1,2} = \frac{-b_r \pm \sqrt{b_r^2-4Jk_r}}{2J} = \omega_n \left(-\zeta \pm \sqrt{\zeta^2-1}\right)$</td>
</tr>
<tr>
<td><strong>overdamped:</strong> (real &amp; distinct, (\zeta &gt; 1))</td>
<td>$b &gt; 2\sqrt{mk}$</td>
<td>$b_r &gt; 2\sqrt{Jk_r}$</td>
</tr>
<tr>
<td></td>
<td>$s_{1,2} = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}} = \omega_n \left(-\zeta \pm \sqrt{\zeta^2-1}\right)$</td>
<td>$s_{1,2} = -\frac{b_r}{2J} \pm \sqrt{\frac{b_r^2}{4J^2} - \frac{k_r}{J}} = \omega_n \left(-\zeta \pm \sqrt{\zeta^2-1}\right)$</td>
</tr>
<tr>
<td><strong>critically damped:</strong> (repeated real, (\zeta = 1))</td>
<td>$b = 2\sqrt{mk}$</td>
<td>$b_r = 2\sqrt{Jk_r}$</td>
</tr>
<tr>
<td></td>
<td>$s_1 = s_2 = -\frac{k}{m} = -\omega_n$</td>
<td>$s_1 = s_2 = -\frac{k_r}{J} = -\omega_n$</td>
</tr>
<tr>
<td><strong>underdamped:</strong> (complex conjugate, (0 \leq \zeta &lt; 1))</td>
<td>$b &lt; 2\sqrt{mk}$</td>
<td>$b_r &lt; 2\sqrt{Jk_r}$</td>
</tr>
<tr>
<td></td>
<td>$s_{1,2} = -\frac{b}{2m} \pm j\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = -\zeta \omega_n \pm j\omega_d$</td>
<td>$s_{1,2} = -\frac{b_r}{2J} \pm j\sqrt{\frac{k_r}{J} - \frac{b_r^2}{4J^2}} = -\zeta \omega_n \pm j\omega_d$</td>
</tr>
<tr>
<td><strong>undamped:</strong> (imaginary conjugate, (\zeta = 0))</td>
<td>$b = 0$</td>
<td>$b_r = 0$</td>
</tr>
<tr>
<td></td>
<td>$s_{1,2} = \pm j\sqrt{\frac{k}{m}} = \pm j\omega_n$</td>
<td>$s_{1,2} = \pm j\sqrt{\frac{k_r}{J}} = \pm j\omega_n$</td>
</tr>
</tbody>
</table>
### 3-2 1-DOF MECHANICAL SYSTEMS (2<sup>ND</sup>-ORDER, SPRING-INERTIA-DAMPER) (continued)

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<tr>
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<th>Translational System</th>
<th>Rotational System</th>
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</thead>
<tbody>
<tr>
<td>free (or natural or initial) response – overdamped</td>
<td>[ x(t) = \left[ \frac{(-\zeta + \sqrt{\zeta^2 - 1}) x_0}{2\sqrt{\zeta^2 - 1}} + \frac{v_0}{2\omega_n \sqrt{\zeta^2 - 1}} \right] e^{(-\zeta - \sqrt{\zeta^2 - 1}) \omega_n t} ]</td>
<td>[ \theta(t) = \left[ \frac{(-\zeta + \sqrt{\zeta^2 - 1}) \theta_0}{2\sqrt{\zeta^2 - 1}} - \frac{\omega_0}{2\omega_n \sqrt{\zeta^2 - 1}} \right] e^{(-\zeta - \sqrt{\zeta^2 - 1}) \omega_n t} ]</td>
</tr>
<tr>
<td>free (or natural or initial) response – critically damped</td>
<td>[ x(t) = [x_0 + (\omega_n x_0 + v_0) t] e^{-\omega_d t} ]</td>
<td>[ \theta(t) = [\theta_0 + (\omega_n \theta_0 + \omega_0) t] e^{-\omega_d t} ]</td>
</tr>
<tr>
<td>free (or natural or initial) response – underdamped</td>
<td>[ x(t) = e^{-\omega_d t} \left[ \left( \frac{\zeta x_0}{\sqrt{1 - \zeta^2}} + \frac{v_0}{\omega_d} \right) \sin \omega_d t + x_0 \cos \omega_d t \right] ]</td>
<td>[ \theta(t) = e^{-\omega_d t} \left[ \left( \frac{\zeta \theta_0}{\sqrt{1 - \zeta^2}} + \frac{\omega_0}{\omega_d} \right) \sin \omega_d t + \theta_0 \cos \omega_d t \right] ]</td>
</tr>
<tr>
<td>free (or natural or initial) response – undamped</td>
<td>[ x(t) = x_0 \cos \omega_d t + \frac{v_0}{\omega_n} \sin \omega_d t ]</td>
<td>[ \theta(t) = \theta_0 \cos \omega_d t + \frac{\omega_0}{\omega_n} \sin \omega_d t ]</td>
</tr>
</tbody>
</table>
### 3-2 1-DOF Mechanical Systems (2nd-Order, Spring-Inertia-Damper) (continued)

