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FINITE-DIFFERENCE APPROACH TO THE HODGE THEORY OF HARMONIC FORMS.*

By JOZEF DODZIUK.

Table of Contents

0. Introduction. Let K be a finite simplicial complex. Eckmann (see [3]) observed that any inner product in real cochain spaces of K gives rise to a combinatorial Hodge theory. We show that if K is a smooth triangulation of a compact oriented, Riemannian manifold X, then the combinatorial Hodge theory (for a suitable choice of inner product) is an approximation of the Hodge theory of forms on X.

Before giving a more detailed description of our results we introduce some notation and formulas. Thus let X be a compact, oriented, C^∞ Riemannian manifold of dimension N, whose boundary consists of two disjoint closed submanifolds M_1 and M_2. We do not exclude the possibility that M_1, M_2 or both are empty. The Riemannian metric provides the space $\Lambda = \sum \Lambda^q$ of C^∞ differen-

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tial forms on X with an inner product

$$(f,g)=\int_X f\wedge *g, \qquad f,g\in\Lambda. \tag{0.1}$$

The completion of Λ^q with respect to this inner product will be denoted by $L^2\Lambda^q$. Let $d\colon \Lambda^q\to\Lambda^{q+1}$ be the exterior derivative and $\delta=(-1)^{Nq+N+1}*\delta*$ the formal adjoint of d on Λ^q. By Stokes' theorem

$$(df,g)-(f,\delta g)=\int_{M_1\cup M_2} f\wedge *g \tag{0.2}$$

for $f\in\Lambda^q$, $g\in\Lambda^{q+1}$.

At every boundary point of X a differential form f can be decomposed into its normal and tangential components: $f=f_{\tan}+f_{\mathrm{norm}}$. The operator $*$ maps every covector corresponding to a subspace into a covector corresponding to its orthogonal complement and therefore $(*f)_{\tan}=*(f_{\mathrm{norm}})$. Using this we can write the boundary terms in (0.2) as follows:

$$\begin{aligned}\int_{M_1\cup M_2} f\wedge *g &= \int_{M_1\cup M_2}(f\wedge *g)_{\tan}=\int_{M_1\cup M_2} f_{\tan}\wedge(*g)_{\tan}\\ &=\int_{M_1\cup M_2} f_{\tan}\wedge *(g_{\mathrm{norm}}).\end{aligned} \tag{0.3}$$

We will denote by $\Lambda_1=\Sigma\Lambda_1^q$ the space of C^∞ forms on X which satisfy the boundary conditions $f_{\tan}=0$ on M_1 and $f_{\mathrm{norm}}=0$ on M_2. Then, by (0.2) and (0.3),

$$(df,g)=(f,\delta g) \qquad f\in\Lambda_1^q,\quad g\in\Lambda_1^{q+1}. \tag{0.4}$$

Finally, let K be a finite simplicial complex of a C^∞ triangulation of X which contains subcomplexes L_1 and L_2 triangulating M_1 and M_2 respectively.

In Section 1 we describe an operation, due to Whitney [8], which assigns an L^2 form on X to every simplicial cochain of K. More precisely, we define the linear mapping $W\colon C^q(K)\to L^2\Lambda^q$ of real cochain spaces of K into the spaces of L^2 forms on X, where $q=0,1,\dots,N$.

In Section 2, we describe standard subdivisions S_nL of an arbitrary simplicial complex L, also introduced by Whitney in [8].

Section 3 is devoted to the proof of crucial approximation theorem. Let $W_n\colon C^q(S_nK)\to L^2\Lambda^q$ be the Whitney mapping for the complex S_nK. Let $R_n\colon\Lambda^q\to C^q(S_nK)$ be the de Rham mapping defined by integration of forms over chains of S_nK. The approximation theorem asserts that, for large n, W_nR_nf is a good approximation of f.

In Section 4 we define the inner product in cochain spaces $C^q(S_nK)$ and discuss resulting combinatorial Hodge theory. We then show that if $f = dg + h + \delta k$ is the Hodge decomposition of a C^∞ form $f \in \Lambda_1$ and $R_n f = d_n g_n + h_n + \delta_n k_n$ is the combinatorial Hodge decomposition of the cochain $R_n f \in C^q(S_nK)$, then $W_n d_n g_n \to dg$, $W_n h_n \to h$, and $W_n \delta_n k_n \to \delta k$ in $L^2\Lambda^q$. This is our main result.

We also discuss in section 4 the differences between classical finite-difference techniques for solving partial differential equations and our methods.

Finally, in Section 5, we prove that the eigenvalues of combinatorial Laplacians converge to eigenvalues of the continuous Laplacian, at least in dimension 0, i.e., for 0-cochains and functions. Our proof is an application of the classical Rayleigh–Ritz method (see Gould [4]). As a corollary we obtain that the zeta functions of combinatorial Laplacians converge to the zeta function of the continuous Laplacian. We conjecture that the results of this section are true in all dimensions $q = 0, 1, \ldots, N$; but we were unable to prove it.

This eigenvalue problem is related to a question of equality of analytic torsion and Reidemeister-Franz torsion raised by Ray and Singer [7]. The proof that zeta functions converge in all dimensions would be a step toward proving that the two torsions are equal.

1. Whitney Forms. The definition of Whitney forms and all results of this section, except for (1.5), (1.6), (1.8), are from Whitney [8].

We now define a linear mapping W of the real cochain groups $C^q(K)$ into $L^2\Lambda^q$. To do so we identify K with X and fix some ordering of the set of vertices of K. For a vertex p of K, we denote by μ_p the pth barycentric coordinate in K. Since K is a finite complex we can identify chains and cochains and write every cochain $c \in C^q(K)$ as the sum $c = \sum c_\tau \cdot \tau$ with $c_\tau \in \mathbf{R}$ and τ running through all q-simplexes $[p_0, p_1, \ldots, p_q]$ of K whose vertices form an increasing sequence with respect to the ordering of K. It follows, that it suffices to define W_τ for such simplexes τ.

Definition 1.1. *Let* $\tau = [p_0, p_1, \ldots, p_q]$, *where* $p_0, p_1, \ldots, p_q$ *is an increasing sequence of vertices of* K. *Define* $W\tau \in L^2\Lambda^q$ *by the formula*

$$W\tau = q! \sum_{i=0}^{q} (-1)^i \mu_{p_i} d\mu_{p_0} \wedge d\mu_{p_1} \wedge \cdots \wedge d\mu_{p_{i-1}} \wedge d\mu_{p_{i+1}} \wedge \cdots \wedge d\mu_{p_q}. \quad (1.2)$$

We remark that the above definition makes sense even though the barycentric coordinates are not C^1 functions on X. However, since the triangulation is of class C^∞, the barycentric coordinates are C^∞ on the complement of $(n-1)$-dimensional skeleton of K. This allows us to apply the exterior derivative d in (1.2) and the resulting form is a well defined element of $L^2\Lambda^q$. Also, (1.2) holds for every simplex $\tau=[p_0,\dots,p_q]$ (the vertices need not form an increasing sequence) because both sides of (1.2) are alternating in the subscripts 0, 1, $2,\dots,q$.

For a cochain $c\in C^q(K)$, we call Wc the Whitney form associated with c. We now state and verify some properties of Whitney forms.

$$W\tau=0 \quad \text{on} \quad X\setminus\overline{\mathrm{St}(\tau)}\ . \tag{1.3}$$

for every simplex τ of K. $\mathrm{St}(\tau)$ denotes the open star of τ.

Proof. Suppose $\tau=[p_0,p_1,\dots,p_q]$. Since $\mathrm{St}(\tau)=\{p\in X\mid \mu_{p_i}(p)\neq 0,\ i=0,\dots,q\}$, (1.3) follows from the definition of $W\tau$.

$$Wdc=dWc \quad \text{for} \quad c\in C^q(K). \tag{1.4}$$

Here $dc\in C^{q+1}(K)$ is the simplicial coboundary of c and dWc denotes the exterior derivative applied to Wc on the complement of $(n-1)$-dimensional skeleton of K. (See the remark following Definition 1.1.)

