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Yang-Mills Theories as Deformations of Massive Integrable Models

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AS DEFORMATIONS

OF MASSIVE INTEGRABLE MODELS

by

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ABSTRACT

YANG-MILLS THEORIES AS DEFORMATIONS OF MASSIVE INTEGRABLE MODELS

by

Axel Cortés Cubero

Advisor: Prof. Peter Orland

Yang Mills theory in 2+1 dimensions can be expressed as an array of coupled (1+1)-dimensional principal chiral sigma models. The $SU(N) \times SU(N)$ principal chiral sigma model in 1+1 dimensions is integrable, asymptotically free and has massive excitations. We calculate all the form factors and two-point correlation functions of the Noether current and energy-momentum tensor, in 't Hooft's large-N limit (some form factors can be found even at finite N). We use these new form factors to calculate physical quantities in (2+1)-dimensional Yang-Mills theory, generalizing previous $SU(2)$ results from references [1], [2], [3], to $SU(N)$. The anisotropic gauge theory is related to standard isotropic one by a Wilsonian renormalization group with ellipsoidal cutoffs in momentum. We calculate quantum corrections to the effective action of QED and QCD, as the theory flows from isotropic to anisotropic. The exact principal chiral sigma model S-matrix is used to examine the spectrum of (1+1)-dimensional massive Yang Mills theory.

To my cousin, Juanma.

I am still just trying to be more like you, when I grow up.

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Contents

Chapter 1

Introduction

1.1 Introduction

Quantum chromodynamics (QCD) is the quantum field theory describing the interaction of quarks and gluons. The dynamics of hadrons and nuclei emerges from QCD. In QCD the interactions between particles are strong and the observed physical states are bound states of fundamental particles. Therefore to fully understand QCD, one needs methods beyond perturbation theory.

The most successful non-perturbative tool in QCD is numerical lattice simulation. Numerical methods provide accurate results in scenarios where perturbation theory fails. Unfortunately, we lack a mathematical route to complement them. Numerical simulations are invaluable for finding bound state masses and string tensions, but they do not elucidate qualitatively the mechanism of how bound states form or how quarks are confined.

In this thesis, we develop analytical methods to examine field theories, with an eye towards QCD. We discuss an anisotropic version of QCD, where the constants coupling the different vector components of the gluon field have different strengths. This is understood as a longitudinal rescaling of the coordinates. If two of the four space-time coordinates in QCD (say one space direction, x^3, and the time direction, x^0) are rescaled by a parameter λ (replacing them by λx^3 and λx^0, respectively), we obtain an action with new couplings dependent on λ. We consider the limit where this parameter λ goes to zero. We can perturb in powers of λ instead of powers of the coupling constant.

The anisotropic regime of QCD has important applications in the physics of heavy-ion collisions at RHIC and the LHC. Verlinde and Verlinde [5] have argued that the longitudinally rescaled action yields BFKL theory [6]. McLerran and Venugopalan used a similar idea to derive the Color-Glass Condensate picture [7].

We view the anisotropic model as an array of two-dimensional field theories, coupled together to form a higher-dimensional theory. The strength of the coupling between these two-dimensional models depends on the rescaling parameter λ. The two-dimensional theory is the principal chiral sigma model (PCSM) [4]. The PCSM is known to be integrable, and this property has been exploited to find exact results. The main goal of our program is to use exact results from the PCSM to calculate physical quantities in anisotropic QCD, finding corrections for small λ.

This approach is especially interesting for $(2+1)$-dimensional QCD, since there are two different coupling constants for the gluon field, but they are both small compared to the cutoff. This makes our approach fundamentally different from other analytic studies of $(2+1)$-dimensional QCD (which are generally at large dimensionless coupling, far from the continuum limit) [8], [9].

In the remainder of this chapter, we give a brief introduction to QCD and Yang-Mills theory in 3+1, and 2+1 space-time dimensions, both in the continuum and on a lattice.

In Chapter 2, we discuss the Hamiltonian of longitudinally-rescaled QCD. In 2+1 dimensions, we show how the anisotropic Hamiltonian is equivalent to an array of PCSM's.

In Chapter 3, we present a short introduction to the integrable bootstrap program in 1+1 dimensions. We discuss the general procedure of calculating exact S-matrices using elasticity and factorization. The S-matrix and some other assumptions can be used to calculate exact form factors and correlation functions.