<table>
<thead>
<tr>
<th>Energy Balance (or thermodynamics 1st law) (or Lagrangian dynamics)</th>
<th>Translational System</th>
<th>Rotational System</th>
</tr>
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<tbody>
<tr>
<td>$\Delta E = W_{\text{input}} + Q_{\text{input}} \rightarrow \Delta (K + U) = \int P_{\text{input}} , dt - \int D_{\text{output}} , dt \rightarrow \dot{K} + \dot{U} = P_{\text{input}} - D_{\text{output}}$</td>
<td>$K = \frac{1}{2} m \left[ v(t) \right]^2 = \frac{1}{2} m \left[ \dot{x}(t) \right]^2$</td>
<td>$K = \frac{1}{2} J \left[ \omega(t) \right]^2 = \frac{1}{2} J \left[ \dot{\theta}(t) \right]^2$</td>
</tr>
<tr>
<td>where $\Delta E = \text{change in internal energy}$</td>
<td>$\dot{K} = \frac{dK}{dt} = \frac{dK}{dx} \frac{dx}{dt} = \frac{d}{dt} \left( \frac{1}{2} mx^2 \right) \dot{x}(t) = m \ddot{x}(t) \dot{x}(t)$</td>
<td>$\dot{K} = \frac{dK}{dt} = \frac{dK}{d\theta} \frac{d\theta}{dt} = \frac{d}{d\theta} \left( \frac{1}{2} J \dot{\theta}^2 \right) \dot{\theta}(t) = J \ddot{\theta}(t) \dot{\theta}(t)$</td>
</tr>
<tr>
<td>$\Delta K = \text{change in kinetic energy}$</td>
<td>$\Delta U = \text{change in potential energy}$</td>
<td>$U = \frac{1}{2} k \left[ \theta(t) \right]^2$</td>
</tr>
<tr>
<td>$W_{\text{input}} = \text{work input}$</td>
<td>$Q_{\text{input}} = \text{heat input}$</td>
<td>$U = \frac{1}{2} k, \dot{\theta}(t) \dot{\theta}(t)$</td>
</tr>
<tr>
<td>$P_{\text{input}} = \text{power input}$</td>
<td>$D_{\text{output}} = \text{dissipated power}$</td>
<td></td>
</tr>
<tr>
<td>$D_{\text{output}} = f_b \dot{x}(t) = b \left[ \dot{x}(t) \right]^2$</td>
<td>$T(t) = T(t) \omega(t) = T(t) \dot{\theta}(t)$</td>
<td></td>
</tr>
<tr>
<td>Governing Equation $m \ddot{x}(t) + k \dot{x}(t) \dot{x}(t) = f(t) \dot{x}(t) - b \left[ \dot{x}(t) \right]^2$</td>
<td>$J \ddot{\theta}(t) + k, \dot{\theta}(t) \dot{\theta}(t) = T(t) \dot{\theta}(t) - b \left[ \dot{\theta}(t) \right]^2$</td>
<td></td>
</tr>
<tr>
<td>$\rightarrow m \ddot{x}(t) + b \dot{x}(t) + k \dot{x}(t) = f(t)$</td>
<td>$\rightarrow J \ddot{\theta}(t) + b \dot{\theta}(t) + k, \dot{\theta}(t) = T(t)$</td>
<td></td>
</tr>
</tbody>
</table>
### 3-3 Rotation-Translation Analogy