Proof. It suffices to prove (1.4) for $c=\tau=[p_0,\dots,p_q]$. Let $\mu_0,\dots,\mu_q$ be the barycentric coordinates corresponding to $p_0,\dots,p_q$. Observe first that

$$dW\tau=d\left(q!\sum_{i=0}^{q}(-1)^i\mu_i\wedge d\mu_0\wedge\dots\overset{i}{\vee}\dots\wedge d\mu_q\right)$$

$$=q!\sum_{i=0}^{q}d\mu_0\wedge d\mu_1\wedge\dots\wedge d\mu_q=(q+1)!\,d\mu_1\wedge\dots\wedge d\mu_q$$

On the other hand, $d\tau=\Sigma[p,p_0,\dots,p_q]$ where summation is extended over all vertices p of K such that $[p,p_0,\dots,p_k]$ is a simplex of K. In computing $Wd\tau$ we shall use the following facts:

$$\sum_p \mu_p=1, \qquad \sum_p d\mu_p=0,$$

$$\mu_p=0 \quad \text{on} \quad W\setminus St(p),$$

$$df\wedge df=0 \qquad \text{for any function } f.$$

Using these we have

$$\frac{1}{(q+1)!}Wd\tau=\frac{1}{(q+1)!}W\Big(\sum_p{}'[\,p,p_0,\ldots,p_q]\Big)$$

$$=\sum_p{}'\Big(\mu_p d\mu_0\wedge\cdots\wedge d\mu_q+\sum_{j=0}^{q}(-1)^{j+1}\mu_j d\mu_p\wedge d\mu_0\wedge\cdots\overset{j}{\vee}\cdots\wedge d\mu_q\Big)$$

$$=\sum_p{}'\mu_p d\mu_0\wedge\cdots\wedge d\mu_q+\sum_{j=0}^{q}(-1)^{j+1}\mu_j d\Big(\sum_p{}'\mu_p\Big)\wedge d\mu_0\cdots\overset{j}{\vee}\cdots\wedge d\mu_q$$

$$=\sum_p{}'\mu_p d\mu_0\wedge\cdots\wedge d\mu_q+\sum_{j=0}^{q}(-1)^{j}\mu_j d\Big(\sum_{i=0}^{q}\mu_i\Big)\wedge d\mu_0\wedge\cdots\overset{j}{\vee}\cdots\wedge d\mu_q$$

$$=\sum_p{}'\mu_p d\mu_0\wedge\cdots\wedge d\mu_q+\sum_{j=0}^{q}\mu_j d\mu_0\wedge\cdots\wedge d\mu_q=d\mu_0\wedge\cdots\wedge d\mu_q,$$

as required. In the above calculations $\Sigma_p{}'$ denotes the summation over all vertices p such that $[p,p_0,\ldots,p_q]$ is a $(q+1)$-simplex of K.

To state the next property we observe that the barycentric coordinates are C^∞ on every closed simplex of K. Thus, for every closed N-dimensional simplex σ and every cochain $c\in C^q(K)$, $Wc|\sigma$ has the unique C^∞ extension to $\bar\sigma$ denoted by $Wc|\bar\sigma$. If $c=\tau$ is a simplex, $Wc|\bar\sigma$ is given by (1.2). However, if ρ is a simplex on the boundary of more than one N-simplex, say $\rho\subset\bar\sigma\cap\bar\sigma'$, the values on ρ of the two extensions need not agree. Nevertheless the restrictions (as forms) of $Wc|\bar\sigma$ and $Wc|\bar\sigma'$ to ρ are equal. Let $i:\rho\subset\bar\sigma$, $i':\rho\subset\bar\sigma'$ be the inclusion maps. We have

$$i^*(Wc|\bar\sigma)=i'^*(Wc|\bar\sigma') \tag{1.5}$$

Proof. We can assume that $c=\tau$ is a simplex. Then Wc is given by (1.2). Since the restriction commutes with exterior product it suffices to prove that, if μ is a barycentric coordinate corresponding to a vertex of K, then μ and $d\mu$ satisfy

$$i^*\mu=i'^*\mu,\qquad i^*d\mu=i'^*d\mu.$$

But $i^*\mu=\mu|\rho=i'^*\mu$ because μ is continuous and $i^*d\mu=di^*\mu=d(\mu|\rho)=di'^*\mu=i'^*d\mu$, because exterior derivative commutes with restriction.

Let $j:M_1\subset W$ be the inclusion map. Let $c\in C^q(K,L_1)$, i.e., c evaluated on every simplicial chain of L_1 is zero. Then

$$j^* Wc = 0. \tag{1.6}$$

Proof. We can assume $c = \tau = [p_0, p_1, \ldots, p_q]$ with $p_0 \notin M_1$. The barycentric coordinate μ_{p_0} vanishes on M_1 and (1.6) follows from (1.2).

Let $C_q(K)$ be the group of real simplicial q-chains of K and let $\langle\ ,\ \rangle$ denote the standard pairing of $C_q(K)$ and $C^q(K)$.

$$\int_a Wc = \langle c, a\rangle \tag{1.7}$$

for every $c \in C^q(K)$, $a \in C_q(K)$. The integral above is well defined by (1.5).

Proof. The proof proceeds by induction on q. For $q = 0$, c and a can be written as

$$c = \sum c_p \cdot p, \qquad a = \sum a_p \cdot p.$$

By definition

$$Wc = \sum c_p \mu_p$$

$$\int_a Wc = \sum_{p,p'} c_{p'} \cdot a_p \cdot \mu_{p'}(p) = \sum_p a_p \cdot c_p = \langle c, a\rangle.$$

We now assume that (1.7) holds for $q-1$. Let $\tau_1, \ldots, \tau_s$ be the basis of $C^q(K)$ consisting of simplexes of K. We have to show that

$$\int_{\tau_i} W\tau_j = \delta_{ij}.$$

If $i \neq j$

$$\int_{\tau_i} W\tau_j = 0$$

by (1.3) and (1.5). If $i = j$, choose a $(q-1)$-simplex ρ such that ρ is a face of τ_i. Then

$$d\rho = \tau_i + \sum_{\tau' \neq \tau_i} \tau'$$

and

$$\int_{\tau_i} W\tau_i = \int_{\tau_i} Wd\rho = \int_{\tau_i} dW\rho = \int_{\partial\tau_i} W\rho = \int_\rho W\rho + \sum_{\tau'' \neq \rho} \int_{\tau''} W\rho = 1$$

by induction hypothesis, (1.4), and Stokes' theorem.

The following Lemma will be very useful.

LEMMA 1.8. *Let f be a $C^\infty(q+1)$-form on X such that $f_{\text{norm}}=0$ on M_2. Let $c \in C^q(K, L_1)$. Then*

$$(dWc, f) = (Wc, \delta f).$$

Proof.

$$(dWc, f) = \sum_\sigma \int_{\bar\sigma} dWc \wedge *f,$$

where the summation is over all N-dimensional simplexes of K. Moreover

$$\int_{\bar\sigma} dWc \wedge *f = \int_{\bar\sigma} Wc \wedge *\delta f + \int_{\partial\bar\sigma} Wc \wedge *f.$$

Therefore

$$(dWc, f) - (Wc, \delta f) = \sum_\sigma \int_{\partial\bar\sigma} Wc \wedge *f.$$

The last sum can be written as the sum of the integrals over $(N-1)$-dimensional simplexes. The integrals over simplexes of L_1 vanish by (1.6). Similarly, the integrals over simplexes of L_2 vanish because $(*f)_{\tan} = *f_{\text{norm}} = 0$ on M_2. Finally, for every simplex τ in the interior of X, we get two integrals which cancel by (1.5).

2. Standard Subdivisions of a Complex. In this section we describe a method of subdividing a simplicial complex. This method was introduced by Whitney in [8]. It is very well suited to our purposes. The resulting subdivision of a given complex is called the standard subdivision.