In Chapter 4, we apply the integrable bootstrap program to the PCSM. This is very difficult, in general, unless we take 't Hooft's large-N limit. We calculate all the exact form factors of the Noether current and energy-momentum tensor operators. These form factors are then used to calculate the exact two-point function of these operators. For finite N, we are only able to find the first nontrivial form factor.

In Chapter 5, we study the $(1+1)$-dimensional massive Yang-Mills theory. This model can be understood as a gauged PCSM. Using the exact PCSM S-matrix, we calculate the mass spectrum.

In Chapter 6, we use the PCSM S-matrix and the exact finite-N current form factor to compute some physical quantities in (2+1)-dimensional anisotropic Yang-Mills theory. We first compute the low-lying glueball spectrum (which is very similar to the massive spectrum from Chapter 5). We then compute the string tension for a static quark-antiquark pair. Because the theory is anisotropic, the string tension will be different if the quarks are separated in the x^1- or the x^2-direction. These $SU(N)$ results generalize those obtained by Orland for $N=2$.

In Chapter 7, we explore the quantum effects of longitudinal rescaling. Rescaling the coordinates changes the ultraviolet cutoffs of the theory. We therefore need a renormalization group to see how the couplings run in the quantum theory as we go from isotropic to anisotropic momentum cutoffs. To illustrate this idea, we first examine quantum electrodynamics in 3+1 dimensions, with massless electrons. We then apply the anisotropic-renormalization group to (3+1)-dimensional QCD, with massless quarks.

We summarize our results and discuss possible future projects in the last chapter.

1.2 QCD and Yang-Mills theory

In this section, we present the QCD action and Hamiltonian, as well as some notation and conventions that will be used throughout this thesis.

The Dirac action for N free massless fermions in $3+1$ dimensions is

$$S_{\rm Dirac} = i \int d^4x \, \bar{\psi}^a \gamma^\mu \partial_\mu \psi^a, \tag{1.2.1}$$

where γ^μ are the Dirac matrices, which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$, where $\eta^{\mu\nu}$ is the Minkowski metric with $\mu, \nu = 0, 1, 2, 3$, and $a = 1, 2, \ldots, N$ (where $N = 3$ in quantum chromodynamics). The action (1.2.1) is invariant under the global $SU(N)$ transformation, $\psi(x) \to V\psi(x)$, where $V \in SU(N)$. This symmetry is promoted to a local gauge symmetry by introducing an $SU(N)$-algebra-valued gauge field $A_\mu(x)^{ab}$ that transforms as $A_\mu(x)^{ab} \to V^\dagger(x)^{ca} A_\mu(x)^{cd} V(x)^{db} - \frac{i}{g_0} V^\dagger(x)^{ca} \delta^{cd} \partial_\mu V(x)^{db}$. Introducing the covariant derivative $D_\mu(x)^{ab} = \delta^{ab}\partial_\mu - ig_0 A_\mu(x)^{ab}$, the gauged Dirac action is

$$S_{\rm Dirac} = i \int d^4x \, \bar{\psi}^a \gamma^\mu D_\mu^{ab} \psi^b. \tag{1.2.2}$$

The gauge field can be made dynamical by adding a Yang-Mills term to the action:

$$S_{\mathrm{YM}} = -\int d^4x \frac{1}{4} \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}, \tag{1.2.3}$$

where $F_{\mu\nu} = \frac{i}{g_0}[D_\mu, D_\nu]$. The gauge field can be expressed in terms of the generators of $SU(N)$, $A_{ab} = A^\alpha\, t^\alpha_{ab}$, where $\alpha = 1, 2, \ldots, N^2 - 1$, and the generators satisfy the lie algebra, $[t^\alpha, t^\beta] = i f^{\alpha\beta\gamma} t^\gamma$, where $f^{\alpha\beta\gamma}$ are the $SU(N)$ structure constants. We normalize the generators by $\mathrm{Tr}\, t^\alpha\, t^\beta = \delta^{\alpha\beta}$. Henceforth we will rescale the gauge field as $A_\mu \to \frac{1}{g_0} A_\mu$, so that $S_{\mathrm{YM}} = -\int d^4x \frac{1}{4g_0^2} \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$

We now find the Hamiltonian that corresponds to the Yang-Mills action (1.2.3). We will work in the temporal gauge, $A_0 = 0$. This is an incomplete gauge fixing, which still allows time-independent gauge transformations. In the next chapter we restrict this remaining gauge freedom, by imposing the additional axial gauge condition, $A_1 = 0$ on the Hilbert space.