<table>
<thead>
<tr>
<th>Rotational Mechanical System (Gear train)</th>
<th>Translation Mechanical System (Lever linkage)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_1, \phi_1, n_1, R_1$</td>
<td>$m_1$</td>
</tr>
<tr>
<td>$k_r \Delta f, x$</td>
<td>$L_1$</td>
</tr>
<tr>
<td>$J_m T, \theta_m$</td>
<td>$M$</td>
</tr>
<tr>
<td>$i$</td>
<td>$y_m$</td>
</tr>
<tr>
<td>$+ e -$</td>
<td>$f$</td>
</tr>
</tbody>
</table>

- **applied torque**: $T$
- **driver inertia**: $J_m$
- **driver rotation**: $\theta_m$
- **rotational spring**: $k_r$
- **1st gear inertia**: $J_1$
- **1st gear rotation**: $\phi_1$
- **1st gear tooth number**: $n_1$
- **1st gear radius**: $R_1$
- **belt differential force**: $\Delta f = f_1 - f_2$
- **belt translation**: $x$
- **2nd gear inertia**: $J_2$
- **2nd gear rotation**: $\phi_2$
- **2nd gear tooth number**: $n_2$
- **2nd gear radius**: $R_2$
- **gear-train relation**: $\begin{cases} \frac{n_1}{R_1} = \frac{n_2}{R_2} \\ x = R_1 \phi_1 = R_2 \phi_2 \end{cases}$
- **rotational damper**: $b_r$

| applied force | $F$
|---------------|---------------|
| driver mass   | $M$
| driver translation | $y_m$
| linear spring | $k$
| 1st block mass | $m_i$
| 1st block translation | $x_i$
| length between 1st block and the lever pivot | $L_i$
| lever differential torque | $\Delta T = T_1 - T_2$
| level rotation | $\theta$
| 2nd block mass | $m_2$
| 2nd block translation | $x_2$
| length between 1st block and the lever pivot | $L_2$
| lever relation | $\theta = \frac{x_1}{L_1} = \frac{x_2}{L_2}$
| linear damper | $b$

#### Questions:
1. How is the lever differential torque: $\Delta T = T_1 - T_2$ generated?
2. What is the limitation of the lever relation: $\theta = \frac{x_1}{L_1} = \frac{x_2}{L_2}$? **Hint**: $\sin \theta = \theta$. 

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**ME 41100: System Dynamics & Control**

**Topic 3: Mechanical Systems**
As shown in the figure at right, a concentrated mass $m_c$ is mounted on a cantilever beam at a distance $a$ from the free end. The beam has a density $\rho$, Young’s modulus $E$, uniform cross-sectional area $A$, 2nd moment of inertia $I$ and length $L$. In the figure, $(u,v)$ are the displacements along the Cartesian coordinates $(x,y)$, respectively. Note that the boundary conditions at a free end are prescribed bending moment and shear force. For this problem, they are: $M(x)|_{x=0} = V(x)|_{x=0} = 0$. On the other hand, the boundary conditions at a clamped (or fixed or built-in) end are zero deflection and rotation/slope; that is, $v(x)|_{x=L} = \theta(x)|_{x=L} = 0$.

a. **Statically determinate vs indeterminate beam.** Replace the concentrated mass by a concentrated force $F$ and determine if the beam is statically determinate or indeterminate.

b. **Beam deflection, rotation/slope, bending moment & shear force.** Solve the 4th-order differential equation governing the beam behavior to find the beam deflection $v(x)$, rotation/slope $\theta(x)$, bending moment $M(x)$ and shear force $V(x)$. If needed, use Symbolic Math software to help solve for the resulting simultaneous linear algebraic equations.

c. **Moment & shear diagrams.** Plot the moment and shear diagrams within the beam length: $0 \leq x \leq L$.

d. **Reactions.** Find the reaction forces and moments at the two ends $A$ and $B$.

e. **Deflections & rotations/slopes.** Find the beam deflections and rotations/slopes at the two ends $A$ and $B$ and at the loading point: $x = a$, respectively.

f. **Equivalent spring constant & mass.** The dynamics of the beam-mass system can be viewed as an equivalent mass-spring system $m_i\ddot{\Delta}(t) + k_e\Delta(t) = 0$ where $m_i = m_c + m_e$

here $\Delta(t)$ is the time-history of the deflection at the location where the mass is mounted (i.e., $x = a$) and $k_e$ and $m_e$ are the equivalent spring constant and mass of the beam, respectively. Find the proper expressions of $k_e$ for an arbitrary $a$. Determine also $(k_e, m_e)$ for two special cases: $a = 0$ and $a = \frac{L}{2}$, respectively. If needed, use Symbolic Math software to help perform integrations.