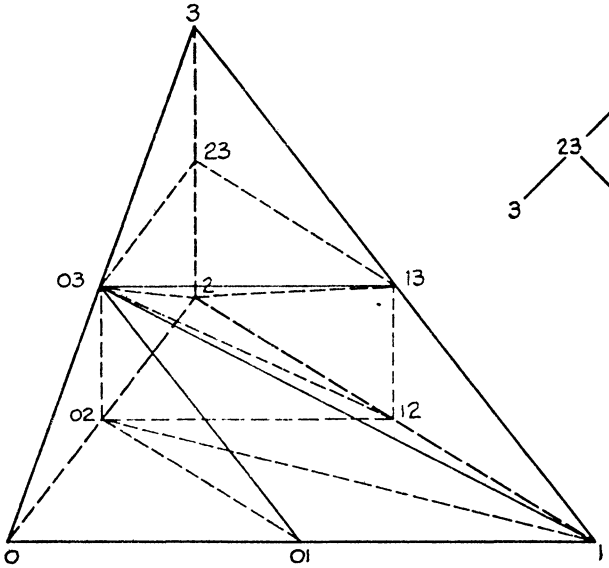
We first discuss the standard subdivision of a simplex. Thus let $\sigma = [p_0, p_1, \ldots, p_m]$ be a simplex in $\mathbf{R}^k$, $k \geqslant m$. The vertices of $S\sigma$, the standard subdivision of σ, are the points

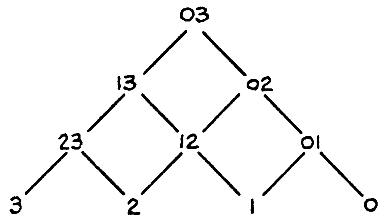
$$p_{ij} = \tfrac{1}{2}(p_i + p_j) \qquad i \leqslant j. \tag{2.1}$$

We define partial ordering of the vertices of $S\sigma$ by setting

$$p_{ij} \leqslant p_{kl} \qquad \text{if } i \geqslant k \text{ and } j \leqslant l \quad (\text{sic}). \tag{2.2}$$

The simplexes of $S\sigma$ are the increasing sequences of vertices with respect to the above ordering. There are 2^m m-dimensional simplexes in $S\sigma$, which can be seen as follows. The last vertex is $p_{0,r}$. It is preceded by $p_{1,r}$ and $p_{0,r-1}$. In general, p_{ij} is preceded by $p_{i+1,j}$ and $p_{j,i-1}$. The interiors of these simplexes are disjoint and $S\sigma$ is a simplicial complex. The diagram below shows standard subdivision of a tetrahedron. The simplexes of the subdivision correspond to paths going upwards in the graph.





Let $\tau_i = [p_0, p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_m]$ be an $(m-1)$-dimensional face of σ. The simplexes of $S\sigma$ which are contained in τ_i form a subdivision of τ_i. This subdivision is precisely $S\tau_i$, the standard subdivision of τ_i.

The last remark allows us to define the standard subdivision SL of any simplicial complex L. First we fix some ordering of the vertices of L. This gives an ordering of vertices of any simplex σ of L. We subdivide each simplex σ of L separately and get a subdivision SL of L. Each simplex of SL has ordered vertices and we can subdivide again. Inductively, we define

$$S_0 L = L, \qquad S_{n+1} L = S(S_n L) \tag{2.3}$$

Definition 2.4. *Let* $\sigma = [p_0, p_1, \ldots, p_m]$, $\sigma' = [p_0', \ldots, p_m']$ *be two simplexes in* $\mathbf{R}^m$. *We say that* σ *is strongly similar to* σ' *if there exists* $\lambda > 0$ *such that*

$$\lambda(\sigma - p_0) = \sigma' - p_0',$$

i.e., if σ' *can be obtained from* σ *by translation, multiplication by positive constant, and translation.*

Obviously, strong similarity is an equivalence relation. The following Lemma explains why standard subdivisions are better than barycentric subdivisions.

LEMMA 2.5. *Let $\mathfrak{A}$ be the set of all m-dimensional simplexes occurring in all complexes $S_n\sigma$, $n=0,1,2,\ldots$, where σ is an m-dimensional simplex in $\mathbf{R}^m$. Then, there are finitely many classes of strongly similar simplexes in $\mathfrak{A}$.*

Proof. For $\tau=[q_0,\ldots,q_m]\subset\mathbf{R}^m$ we define the sequence of edge vectors of τ by setting

$$w_i=q_{i+1}-q_i,\qquad 0\leqslant i<m. \tag{2.6}$$

If $w_0',\ldots,w_m'$ is the sequence of edge vectors of another simplex σ', then the two simplexes are strongly similar if and only if there exists $\lambda>0$ such that $w_i=\lambda w_i'$ for $i=0,1,\ldots,m-1$.

Suppose $\sigma=[p_0,\ldots,p_m]$. Its edge vectors are v_i. Let τ be any m-simplex of $S\sigma$. The edge vectors of τ belong to the set $\{\pm(1/2)v_1,\pm(1/2)v_2,\ldots,\pm(1/2)v_{m-1}\}$ by (2.1), (2.2), and (2.6). Inductively, we see that, if τ is an m-simplex of $S_n\sigma$, the edge vectors of τ are contained in $\{\pm(1/2^n)v_1,\pm(1/2^n)v_2,\ldots,\pm(1/2^n)v_{m-1}\}$. This finishes the proof.

COROLLARY 2.7 (*of the proof*): *Fix an inner product in $\mathbf{R}^m$. Let*

$$\epsilon_n(\sigma)=\sup\operatorname{diam}\tau, \tag{2.8}$$

where sup is taken over all simplexes τ of $S_n\sigma$. Then

$$\lim_{n\to\infty}\epsilon_n(\sigma)=0.$$

Proof. $\epsilon_n(\sigma)\leqslant(1/2^n)\Sigma_{i=0}^{m-1}\|v_i\|$.

3. Approximation Theorem. We return to the setting of Section 1. Observe that we can define the Whitney map $W_n:C^q(S_nK)\rightarrow L^2\Lambda^q$ for the complex S_nK. The only properties of K required in the construction of W were that it be a complex of C^∞ triangulation of X and that the vertices of every simplex be ordered. S_nK has these properties for all $n>0$. Observe further that the results of section 1 hold for every W_n.

Let $R_n:\Lambda^q\rightarrow C^q(S_nK)$ be the de Rham map

$$(R_nf,a)=\int_a f,\qquad f\in\Lambda^q,\quad a\in C_q(K). \tag{3.1}$$

We want to show that, for any $f\in\Lambda^q$, W_nR_nf is a good approximation to f provided n is large.

Definition 3.2. *We say that an m-dimensional simplex σ in $\mathbf{R}^m$ is well placed if it is strongly similar to the simplex $[0,e_1,\ldots,e_m]$, where $e_1,\ldots,e_m$ is the standard basis of $\mathbf{R}^m$.*

Let (U,φ) be a coordinate chart of X, i.e., $U\subset X$ is open and $\varphi: U\to\mathbf{R}^N$ is a diffeomorphism onto its image.

Definition 3.3. *We say that an N-simplex σ of a smooth triangulation of X is well placed in a coordinate chart (U,φ) if*

(a) $\bar\sigma\subset U$
(b) $\varphi|\bar\sigma:\bar\sigma\to\mathbf{R}^N$ *is linear*
(c) $\varphi(\sigma)$ *is well placed in* $\mathbf{R}^N$.

The following lemma is a consequence of Lemma 2.5.

LEMMA 3.4. *There exists a finite set $\mathfrak{U}$ of coordinate charts of X with the following property. For every integer $n\geq 0$ and every N-dimensional simplex τ of S_nK there exist a coordinate chart $(U,\varphi)\in\mathfrak{U}$ and an N-simplex σ of K such that*

(a) τ *is well placed in* (U,φ)
(b) $\tau\subset\bar\sigma\subset U$.