The Yang-Mills action in temporal gauge is

$$S_{\mathrm{YM}} = \int d^4x \left(\sum_{i=1}^{3} \frac{1}{2g_0^2} \mathrm{Tr}\, \partial_0 A_i \partial_0 A^i - \sum_{i,j=1}^{3} \frac{1}{4g_0^2} F_{ij}F^{ij} \right) \tag{1.2.4}$$

The conjugate-momentum field or electric field is $E_i = -i\delta/\delta A_i$. The Hamiltonian is

$$H_{\mathrm{YM}} = \int d^3x \sum_{i=1}^{3} \left(\frac{g_0^2}{2} E_i^2 + \frac{1}{2g_0^2} B_i^2 \right),$$

where $B_i = \frac{1}{2}\epsilon^{ijk} F_{jk}$ is the magnetic field. In the temporal gauge, we enforce the Gauss-law constraint on physical wave functionals Ψ:

$$\left(\sum_{i=1}^{3} D_i E^i - \rho\right)\Psi = 0,$$

where ρ is the charge density of the Dirac field.

In 2+1 dimensions, there is only one component of the magnetic field, namely $B = \frac{1}{2}\epsilon^{ij}F_{ij}$, where $i, j = 1, 2$. The $(2+1)$-dimensional Yang-Mills Hamiltonian is

$$H_{\mathrm{YM}}^{(2+1)} = \int d^2x \left(\sum_{i=1}^{2} \frac{g_0^2}{2} E_i^2 + \frac{1}{2g_0^2} B^2\right).$$

1.3 Lattice gauge theory

In this section we present the action of the lattice Yang-Mills theory, first formulated by Wilson [10]. We find the Hamiltonian as the x^0 direction becomes continuous, as first discussed by Kogut and Susskind [11].

Wilson's lattice gauge theory is formulated on a 4-dimensional, hypercubic lattice, in Euclidean space. The distance between two neighboring lattice sites is a, which we call the lattice spacing. We denote a unit vector in the μ direction by $\hat{\mu}$. We label the link between the nearest-neighbor space-time points x and $x + a\hat{\mu}$, by (x, μ).

We assign the $SU(N)$-group-valued gauge field on the link (x, μ), written as $U(x)_\mu \in SU(N)$. This field is related to the continuum $SU(N)$-algebra valued field by

$$U(x)_\mu = \mathcal{P} \exp i \int_0^a dt\, A_\mu(x + t\hat{\mu}),$$

where $\mathcal{P}$ denotes path ordering. Under a local gauge transformation, the gauge field transforms as $U_{x,\mu} \to V(x)U(x)_\mu V^\dagger(x+a\hat{\mu})$, where $V(x) \in SU(N)$. The simplest gauge-invariant object we can construct out of the gauge fields is defined at a plaquette or elementary square made of the four links (x,μ), $(x+a\hat{\mu},\nu)$, $(x+a\hat{\nu},\mu)$ and (x,ν). This object is $\mathrm{Tr}\, U^\Box(x)_{\mu\nu}$, where

$$U^\Box(x)_{\mu\nu} = U(x)_\mu U(x+a\hat{\mu})_\nu U^\dagger(x+a\hat{\nu})_\mu U^\dagger(x)_\nu.$$

The simplest gauge-invariant action one can construct is

$$S_W = C \sum_{x,\mu,\nu} \mathrm{Tr}\, \left[U^\Box(x)_{\mu\nu} + U^\Box(x)^\dagger_{\mu\nu}\right], \tag{1.3.1}$$

which is called the Wilson action. The Yang-Mills action is the continuum limit $a \to 0$, of the Wilson action, with $C = -a^2/g_0^2$.

A detailed introduction to Hamiltonian lattice gauge theory is given in References [12], [13]. The Kogut-Susskind lattice Hamiltonian is usually derived in the temporal gauge, $U(x)_0 = 1$.