**Sol:**

a. **Statically determinate vs indeterminate beam.**

Once the concentrated mass has been replaced by a concentrated force $F$, the structure and its corresponding free-body diagram can be represented by the figures shown below, where $A$ and $B$ are the free and fixed ends, respectively. Since the two unknown reactions (namely, the reaction force $R_y$ and the reaction moment
\( M_B \) can be determined using two equilibrium equations: one for vertical force and the other for moment, i.e., \(+ \sum F_y = 0\) and \(+ \sum M_x = 0\). Thus, the beam is \textit{statically determinate}.

**b. Beam deflection, rotation/slope, bending moment & shear force.**

4\textsuperscript{th}-order ODE for beam deflection:

\[
iv(x) = \frac{p(x)}{EI}
\]  

(\textbf{b1})

total solution = homogeneous solution + particular solution \( \rightarrow v(x) = v_h(x) + v_p(x) \)

where \( v_p(x) \) is the particular solution due to the lateral load \( p(x) \).

no distributed load \( \rightarrow p(x) = 0 \) \( \rightarrow v_p(x) = 0 \) \( \rightarrow v(x) = 0 \)

\[
\begin{align*}
\text{deflection:} & \quad v(x) = C_{3L} x^3 + C_{2L} x^2 + C_{1L} x + C_{0L} \\
\text{rotation/slope:} & \quad \theta(x) = \frac{d}{dx} v(x) = 3C_{3L} x^2 + 2C_{2L} x + C_{1L} \\
\text{bending moment:} & \quad M(x) = EI \frac{d^2}{dx^2} (x) = 6EIC_{3L} x + 2EIC_{2L} \\
\text{shear force:} & \quad V(x) = EI \frac{d}{dx} (x) = 6EIC_{3L}
\end{align*}
\]  

(\textbf{b2})

and \( a < x \leq L \):

\[
\begin{align*}
\text{deflection:} & \quad v(x) = C_{3R} (L-x)^3 + C_{2R} (L-x)^2 + C_{1R} (L-x) + C_{0R} \\
\text{rotation/slope:} & \quad \theta(x) = \frac{d}{dx} v(x) = -3C_{3R} (L-x)^2 - 2C_{2R} (L-x) - C_{1R} \\
\text{bending moment:} & \quad M(x) = EI \frac{d^2}{dx^2} (x) = 6EIC_{3R} (L-x) + 2EIC_{2R} \\
\text{shear force:} & \quad V(x) = EI \frac{d}{dx} (x) = -6EIC_{3R}
\end{align*}
\]  

(\textbf{b3})

\[
\begin{align*}
\text{B.C.’s @ free end (} x = 0 \text{):} & \quad \left\{ \begin{array}{l} 
\text{no bending moment:} \quad M(x)\bigg|_{x=0} = 0 \\
\text{no shear force:} \quad V(x)\bigg|_{x=0} = 0
\end{array} \right. \\
\rightarrow & \quad \left\{ \begin{array}{l} 
2EIC_{2L} = 0 \\
6EIC_{3L} = 0
\end{array} \right. \quad (\textbf{Note:} \quad C_{3L} = C_{2L} = 0)
\end{align*}
\]  

(\textbf{b4})

\[
\begin{align*}
\text{B.C.’s @ fixed end (} x = L \text{):} & \quad \left\{ \begin{array}{l} 
\text{no deflection:} \quad v(x)\bigg|_{x=L} = 0 \\
\text{no rotation/slope:} \quad \theta(x)\bigg|_{x=L} = 0
\end{array} \right. \\
\rightarrow & \quad \left\{ \begin{array}{l} 
C_{0R} = 0 \\
-C_{1R} = 0
\end{array} \right. \quad (\textbf{Note:} \quad C_{1R} = C_{0R} = 0)
\end{align*}
\]  

(\textbf{b5})
The compatibility conditions at loading point \((x = a)\):

\[
\begin{align*}
v(a^-) &= v(a^+) \\
\theta(a^-) &= \theta(a^+) \\
M(a^-) &= M(a^+) \\
V(a^-) &= V(a^+) + F
\end{align*}
\]