Proof. By definition of C^∞ triangulation there exists a homeomorphism $\chi: K\to X$ such that for every N-simplex of K there exists a coordinate chart $(U_\sigma,\varphi_\sigma)$ such that $\chi(\bar\sigma)\subset U_\sigma$ and $\varphi_\sigma\circ\chi$ maps $\bar\sigma$ into $\mathbf{R}^N$ linearly. We identify K with X via χ. Since K is a finite complex, it suffices to construct a finite set $\mathfrak{U}_\sigma$ of coordinate charts (U_σ,φ) such that, for every N-simplex τ of $S_n\sigma$, τ will be well placed in some $(U_\sigma,\varphi)\in\mathfrak{U}_\sigma$. We then would set $\mathfrak{U}=\cup_\sigma\mathfrak{U}_\sigma$. The existence of $\mathfrak{U}_\sigma$ is just a restatement of Lemma 2.5.

Definition 3.5. *Let*

$$\eta_n=\sup_{\sigma\in S_nK}\operatorname{diam}\sigma$$

where $\operatorname{diam}\sigma$ is measured in the metric induced by the euclidean distance in a coordinate neighborhood in which σ is well placed. We call η_n the mesh of S_nK.

LEMMA 3.6. $\lim_{n\to\infty}\eta_n=0$.

Proof. Since both K and $\mathfrak{U}$ are finite, this is a consequence of 2.7.

Let $\Lambda^qT^*(X)_p$ be the qth exterior power of a cotangent space to X at p. The Riemannian structure induces inner product $(\ ,\ )_p$ and norm $\|\ \ \|_p$ on $\Lambda^qT^*(X)_p$.

We are now ready to state the approximation theorem.

THEOREM 3.7. *Let f be a C^∞ q-form on X. There exists a constant C_f independent of n such that*

$$\|f(p) - W_n R_n f(p)\|_p \leqslant C_f \cdot \eta_n$$

almost everywhere on X.

Proof. We fix n. The $(N-1)$-dimensional skeleton of $S_n K$ has measure zero and, therefore, we can assume that p lies in the interior of a unique N-simplex σ of $S_n K$. Let (U, φ) be a coordinate chart in which σ is well placed. In U

$$f = \sum a_{i_1 \cdots i_q} dx_{i_1} \wedge \cdots \wedge ds_{i_q} \tag{3.8}$$

$$W_n R_n f = \sum a'_{i_1 \cdots i_q} dx_{i_1} \wedge \ldots \wedge dx_{i_q}. \tag{3.9}$$

In view of finiteness of U, it suffices to prove that there exist a constant C'_f independent of n and p such that

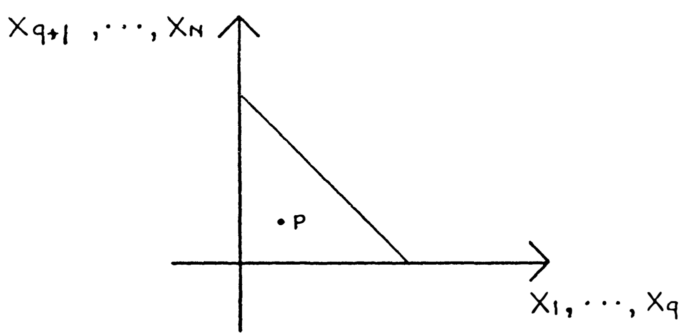
$$|a_{i_1 \ldots i_q}(p) - a'_{i_1 \ldots i_q}(p)| \leqslant C'_f \cdot \eta_n, \tag{3.10}$$

for all $i_1 \ldots i_q$. Furthermore, we can assume that $f = a_{i_1 \ldots i_q} dx_{i_1} \wedge \ldots \wedge dx_{i_q}$ for some $i_1 \ldots i_q$. Renumbering, if necessary, we can assume that

$$f = a\, dx_1 \wedge \ldots \wedge dx_q \tag{3.11}$$

in U. We identify U with a subset of $\mathbf{R}^N$ by means of φ. The simplex σ is well placed in $\mathbf{R}^N$ and, translating to the origin if necessary, we can assume that

$$\sigma = [0, he_0, \ldots, he_N] \qquad \text{for some} \quad h > 0. \tag{3.12}$$



Now we have to compute W_nR_nf explicitly in terms of local coordinates. By (1.3) the values of W_nR_nf on σ depend only on the values of R_nf on the faces of σ. The only q-dimensional faces τ of σ such that

$$\langle R_nf,\tau\rangle=\int_\tau f\neq 0 \tag{3.13}$$

are

$$\begin{aligned}\tau_0&=[0,he_1,\dots,he_q]\\ \tau_s&=[he_s,he_1,\dots,he_q]\qquad s=q+1,\dots,N\end{aligned} \tag{3.14}$$

Set

$$\begin{aligned}\alpha_0&=\int_{\tau_0}f=\int_{\tau_0}a(x_1,\dots,x_q,0,\dots,0)\,dx_1\,dx_2\dots dx_q\\ \alpha_s&=\int_{\tau_s}f=\int_{\tau_0}a\Big(x_1,\dots,x_q,0,\dots,0,\overbrace{h\cdot\Big(1-\frac{1}{h}\sum_{i=1}^{q}x_i\Big)}^{s},0,\dots,0\Big)\,dx_1\,dx_2\dots dx_q\end{aligned} \tag{3.15}$$

Thus

$$W_nR_nf=\alpha_0W_n\tau_0+\sum_{s=q+1}^{N}\alpha_sW_n\tau_s\quad\text{on}\quad\sigma. \tag{3.16}$$

The barycentric coordinates corresponding to the vertices $0,he_1,\dots,he_n$ of σ are

$$\begin{aligned}\mu_0&=1-\frac{1}{h}\sum_{i=1}^{N}x_i\\ \mu_i&=\frac{x_i}{h}\qquad i=1,2,\dots,N\end{aligned} \tag{3.17}$$

respectively. We now introduce some notation. Let

$$\begin{aligned}dx^q&=dx_1\wedge\dots\wedge dx_q\\ dx^{i,s}&=dx_1\wedge\dots\wedge dx_{i-1}\wedge dx_{i+1}\wedge\dots\wedge dx_q\wedge dx_s,\\ &\qquad 1\leq i\leq q<s\leq N\\ A&=\frac{q!}{h^q}=\Big(\int_{\tau_0}dx_1\,dx_2\dots dx_q\Big)^{-1}.\end{aligned} \tag{3.18}$$

By (1.2)

$$W_n\tau_0 = q!\cdot\left(1-\frac{1}{h}\sum_{i=1}^{N}x_i\right)\cdot\frac{1}{h^q}dx_1\wedge\ldots\wedge dx_q$$

$$+q!\sum_{i=1}^{q}(-1)^i\frac{x_i}{h}\left(-\frac{1}{h}\sum_{i=1}^{N}dx_i\right)\frac{1}{h^{q-1}}dx_1\wedge\ldots\overset{i}{\vee}\ldots\wedge dx_q$$

$$=A\cdot dx^q - A\cdot\frac{1}{h}\cdot\sum_{i=1}^{N}x_i dx^q$$

$$+A\cdot\sum_{i=1}^{q}(-1)^{i+1}\frac{x_i}{h}\cdot\left((-1)^{i-1}dx^q+\sum_{s=q+1}^{N}(-1)^{q-1}dx^{i,s}\right)$$

$$=A\cdot dx^q - A\cdot\sum_{s=q+1}^{N}\frac{x_s}{h}dx^q + A\cdot\sum_{i=1}^{q}\sum_{s=q+1}^{N}(-1)^{i+q}\frac{x_i}{h}dx^{i,s} \tag{3.19}$$

Similarly

$$W_n\tau_s = q!\cdot\left[\frac{x_s}{h}\frac{1}{h^q}dx^q+\sum_{i=1}^{q}(-1)^q\frac{x_i}{h}\frac{1}{h^q}dx_s\wedge dx_1\wedge\ldots\overset{i}{\vee}\ldots\wedge dx_q\right]$$

$$=A\cdot\frac{x_s}{h}dx^q + A\cdot\sum_{i=1}^{q}(-1)^{q+s-1}\frac{x_i}{h}dx^{i,s}\qquad q<s\leqslant N. \tag{3.20}$$

Substituting (3.19) and (3.20) into (3.16) we get

$$W_nR_nf=\alpha_0 A dx^q+\sum_{s=q+1}^{N}A\cdot(\alpha_s-\alpha_0)\frac{x_s}{h}dx^q$$

$$+\sum_{i=1}^{q}\sum_{s=q+1}^{N}(-1)^{i+s}A\cdot(\alpha_0-\alpha_s)\cdot\frac{x_i}{h}dx^{i,s} \tag{3.21}$$

We now have to compare (3.21) with (3.11). We show that $\alpha_0\cdot A$ is a good approximation of $a(p)$ and that the numbers $A\cdot(\alpha_0-\alpha_s)\cdot(x_i/h)$ are small. Lemma 3.4(a) implies that we can find bounds for derivatives of the function $a(x_1,\ldots,x_N)$ in the neighborhood of p and these bounds are independent of n.