The equivalent spring/mass (cantilever beam):

\[
\begin{align*}
C_{3L}a^3 + C_{2L}a^2 + C_{1L}a + C_{0L} &= C_{3R}(L-a)^3 + C_{2R}(L-a)^2 + C_{1R}(L-a) + C_{0R} \\
3C_{3L}a^2 + 2C_{2L}a + C_{1L} &= -3C_{3R}(L-a)^2 - 2C_{2R}(L-a) - C_{1R} \\
6EIC_{3L}a + 2EIC_{2L} &= 6EIC_{3R}(L-a) + 2EIC_{2R} \\
6EIC_{3L} &= -6EIC_{3R} + F
\end{align*}
\]

(b6)

Notes:
1. The conventions for positive internal forces (i.e., shear force and bending moment) are defined as in the figure at left below.
2. The compatibility condition in shear force, Eq (b6.4), can be obtained by considering the free-body diagram shown in the figure at right below.

Question: What will the compatibility condition in bending moment be, if the concentrated force \(F\) is replaced by a concentrated moment \(M_0\)?

The 8 linear simultaneous algebraic eqs: (b4) to (b6) are solved using Symbolic Math software, which is attached at the end of this article, and we obtain

| \(C_{3L}\) | 0 | \(C_{3R}\) | \(\frac{F}{6EI}\) |
| \(C_{2L}\) | 0 | \(C_{2R}\) | \(-\frac{F(L-a)}{2EI}\) |
| \(C_{1L}\) | \(\frac{F(L-a)^2}{2EI}\) | \(C_{1R}\) | 0 |
| \(C_{0L}\) | \(-\frac{F(2L+a)(L-a)^2}{6EI}\) | \(C_{0R}\) | 0 |

Substitute these \(C\) values into Eqs (b2,b3) we have

Deflection:
\[
v(x) = \begin{cases} 
\frac{F(L-a)^2}{6EI}[3x-(2L+a)] & \text{for } 0 \leq x \leq a \\
\frac{F}{6EI} \left[ (L-x)^3 - 3(L-a)(L-x)^2 \right] & \text{for } a \leq x \leq L 
\end{cases}
\]

(b7)
rotation/slope: \( \theta(x) = v'(x) = \begin{cases} \frac{F}{2EI} (L-a)^2 & \text{for } 0 \leq x \leq a \\ \frac{F}{2EI} \left[-(L-x)^2 + 2(L-a)(L-x)\right] & \text{for } a \leq x \leq L \end{cases} \) (b8)

bending moment: \( M(x) = E I v''(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \\ -F(x-a) & \text{for } a \leq x \leq L \end{cases} \) (b9)

shear force: \( V(x) = E I v''(x) = \begin{cases} 0 & \text{for } 0 \leq x < a \\ -F & \text{for } a < x \leq L \end{cases} \) (b10)

c. Moment & shear diagrams.

d. Reactions.

<table>
<thead>
<tr>
<th>Moment Diagram</th>
<th>Shear Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M(x) )</td>
<td>( V(x) )</td>
</tr>
<tr>
<td>( -F(L-a) )</td>
<td>( -F )</td>
</tr>
<tr>
<td>( a )</td>
<td>( a )</td>
</tr>
<tr>
<td>( L )</td>
<td>( L )</td>
</tr>
<tr>
<td>( x )</td>
<td>( x )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reactions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Free End</strong> ( x = 0 )</td>
</tr>
<tr>
<td>( \begin{cases} R_A = V(0) = 0 \ M_A = M(0) = 0 \end{cases} )</td>
</tr>
</tbody>
</table>
### Example 3-1: Equivalent Spring/Mass (Cantilever Beam)

<table>
<thead>
<tr>
<th>Free End: ( x = 0 )</th>
<th>( v(0) )</th>
<th>( \frac{F(L-a)^2(2L+a)}{6EI} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotation/Slope</td>
<td>( \theta(0) )</td>
<td>( \frac{F(L-a)^2}{2EI} )</td>
</tr>
<tr>
<td>Loading Location: ( x = a )</td>
<td>( v(a) = -\Delta )</td>
<td>( \frac{F(L-a)^3}{3EI} )</td>
</tr>
<tr>
<td>Rotation/Slope</td>
<td>( \theta(a) )</td>
<td>( \frac{F(L-a)^2}{2EI} )</td>
</tr>
<tr>
<td>Clamped End: ( x = L )</td>
<td>( v(L) )</td>
<td>0</td>
</tr>
<tr>
<td>Rotation/Slope</td>
<td>( \theta(L) )</td>
<td>0</td>
</tr>
</tbody>
</table>