Let $p=(\mathring{x}_1,\ldots,\mathring{x}_N)$. By the mean value theorem and (3.18)

$$A\cdot\alpha_0=\frac{\int_{\tau_0}a(x_1,\ldots,x_q,0,\ldots,0)\,dx_1\ldots dx_q}{\int_{\tau_0}dx_1\ldots dx_q}=a(x_1',\ldots,x_q',0,\ldots,0) \tag{3.22}$$

for some point $(x_1',\ldots,x_q',0,\ldots,0)\in\overline{\tau_0}$. Therefore

$$|a(\mathring{x}_1,\ldots,\mathring{x}_N)-A\cdot\alpha_0| =|a(\mathring{x}_1,\ldots,\mathring{x}_N)-a(x_1',\ldots,x_q',0,\ldots,0)|\leqslant c_1\cdot\eta_n, \tag{3.23}$$

where the constant c_1 is independent of n.

Similarly

$$\begin{aligned}\left|\frac{A\cdot(\alpha_0-\alpha_s)}{h}\right| &=\left|\frac{\int_{\tau_0}a(x_1,\ldots,x_q,0,\ldots,0)-a\left(x_1,\ldots,x_q,0,\ldots,\overset{s}{\overbrace{h\cdot\left(1-\frac{1}{h}\sum_i^q x_i\right)}},\ldots,0\right)}{\int_{\tau_0}dx_1\ldots dx_q}\right|\\ &=\left|\frac{a(x_1',\ldots,x_q',0,\ldots,0)-a\left(x_1',\ldots,x_q',0,\ldots,0,\overset{s}{\overbrace{h\cdot\left(1-\frac{1}{h}\sum_1^q x_i\right)}},0,\ldots,0\right)}{h}\right|\\ &\leqslant c_2.\end{aligned} \tag{3.24}$$

On $\bar{\sigma}\leqslant x_i\leqslant h\leqslant\eta_n$ and

$$\left|\frac{A\cdot(\alpha_0-\alpha_s)}{h}\cdot x_1\right|\leqslant c_2\cdot\eta_n. \tag{3.25}$$

This implies (3.10) which proves the theorem.

We recall that $L^2\Lambda^q$ is a Hilbert space whose norm $\|\ \ \|$ is given by

$$\|f\|=\left(\int_X f\wedge *f\right)^{1/2}=\left(\int_X\|f(p)\|_p^2dV\right)^{1/2},\qquad f\in L^2\Lambda^q, \tag{3.26}$$

where dV is the volume element of X.

COROLLARY 3.27. *Let $f \in \Lambda^q$. There exists a constant c_f independent of n such that.*

$$\|f - W_n R_n f\| \leqslant c_f \cdot \eta_n.$$

Proof. By (3.7) and (3.26) we have

$$\|f - W_n R_n f\|^2 = \int_X \|f(p) - W_n R_n f(p)\|_p^2 dV$$
$$\leqslant \left(\int_X dV\right) \cdot C_f^2 \cdot \eta_n^2.$$

Setting $c_f = c_f \cdot (\int_X dV)^{1/2}$ and taking square root we get

$$\|f - W_n R_n f\| \leqslant c_f \cdot \eta_n.$$

Remark. (1) The proof shows that approximation theorem holds if we assume that f, X, and the triangulation are of class C^2, and the Riemannian metric is merely continuous.

(2) The constant C_f is a product of a universal constant (depending only on the manifold and initial triangulation K) and the maximum of absolute values of first derivatives of the components of f in coordinate systems of $\mathfrak{U}$. This is also a consequence of the proof.

(3) Only Corollary 3.27 will be used in what follows.

4. Inner Product in Cochain Spaces. Combinatorial and Continuous Hodge Theories. In this section we define an inner product in cochain spaces $C^q(S_n K, S_n L_1)$. This gives rise to a combinatorial Hodge theory in every complex $C^*(S_n K, S_n L_1)$. This idea goes back to Eckmann [3]. Next we show that the Hodge theory of forms in $\Lambda_1 = \Sigma \Lambda_1^q$ is, in a sense, the limit of combinatorial Hodge theory.

Definition 4.1. *Let c, c' be elements of $C^q(S_n K, S_n L_1)$. We define their inner product (c, c') by*

$$(c, c') = \int_S Wc \wedge * Wc' = (Wc, Wc').$$

We do not indicate dependence on n and q, and we use the same symbol to denote the inner products in $C^*(S_n K, S_n L_1)$ and Λ because it will not cause any confusion.

Note that (4.1) indeed defines an inner product. Only nondegeneracy is not obvious but it is a consequence of (1.7).

Consider the complex $C^*(S_nK, S_nL_1)$ for a fixed n.

$$0 \longrightarrow C^0(S_nK, S_nL_1) \xrightarrow{d_n} \cdots \xrightarrow{d_n} C^N(S_nK, S_nL_1) \longrightarrow 0, \tag{4.2}$$

where d_n denotes the simplicial coboundary. Let δ_n be the adjoint of d_n,

$$(d_nc, c') = (c, \delta_nc'), \qquad c \in C^q(S_nK, S_nL_1) \quad c' \in C^{q+1}(S_nK, S_nL_1). \tag{4.3}$$

Now set

$$\Delta_n = d_n\delta_n + \delta_nd_n \tag{4.4}$$

on $C^q(S_nK, S_nL_1)$, $q = 0, 1, \ldots, N$. Let H_n^q, the space of harmonic cochains in $C^q(S_nK, S_nL_1)$, be the kernel of $\Delta_n | C^q(S_nK, S_nL_1)$.

The following statements are well known and very easy to verify.

$$H_n^q = \{c \in C^q(S_nK, S_nL_1) | d_nc = 0, \delta_nc = 0\} \tag{4.5}$$

For $0 \leqslant q \leqslant N$

$$C^q(S_nK, S_nL_1) = d_n(C^{q-1}(S_nK, S_nL_1)) \oplus H_n^q \oplus \delta_n(C^{q+1}(S_nK, S_nL_1)), \tag{4.6}$$

and this direct sum is orthogonal.

$$H^q(X, M_1; \mathbf{R}) \cong H^q(C^*(S_nK, S_nL_1)) \cong H_n^q \qquad \text{for} \quad 0 \leqslant q \leqslant N. \tag{4.7}$$

The above setup is formally analogous to the Hodge theory of forms on X which we now describe (for precise statements and proofs see [7], particularly Corollary 5.7, p. 178).

First, we recall that Λ_1^q is the space of C^∞ q-forms f on X which satisfy boundary conditions $f_{\tan} = 0$ on M_1 and $f_{\text{norm}} = 0$ on M_2.

If $f \in \Lambda_1^q$ then f has the following Hodge decomposition

$$f = dg + h + \delta k, \tag{4.8}$$

where $g \in \Lambda_1^{q-1}$, $k \in \Lambda_1^{q+1}$, $dg \in \Lambda_1^q$, $\delta k \in \Lambda_1^q$, $h \in \Lambda_1^q$ and h is harmonic, i.e., $\delta h = dh = 0$. The summands in (4.8) are mutually orthogonal and the space H^q of harmonic q-forms is mapped one-to-one onto a linear space of cocycles representing $H^q(C^*(S_nK, S_nL_1)$ by the de Rham map R_n.