**e. Deflections & rotations/slopes.**

<table>
<thead>
<tr>
<th><strong>Deflection</strong></th>
<th><strong>Rotation/Slope</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(0) )</td>
<td>( \frac{F(L-a)^2(2L+a)}{6EI} )</td>
</tr>
<tr>
<td>( \theta(0) )</td>
<td>( \frac{F(L-a)^2}{2EI} )</td>
</tr>
<tr>
<td>( v(a) = -\Delta )</td>
<td>( \frac{F(L-a)^3}{3EI} )</td>
</tr>
<tr>
<td>( \theta(a) )</td>
<td>( \frac{F(L-a)^2}{2EI} )</td>
</tr>
<tr>
<td>( v(L) )</td>
<td>0</td>
</tr>
<tr>
<td>( \theta(L) )</td>
<td>0</td>
</tr>
</tbody>
</table>

**f. Equivalent spring constant & mass.**

Since the deflection at \( x = a \) is:

\[
\Delta = -v(x)\Big|_{x=a} = \frac{F(L-a)^3}{3EI} \rightarrow F = \frac{3EI}{(L-a)^3}\Delta \tag{f1}
\]

Consider the cantilever beam as a spring oscillating under the action of the concentrated mass. From Hooke’s law: \( F = k_c \Delta \), the equivalent spring constant should be:

\[
k_c = \frac{3EI}{(L-a)^3} \tag{f2}
\]

Specifically,

\[
\begin{align*}
\text{when } a &= 0 \rightarrow k_c = \frac{3EI}{L^3} \text{ and } \Delta = \frac{FL^3}{3EI} \\
\text{when } a &= \frac{L}{2} \rightarrow k_c = \frac{24EI}{L^3} \text{ and } \Delta = \frac{FL^3}{24EI}
\end{align*} \tag{f3}
\]

Substitute Eq (f1) into Eq (b7), we get:

\[
v(x) = \begin{cases} 
\frac{\Delta}{2(L-a)}\left[3x-(2L+a)\right] & \text{for } 0 \leq x \leq a \\
\frac{\Delta}{2(L-a)^3}\left[(L-x)^3-3(L-a)(L-x)^2\right] & \text{for } a \leq x \leq L
\end{cases} \tag{f4}
\]

For “\( a = 0 \)” case: \( v(x) = \Delta \left[\frac{1}{2}\left(1-\frac{x}{L}\right)^3 - \frac{3}{2}\left(1-\frac{x}{L}\right)^2\right] \) for \( 0 \leq x \leq L \)

Assume that during vibration the form of this beam deflection (i.e., the elastic curve) is preserved (Justification?), then:

\[
\dot{v}(x,t) = \Delta \left[\frac{1}{2}\left(1-\frac{x}{L}\right)^3 - \frac{3}{2}\left(1-\frac{x}{L}\right)^2\right]
\]

The kinetic energy carried by the beam during vibration is...
K.E. = \[ \frac{1}{2} \left[ \dot{\varphi}(x) \right]^2 dm = \int_0^L \frac{1}{2} \left[ \dot{\varphi}(x) \right]^2 \rho A dx = \frac{1}{2} \rho A L \left\{ \Delta \left[ \frac{1}{2} \left( 1 - \frac{x}{L} \right)^3 - \frac{3}{2} \left( 1 - \frac{x}{L} \right)^2 \right] \right\}^2 dx \]

Change of variable: \( \xi = 1 - \frac{x}{L} \), we have \( dx = -Ld\xi \) and

K.E. = \( -\frac{1}{2} \rho AL \Delta^2 \int_0^L \left( \frac{1}{2} \xi^3 - \frac{3}{2} \xi^2 \right) d\xi = \frac{1}{2} \rho AL \Delta^2 \int_0^L \left( \frac{1}{4} \xi^6 - \frac{3}{2} \xi^5 + \frac{9}{4} \xi^4 \right) d\xi \)

\[ = \frac{1}{2} \rho AL \Delta^2 \left( \frac{1}{28} \xi^7 - \frac{1}{4} \xi^6 + \frac{9}{20} \xi^5 \right) \bigg|_0^L = \frac{1}{2} \rho AL \Delta^2 \left( \frac{1}{28} - \frac{1}{4} + \frac{9}{20} \right) \]

\[ = \frac{1}{2} \rho AL \Delta^2 \cdot \frac{33}{140} \approx 0.236 \left( \frac{1}{2} \rho AL \Delta^2 \right) \]