THEOREM 4.9. *Let $f \in \Lambda_1^q$, so that $R_n f$ is an element of $C^q(S_n K, S_n L_1)$. Let*

$$R_n f = d_n g_n + h_n + \delta_n k_n$$

be the Hodge decomposition of $R_n f$. If (4.8) is the Hodge decomposition of f, then

$$\lim_{n\to\infty} W_n h_n = h, \quad \lim_{n\to\infty} W_n d_n g_n = dg, \quad \lim_{n\to\infty} W_n \delta_n k_n = \delta k,$$

where the limits are in the norm of $L^2\Lambda^q$.

In particular, if $h_{\alpha_n} \in H_n^q$ represents a fixed class $\alpha \in H^q(X, M_1; \mathbf{R})$ and $h_\alpha \in H^q$ also represents α, then

$$\lim_{n\to\infty} W_n h_{\alpha_n} = h_\alpha. \tag{4.10}$$

Proof. Observe first that $R_n f$, is indeed, in $C^q(S_n K, S_n L_1)$ because $f_{\tan} = 0$ on M_1. We first prove (4.9) under the assumption that $f = h$ is harmonic. The cochain $R_n h$ is a cocycle by Stokes' theorem. It represents the same cohomology class α as h. Therefore

$$R_n h = h_{\alpha_n} + d_n g_n \tag{4.11}$$

with $h_{\alpha_n} \in H_q^n$ representing α. By (1.8) and (1.4)

$$(W_n d_n g_n, h) = (dW_n g_n, h) = (W_n g_n, \delta h) = 0. \tag{4.12}$$

On the other hand

$$\|h - W_n R_n h\| \leqslant c_h \cdot \eta_n$$

by (3.27). Therefore

$$\begin{aligned} c_h^2 \cdot \eta_n^2 &\geqslant \|h - W_n R_n h\|^2 = \|h - W_n h_{\alpha_n} - W_n d_n g_n\|^2 \\ &= \|h - W_n h_{\alpha_n}\|^2 + \|W_n d_n g_n\|^2 \end{aligned} \tag{4.13}$$

because $(d_n g_n, h_{\alpha_n}) = (W_n d_n g_n, W_n h_{\alpha_n}) = 0$ by (4.1) and (4.6). This proves the theorem for a harmonic form f and also, proves (4.10).

Now we consider the case of an arbitrary $f \in \Lambda_1^q$. Let P_n be the orthogonal projection of $C^q(S_n K, S_n L_1)$ onto the subspace of cocycles. We first estimate the norm of $P_n R_n \delta k$. Let $c \in C^q(S_n K, S_n L_1)$ be a cocycle. Then

$$\begin{aligned} |(P_n R_n \delta k, c)| &= |(R_n \delta k, c)| = |(W_n R_n \delta k, W_n c)| \\ &= |(W_n R_n \delta k - \delta k, W_n c)| \end{aligned} \tag{4.14}$$

because $(\delta k, W_n c) = 0$ by (1.4) and (1.8). By the Corollary 3.27

$$\begin{aligned} |(P_n R_n \delta k, c)| &\leqslant \| W_n R_n \delta k - \delta k\| \cdot \|c\| \\ &\leqslant c_{\delta k} \cdot \eta_n \cdot \|c\|. \end{aligned} \tag{4.15}$$

This implies that

$$\|P_n R_n \delta k\| \leqslant c_{\delta k} \cdot \eta_n. \tag{4.16}$$

Now we can complete the proof. We have

$$W_n R_n f = W_n R_n dg + W_n R_n h + W_n R_n \delta k \tag{4.17}$$

and

$$\begin{aligned} \| W_n R_n dg - dg\| &\leqslant c_{dg} \cdot \eta_n \\ \| W_n R_n h - h\| &\leqslant c_h \cdot \eta_n \\ \| W_n R_n \delta k - \delta k\| &\leqslant c_{\delta k} \cdot \eta_n. \end{aligned} \tag{4.8}$$

By the first part of the proof (case of harmonic form) we know that

$$R_n h = h_n' + \epsilon_1 \tag{4.19}$$

where $h_n' \in H_n^q$ and $\|\epsilon_1\| \leqslant c_h \cdot \eta_n$. Moreover

$$R_n \delta k = P_n R_n \delta k + R_n \delta k - P_n R_n \delta k \tag{4.20}$$

where $R_n \delta k - P_n R_n \delta k = \delta_n k_n'$ by (4.6) and $\|P_n R_n \delta k\| \leqslant c_{\delta k} \cdot \eta_n$. We write $\epsilon_2 = P_n R_n \delta k$. From (4.8), (4.19), (4.20) we get

$$R_n f = R_n dg + h_n' + \delta_n k_n' + \epsilon_1 + \epsilon_2 \tag{4.21}$$

Subtracting (4.21) from the Hodge decomposition of $R_n f$ we see that

$$(\epsilon_1 + \epsilon_2) = (h_n - h_n') + (\delta_n k_n - \delta_n k_n') + (d_n g_n - R_n dg) \tag{4.22}$$

which implies that

$$\begin{aligned} \|h_n - h_n'\| &\leqslant c_1 \eta_n \\ \|\delta_n k_n - \delta_n k_n'\| &\leqslant c_2 \eta_n \\ \|d_n g_n - R_n dg\| &\leqslant c_3 \eta_n \end{aligned} \tag{4.23}$$

For some constants c_1, c_2, $c_3 > 0$.

Finally

$$\begin{aligned}\|h - W_n h_n\| &\leqslant \|h - W_n R_n h\| + \|W_n R_n h - W_n h_n\| \\ &= \|h - W_n R_n h\| + \|R_n h - h_n\| \\ &\leqslant \|h - W_n R_n h\| + \|h_n' - h_n\| + \|\epsilon_1\| \leqslant c' \cdot \eta_n \end{aligned} \tag{4.24}$$

by (4.19), (4.23), and (3.27).

Similarly

$$\begin{aligned}\|dg - W_n d_n g_n\| &\leqslant \|dg - W_n R_n dg\| + \|W_n R_n dg - W_n d_n g_n\| \\ &= \|dg - W_n R_n dg\| + \|R_n dg - d_n g_n\| \leqslant c'' \cdot \eta_n \end{aligned} \tag{4.25}$$

by (4.23) and (3.27).

$$\begin{aligned}\|\delta k - W_n \delta_n k_n\| &\leqslant \|\delta k - W_n R_n \delta k\| + \|W_n R_n \delta k - W_n \delta_n k_n\| \\ &\leqslant \|\delta k - W_n R_n \delta k\| + \|R_n \delta k - \delta_n k_n\| \\ &\leqslant \|\delta k - W_n R_n \delta k\| + \|\delta_n k_n' - \delta_n k_n\| + \|\epsilon_2\| \\ &\leqslant c''' \cdot \eta_n \end{aligned} \tag{4.26}$$

by (4.20), (4.23), and (3.27).

The inequalities (4.24), (4.25), (4.26) prove the theorem.

Remark. Our methods differ from the classical finite-difference technique of solving partial differential equations in two aspects. In the first place, we use the inner product in cochain spaces to obtain finite-dimensional approximation to the operators δ and Δ. In classical numerical analysis one substitutes difference quotients for the derivatives to obtain such approximations. This brings us to the second difference. Because of the way the approximations are obtained in numerical analysis, it is usually easy to estimate the difference between an operator and its approximation. The approximation is called consistent if this difference is $0(h^n)$ as $h \to 0$, where n is the order of the operator and h is the mesh. Then, one considers only consistent approximations. In contrast to this we have no consistency result. This is one of the reasons why our technique applies only to homogeneous Laplace equations. More precisely, we do not know if, for an arbitrary smooth form f, the solutions of $\Delta_n c_n = R_n f$ in $C^q(S_n K)$ are such that $W_n c_n \to u \in \Lambda^q$, where $\Delta u = f$.

5. Eigenvalues of the Laplacian Acting on Functions. In this section we show that the eigenvalues of the Laplacian Δ acting on smooth functions satisfying certain boundary conditions are limits of the eigenvalues of combinatorial Laplacians $\Delta_n : C^0(S_n K, S_n L_1) \to C^0(S_n K, S_n L_1)$.