Equate this kinetic energy with that of the equivalent system: \( \text{K.E.} = \frac{1}{2} m_\alpha \dot{\Delta}^2 \), we get

\( m_\alpha = \frac{33}{140} m_{\text{beam}} \approx 0.236m_{\text{beam}} \), where \( m_{\text{beam}} = \rho AL \) \( \text{(f6)} \)

For “\( a = \frac{L}{2} \)” case: \( \nu(x) = \left\{ \begin{array}{ll}
\frac{FL}{48EI} \left[ 6 \left( \frac{x}{L} \right) - 5 \right] & \text{for } 0 \leq x \leq \frac{L}{2} \\
\frac{FL}{12EI} \left[ 2 \left( 1 - \frac{x}{L} \right)^3 - 3 \left( 1 - \frac{x}{L} \right)^2 \right] & \text{for } \frac{L}{2} \leq x \leq L
\end{array} \right. \) \( \text{(f8)} \)

\( \nu(x) = \left\{ \begin{array}{ll}
\Delta \left[ 3 \left( \frac{x}{L} \right) - \frac{5}{2} \right] & \text{for } 0 \leq x \leq \frac{L}{2} \\
\Delta \left[ 4 \left( 1 - \frac{x}{L} \right)^3 - 6 \left( 1 - \frac{x}{L} \right)^2 \right] & \text{for } \frac{L}{2} \leq x \leq L
\end{array} \right. \)

Assume that during vibration the form of this beam deflection (i.e., the elastic curve) is preserved (Justification?), then

\( \dot{\nu}(x,t) = \frac{\partial \nu(x,t)}{\partial t} = \left\{ \begin{array}{ll}
\Delta \left[ 3 \left( \frac{x}{L} \right) - \frac{5}{2} \right] & \text{for } 0 \leq x \leq \frac{L}{2} \\
\Delta \left[ 4 \left( 1 - \frac{x}{L} \right)^3 - 6 \left( 1 - \frac{x}{L} \right)^2 \right] & \text{for } \frac{L}{2} \leq x \leq L
\end{array} \right. \) \( \text{(f9)} \)

The kinetic energy carried by the beam during vibration is

K.E. = \[ \int_0^L \frac{1}{2} \left[ \dot{\nu}(x) \right]^2 dm = \int_0^L \frac{1}{2} \left[ \dot{\nu}(x) \right]^2 \rho A dx = \frac{1}{2} \rho A L \left\{ \Delta \left[ \frac{3}{2} \left( \frac{x}{L} \right) - \frac{5}{2} \right]^2 \right\} dx + \int_{\frac{L}{2}}^L \left\{ \Delta \left[ 4 \left( 1 - \frac{x}{L} \right)^3 - 6 \left( 1 - \frac{x}{L} \right)^2 \right]^2 \right\} dx \]
Change of variable: 
\[
\begin{align*}
\xi &= \frac{x}{L} \quad \text{for } 0 \leq x \leq \frac{L}{2} \\
\eta &= 1 - \frac{x}{L} \quad \text{for } \frac{L}{2} \leq x \leq L
\end{align*}
\]
we have 
\[
dx = \begin{cases} 
Ld\xi & \text{for } 0 \leq x \leq \frac{L}{2} \\
-Ld\eta & \text{for } \frac{L}{2} \leq x \leq L
\end{cases}
\]
K.E. = \frac{1}{2} \rho AL\dot{\bar{\lambda}}^2 \left[ \int_{\frac{L}{2}}^{\frac{L}{2}} \left( 3\xi^2 - \frac{5}{2} \right) d\xi - \int_{0}^{\frac{L}{2}} \left( 4\eta^3 - 6\eta^2 \right) d\eta \right]
\]
\[
= \frac{1}{2} \rho AL\dot{\bar{\lambda}}^2 \left[ \int_{0}^{\frac{L}{2}} \left( 9\xi^2 - 15\xi + \frac{25}{4} \right) d\xi + \int_{0}^{\frac{L}{2}} \left( 16\eta^6 - 48\eta^5 + 36\eta^4 \right) d\eta \right]
\]
\[
= \frac{1}{2} \rho AL\dot{\bar{\lambda}}^2 \left[ \left( \frac{3}{8} - \frac{15}{8} + \frac{25}{8} \right) + \left( \frac{1}{56} - \frac{1}{8} + \frac{9}{40} \right) \right]
\]
\[
= \frac{1}{2} \rho AL\dot{\bar{\lambda}}^2 \left( \frac{13}{8} + \frac{33}{280} \right) = \frac{1}{2} \rho AL\dot{\bar{\lambda}}^2 \cdot 61 \approx 1.743 \cdot \left( \frac{1}{2} \rho AL\dot{\bar{\lambda}}^2 \right)
\]