It is known (see [1]) that there exists a complete orthonormal system $\{\varphi_i\}_{i=1}^{\infty}$ in the space $L^2(X)=L^2\Lambda^0$ such that, for all i,

$$\begin{aligned} &\varphi_i \in C^{\infty}(X), \quad \Delta\varphi_i=\lambda_i\varphi_i \\ &\varphi_i|M_1=0, \qquad (d\varphi_i)_{\text{norm}}=0 \quad \text{on} \quad M_2. \end{aligned} \tag{5.1}$$

Each eigenvalue has finite multiplicity and they can be numbered so that $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_n \to \infty$.

The mini-max principle of Courant [2] says that

$$\lambda_i = \sup_{f_1,\ldots,f_{i-1}} \inf_{f \perp f_1, f_2, \ldots, f_{i-1}} \frac{(df,df)}{(f,f)}, \tag{5.2}$$

$f_1,\ldots,f_{i-1}, f$ are smooth functions vanishing on M_1 and $f\neq 0$.

Let $C_1^{\infty}(X)=\{f\in C^{\infty}(X)|f|M_1=0\}$. Since we want to compare the eigenvalues of Δ with the eigenvalues of Δ_n, it would be convenient to have a space which contains both $C_1^{\infty}(X)$ and $W_nC^0(S_nK, S_nL_1)$.

Definition 5.3. *The function* $f \mapsto (f,f)+(df,df)$ *is an inner product on* $C^{\infty}(X)$. *Let* H_1 *be the completion of* $C^{\infty}(X)$ *with respect to the norm given by this inner product. Let* V *be the closure of* $C_1^{\infty}(X)$ *in* H_1.

Remark. Our definition of the Sobolev space H_1 differs from the usual one. However, since X is compact and has smooth boundary, it is equivalent to the usual definition (see [1], Theorems 2.1, 2.2).

Note that the exterior derivative d extends to the mapping $\bar{d}: H_1 \to L^2\Lambda^1$ in the obvious way.

LEMMA 5.4. *For all* $n \geqslant 0$

$$W_nC^0(S_nK, S_nL_1)\subset V.$$

Moreover, for $c\in C^0(S_nK, S_nL_1)$,

$$W_nd_nc=\bar{d}W_nc.$$

We postpone the proof of this lemma since it is rather technical.

Now we can write the mini-max principle in the following equivalent way.

$$\lambda_i = \sup_{f_1,\ldots,f_{i-1}\in V} \inf_{\substack{f\in V\setminus\{0\} \\ (f,f_k)=0,\ k=1,2,\ldots,i-1}} \frac{(\bar{d}f,\bar{d}f)}{(f,f)} \tag{5.5}$$

Let $d(n)$ be the dimension of $C^0(S_nK, S_nL_1)$. The eigenvalues λ_i^n of Δ_n are nonnegative and can be numbered so that

$$0 \leqslant \lambda_1^n \leqslant \lambda_2^n \leqslant \cdots \leqslant \lambda_{d(n)}^n. \tag{5.6}$$

We are now ready to state the main result of this section.

THEOREM 5.7. *Let i be a positive integer. There exists a constant $C_i > 0$ such that*

$$\lambda_i^n - C_i\eta_n \leqslant \lambda_i \leqslant \lambda_i^n$$

whenever $i \leqslant d(n)$.

Proof. The proof proceeds by the classical Rayleigh–Ritz method (see Gould [4]). The inequality $\lambda_i \leqslant \lambda_i^n$ is a special case of general principle due to Poincaré [6] which says roughly that the eigenvalues of a positive semidefinite quadratic form on a subspace are larger than the first eigenvalues of that form on the whole space.

To prove this inequality consider the Rayleigh-Ritz quotient

$$\frac{(\Delta_n c, c)}{(c,c)} = \frac{(d_n c, d_n c)}{(c,c)} = \frac{(W_n d_n c, W_n d_n c)}{(W_n c, W_n c)} \quad \text{on} \quad C^0(S_nK, S_nL_1)\setminus\{0\}. \tag{5.8}$$

By (5.4)

$$\frac{(\Delta_n c, c)}{(c,c)} = \frac{(\bar{d}W_n c, \bar{d}W_n c)}{(W_n c, W_n c)}. \tag{5.9}$$

The finite dimensional mini-max principle gives

$$\begin{aligned} \lambda_i^n &= \sup_{c_1,\ldots,c_{i-1} \in C^0(S_nK, S_nL_1)} \ \inf_{\substack{c \neq 0 \\ (c,c_k)=0,\ k=1,2,\ldots,i-1}} \frac{(\Delta_n c, c)}{(c,c)} \\ &= \sup_{f_1,\ldots,f_{i-1} \in W_nC^0(S_nK, S_nL_1)} \ \inf_{\substack{f \in W_nC^0(S_nK, S_nL_1) \\ f \neq 0,\ (f,f_k)=0,\ k=1,2,\ldots,i-1}} \frac{(\bar{d}f, \bar{d}f)}{(f,f)}. \end{aligned} \tag{5.10}$$

In (5.10) we can even allow $f_1,\ldots,f_{i-1}$ to range over V because we can replace them by their orthogonal projections on $W_nC^0(S_nK, S_nL_1)$.

Comparing (5.5) with (5.10) we see that $\lambda_i \leqslant \lambda_i^n$ because $W_nC^0(S_nK, S_nL_1) \subset V$ and infinum over a smaller set is larger.

To prove the second inequality we consider the space V_i spanned by first i eigenfunctions, $\varphi_1, \ldots, \varphi_i$, of Δ. It follows from Corollary 3.7 that $R_n\varphi_1, \ldots, R_n\varphi_i$ are linearly independent for n large. Also, the same corollary and Remark 2 following it, imply that there exists a constant c_i such that

$$\left| \frac{(d_nR_n\varphi, d_nR_n\varphi)}{(R_n\varphi, R_n\varphi)} - \frac{(d\varphi, d\varphi)}{(\varphi, \varphi)} \right| \leqslant c_i\eta_n \tag{5.11}$$

for $\varphi \in V_i \setminus \{0\}$.

Observe that the largest eigenvalues λ of the form (dc, dc) on R_nV_i is given by

$$\lambda = \sup_{\varphi \in V_i} \frac{(d_nR_n\varphi, d_nR_n\varphi)}{(R_n\varphi, R_n\varphi)}. \tag{5.12}$$

If n is so large that $\dim V_i = i$, the finite-dimensional analog of Poincaré principle gives

$$\lambda_i^n \leqslant \lambda \tag{5.13}$$

On the other hand the supremum in (5.12) is attained for some $\varphi_0 \in V_i$. Thus

$$\lambda = \frac{(d_nR_n\varphi_0, d_nR_n\varphi_0)}{(R_n\varphi_0, R_n\varphi_0)} \leqslant \frac{(d\varphi_0, d\varphi_0)}{(\varphi_0, \varphi_0)} + C_i\eta_n. \tag{5.14}$$

We must have $(d\varphi_0, d\varphi_0)/(\varphi_0, \varphi_0) \leqslant \lambda_i$ since λ_i is the largest eigenvalue of $(d\varphi, d\varphi) = (\Delta\varphi, \varphi)$ on V_i. The inequality $\lambda_i^n - C_i\eta_n \leqslant \lambda_i$ follows because

$$\lambda_i^n \leqslant \lambda \leqslant \lambda_i + C_i\eta_n. \tag{5.15}$$

In order to complete the proof of Theorem 5.7 we have to prove Lemma 5.4.

Proof of Lemma 5.4. Let $c \in C^0(S_nK, S_nL_1)$; c can be written as

$$c = \sum_{i=1}^{s} c_ip_i \qquad c_i \in \mathbf{R}, \quad p_i \in S_nK \setminus S_nL_1. \tag{5.16}$$

Therefore

$$W_nc = \sum_{i=1}^{s} c_i\mu_i, \tag{5.17}$$

where μ_i is the barycentric coordinate corresponding to p_i. The function $W_n c$ is continuous on X, piecewise C^∞ and $W_n c|M_1=0$. Let $U\approx M_1\times[0,2)$ be the collar neighborhood of M_1 in X. We identify points of U with pairs (x,t), $x\in M_1, t\in[0,2)$. In particular M_1 is identified with $M_1\times\{0\}$. $W_n c$ is continuous and piecewise C^∞, therefore Lipschitz on $M_1\times[0,1]$. Therefore there exists a constant C such that

$$|W_n c(x,t)|\leqslant C\cdot t \qquad \text{for all} \quad x\in M_1, \quad t\in[0,1]. \tag{5.18}$$

Let g be a C^∞ function on $\mathbf{R}$ satisfying

$$1\geqslant g\geqslant 0, \qquad g=0 \quad \text{on} \quad (-\infty,0], \qquad g=1 \quad \text{on} \quad [1,\infty). \tag{5.19}$$

For every integer $m>0$ we define $g_m(t)$ by

$$g_m(t)=g(2mt-1). \tag{5.20}$$

Note that

$$\begin{gathered} \operatorname{supp} g'_m\subset\left[\frac{1}{2m},\frac{1}{m}\right] \\ \sup_{t\in\left[\frac{1}{2m},\frac{1}{m}\right]}|g'_m(t)|\leqslant 2m\cdot\sup_{t\in[0,1]}|g'(t)|. \end{gathered} \tag{5.21}$$

For every positive integer m we define a function $\bar{g}_m$ on X by

$$\begin{aligned} &\bar{g}_m(x,t)=g_m(t) && \text{for } x\in M_1, \quad t\in[0,2) \\ &\bar{g}_m(p)=1 && \text{for } p\in X, \quad M_1\times[0,1] \end{aligned} \tag{5.22}$$

Obviously $\bar{g}_m$ is C^∞ on X.
Since V is closed in H_1 it is enough to show that

$$\begin{aligned} &\bar{g}_m\cdot W_n c\in V \qquad \text{for all} \quad m, \\ &\bar{g}_m\cdot W_n c\underset{m\to\infty}{\longrightarrow} W_n c, \\ &\bar{d}\,(\bar{g}_m W_n c)\underset{m\to\infty}{\longrightarrow}\bar{d}\,W_n c \end{aligned} \tag{5.23}$$

in $L^2(X)$ and $L^2\Lambda^1$ respectively.

The function $\bar{g}_m W_n c$ is continuous and piecewise C^∞. The reasoning of Lemma 1.8 can be applied to show that

$$(d(\bar{g}_m\cdot W_n c),f)=(\bar{g}_m\cdot W_n c,\delta f) \tag{5.24}$$

for every C^∞ 1-form whose support is contained in $X\setminus(M_1\cup M_2)$. Therefore the form $d(\bar{g}_m\cdot W_nc)$ is the weak exterior derivative of $\bar{g}_m\cdot W_nc$ in the sense of [1], Definition 1.6. By Theorem 2.2 of [1] $\bar{g}_m\cdot W_nc\in H_1$ and $d(\bar{g}_m\cdot W_nc)=\bar{d}(\bar{g}_m\cdot W_nc)$. But the support of $\bar{g}_m\cdot W_nc$ is contained in $X\setminus M_1\times[0,1/2m)$ which implies that $\bar{g}_m\cdot W_nc\in V$.

The convergence $\bar{g}_m\cdot W_nc\to W_nc$ is obvious since $g_m\leqslant 1$ and $\bar{g}_m\to 1$ almost everywhere on X by (5.19), (5.20). By Leibnitz rule ([1], Theorem 1.13) we have

$$\bar{d}(\bar{g}_m\cdot W_nc)=W_nc\cdot d\bar{g}_m+\bar{g}_m\cdot dW_nc. \tag{5.25}$$

On $X\setminus M_1\times[0,1/m]$ $\bar{g}_m=1$ and $d\bar{g}_m=0$. On $M_1\times[0,1/m]$ we have

$$\bar{d}(\bar{g}_m\cdot W_nc)=\bar{g}_m\cdot dW_nc+g'_m dt\cdot W_nc. \tag{5.26}$$

The proof will be concluded if we show that

$$W_nc\cdot g'_m dt=\bar{d}(\bar{g}_m\cdot W_nc)-\bar{g}_m\cdot dW_nc \underset{m\to\infty}{\longrightarrow} 0. \tag{5.27}$$

By (5.18) and (5.21)

$$|W_nc(x,t)\cdot g'_m(t)|\leqslant C\cdot\frac{1}{m}\cdot 2m=2C \tag{5.28}$$

for $x\in M_1$, $t\in[0,1]$. This means that $W_nc\cdot g'_m$ is bounded and, since we are integrating over sets whose measure tends to zero, it proves (5.27) and finishes the proof of the Lemma.

As an application of Theorem 5.7 we show that the zeta-function of the continuous Laplacian is the limit of zeta-functions of combinatorial Laplacians.

Definition 5.29. *The zeta-function $\zeta(s)$ of the Laplacian Δ is the Dirichlet series*

$$\zeta(s)=\sum_{\lambda_i\neq 0}\lambda_i^{-s}.$$

By analogy we define zeta-function of combinatorial Laplacian Δ_n by

$$\zeta^{(n)}(s)=\sum_{\lambda_i^n\neq 0}(\lambda_i^n)^{-s}.$$

The function $\zeta(s)$ was investigated by Minakshisundaram and Pleijel in [5]. They proved that the abscissa of absolute convergence of the series defining

$\zeta(s)$ is finite, i.e., there exists a number $\sigma_0 \in \mathbf{R}$, $\sigma_0 > 0$ such that $\Sigma_{\lambda_i \neq 0} \lambda_i^{-s}$ converges absolutely for every $s = \sigma + it$ with $\sigma > \sigma_0$. It is well known that in this situation convergence is uniform on every set $\{s = \sigma + it \mid \sigma \geqslant \sigma_0 + \delta, \delta > 0\}$.

THEOREM 5.30. *Let $H = \{s \in \mathbf{C} \mid \operatorname{Re} s > \sigma_0\}$. The sequence $\{\zeta^{(n)}(s)\}$ converges to $\zeta(s)$ uniformly on compact subsets of H.*

Proof. Let $K \subset H$ be compact. Fix $\epsilon > 0$. Let m be a positive integer such that

$$\sum_{i \geqslant m} |\lambda_i^{-s}| = \sum_{i \geqslant m} \lambda_i^{-\operatorname{Re} s} \leqslant \frac{\epsilon}{3} \tag{5.31}$$

for all $s \in K$.

For every n, since $\lambda_i^{(n)} \geqslant \lambda_i$, we have

$$|(\lambda_i^n)^{-s}| = (\lambda_i^n)^{-\operatorname{Re} s} \leqslant \lambda_i^{-\operatorname{Re} s} = |\lambda_i^{-s}|. \tag{5.32}$$

Therefore

$$\left| \sum_{i \geqslant m} \lambda_i^{-s} - \sum_{i=m}^{d(n)} (\lambda_i^n)^{-s} \right| \leqslant \frac{2}{3}\epsilon. \tag{5.33}$$

Also, since $\lambda_i^n \to \lambda_i$ we can find $n(\epsilon)$ such that for $n \geqslant n(\epsilon)$, $s \in K$

$$\left| \sum_{\substack{i \leqslant m \\ \lambda_i \neq 0}} \lambda_i^{-s} - \sum_{\substack{i \leqslant m \\ \lambda_i^n \neq 0}} (\lambda_i^n)^{-s} \right| \leqslant \frac{\epsilon}{3} \tag{5.34}$$

By (5.33) and (5.34)

$$|\zeta(s) - \zeta^{(n)}(s)| \leqslant \epsilon$$

provided $n \geqslant n(\epsilon)$, $s \in K$. This proves the theorem.

As remarked in the introduction, we conjecture that the results of this section hold for all dimensions $q = 0, 1, 2, \ldots, N$. The proofs for $q = 0$ do not apply to higher dimensions since the eigenvalues are critical points of the form $(df, df) + (\delta f, \delta f)$ and we do not know how well δ_n approximates δ.

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Added in proof. V. K. Patodi has recently proved the convergence of eigenvalues for all $q=0,1,2,\ldots,N$.

G. Strang informed us that the techniques used in this paper are very closely related to finite element method of solving partial differential equations numerically.

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