Equate this kinetic energy with that of the equivalent system: 

\[
\text{K.E.} = \frac{1}{2} m_e \dot{\bar{\lambda}}^2 , \quad \text{we have}
\]

\[
m_e = \frac{61}{35} m_{\text{beam}} \approx 1.743 m_{\text{beam}}, \quad \text{where} \quad m_{\text{beam}} = \rho AL \quad (f10)
\]

**Note:** The above integrals can also be evaluated by Symbolic Math software, which is attached at the end of this article.

### SUMMARY

<table>
<thead>
<tr>
<th>a</th>
<th>k_e</th>
<th>m_e</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>\frac{3EI}{L^3}</td>
<td>\frac{33}{140} \quad m_{\text{beam}} \approx 0.236m_{\text{beam}}</td>
</tr>
<tr>
<td>\frac{L}{2}</td>
<td>\frac{24EI}{L^3}</td>
<td>\frac{61}{35} \quad m_{\text{beam}} \approx 1.743m_{\text{beam}}</td>
</tr>
</tbody>
</table>
Symbolic Math (Mathcad)

Equivalent Spring & Mass (Cantilever Beam)

Given

\( 2 \cdot EI \cdot C_{2L} = 0 \)

\( 6 \cdot EI \cdot C_{3L} = 0 \)

\( C_{0R} = 0 \)

\(-C_{1R} = 0 \)

\( C_{3L} \cdot a^3 + C_{2L} \cdot a^2 + C_{1L} \cdot a + C_{0L} = C_{3R} \cdot (L - a)^3 + C_{2R} \cdot (L - a)^2 + C_{1R} \cdot (L - a) + C_{0R} \)

\( 3 \cdot C_{3L} \cdot a^2 + 2 \cdot C_{2L} \cdot a + C_{1L} = -3 \cdot C_{3R} \cdot (L - a)^2 - 2 \cdot C_{2R} \cdot (L - a) - C_{1R} \)

\( 6EI \cdot C_{3L} \cdot a + 2 \cdot EI \cdot C_{2L} = 6 \cdot EI \cdot C_{3R} \cdot (L - a) + 2 \cdot EI \cdot C_{2R} \)

\( 6EI \cdot C_{3L} = -6 \cdot EI \cdot C_{3R} + F \)

\[
\begin{bmatrix}
0 \\
0 \\
\frac{1}{2} F \cdot \left( \frac{L^2 - 2L \cdot a + a^2}{EI} \right) \\
\frac{1}{6} \cdot F \cdot \left( -3 \cdot L^2 \cdot a^3 + 2 \cdot L^3 \right) \frac{1}{EI} \\
\frac{1}{6} \cdot F \cdot \left( -L + a \right) \frac{1}{EI} \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
\frac{1}{2} F \cdot \left( \frac{L^2 - 2L \cdot a + a^2}{EI} \right) \\
\frac{1}{6} \cdot F \cdot \left( -3 \cdot L^2 \cdot a + a^3 + 2 \cdot L^3 \right) \frac{1}{EI} \\
\frac{1}{6} \cdot F \cdot \left( -L + a \right) \frac{1}{EI} \\
0 \\
0
\end{bmatrix}
\]

Find \( C_{3L}, C_{2L}, C_{1L}, C_{0L}, C_{3R}, C_{2R}, C_{1R}, C_{0R} \) →

\[
- \int_{1}^{6} \left( \frac{1}{2} \xi^3 - \frac{3}{2} \xi \right)^2 d\xi \rightarrow \frac{33}{140}
\]

\[
\int_{0}^{1} \left( 3\xi^2 - \xi \right)^2 d\xi \rightarrow \frac{13}{8}
\]

\[
\int_{0}^{1} \left( 4\eta^3 - 6\eta^2 \right)^2 d\eta \rightarrow \frac{33}{280}
\]