In the Hamiltonian formulation, there are electric-field operators $l(x)^b_j$ in the adjoint representation of $SU(N)$, at every space link (x,j), for $j = 1,2,3$ and $b = 1,2,\ldots,N^2-1$. The commutation relations are

$$\begin{aligned}\left[l(x)^b_j, l(y)^c_k\right] &= i\delta_{x\,y}\delta_{j\,k} f^{dbc} l(x)_{j\,d}, \\ \left[l(x)^b_j,\, U(y)_k\right] &= -\delta_{x\,y}\delta_{j\,k}\, t^b U(x)_j.\end{aligned}$$

The Hamiltonian is obtained by taking the continuum limit in the time direction. The

Kogut-Susskind Hamiltonian, inside a box of size $(aL)^3$, is

$$
\begin{aligned}
H &= \sum_{x^1,x^2,x^3=-\frac{L}{2}}^{\frac{L}{2}} \sum_{j=1}^{3} \sum_{b=1}^{N^2-1} \frac{g_0^2}{2a} \left[l(x)_j^b \right]^2 \\
&\quad - \sum_{x^1,x^2,x^3=-\frac{L}{2}}^{\frac{L}{2}} \sum_{b=1}^{3} \frac{1}{4g_0^2\, a} \operatorname{Tr} \left[U^{\Box}(x)_{jk} + \left(U^{\Box}(x)_{jk} \right)^{\dagger} \right],
\end{aligned}
$$

where L is an even integer. The quantities $U^{\Box}(x)_{jk}$, assigned to space-like plaquettes, are associated with the magnetic-field in the continuum limit. They may be expanded in powers of the lattice spacing to yield

$$
U^{\Box}(x)_{jk} = 1 + iaF_{jk}(x) - a^2 F_{jk}(x)^2 + \cdots .
$$

In $2+1$ dimensions, the lattice Hamiltonian is

$$
\begin{aligned}
H &= \sum_{x^1,x^2=-\frac{L}{2}}^{\frac{L}{2}} \sum_{j=1}^{2} \sum_{b=1}^{N^2-1} \frac{g_0^2}{2a} \left[l(x)_j^b \right]^2 \\
&\quad - \sum_{x^1,x^2=-\frac{L}{2}}^{\frac{L}{2}} \sum_{b=1}^{2} \frac{1}{4g_0^2\, a} \operatorname{Tr} \left[U^{\Box}(x)_{12} + \left(U^{\Box}(x)_{21} \right)^{\dagger} \right].
\end{aligned}
$$

In temporal gauge, physical states are those which satisfy Gauss's law:

$$
\sum_{j=1}^{d-1} [\mathcal{D}_j l_j(x)]_b \Psi = 0, \tag{1.3.2}
$$

where

$$[\mathcal{D}_j l_j(x)]_b = l_j(x) - \mathcal{R}_j(x - \hat{j}a)_b{}^c \, l_j(x - \hat{j}a)_c,$$

where $\mathcal{R}_j(x)_b{}^c t_c$ is the adjoint representation of the gauge field,

$$\mathcal{R}_j(x)_b{}^c t_c = U_j(x) t_b U_j^\dagger(x),$$

and d is the number of space-time dimensions [4].

Chapter 2

Longitudinally rescaled Yang-Mills theory

2.1 Longitudinal rescaling of the Yang-Mills action

We seek an effective theory of QCD at large center-of-mass energies. The particle beams in a collider are assumed to travel in the longitudinal x^3-direction. Thus we want (in our effective theory) four-momenta to have large components in the x^0- and x^3-directions, but not in the x^1- and x^2-directions. This effective theory is an anisotropic version of Yang-Mills theory, where two of the coordinates, x^0 and x^3, are rescaled to λx^0 and λx^3, respectively. Under this rescaling, the gauge fields transform as $A_{0,3} \to \lambda^{-1} A_{0,3}$. The remaining transverse space-time coordinates are left unchanged: $x_{1,2} \to x_{1,2},\ A_{1,2} \to A_{1,2}$. Under this rescaling, the Yang-Mills action becomes

$$S \longrightarrow \frac{1}{2g_0^2} \int d^4x \mathrm{Tr} \left(F_{01}^2 + F_{02}^2 - F_{13}^2 - F_{23}^2 + \lambda^{-2} F_{03}^2 - \lambda^2 F_{12}^2 \right).$$

For much of this thesis, we will examine the simpler (2+1)-dimensional Yang-Mills theory. in this case we rescale the longitudinal coordinates as $x^{0,1} \to \lambda x^{0,1}$ and leave the transverse coordinate unchanged, $x^2 \to x^2$. The rescaled (2+1)-dimensional action is

$$S \to \frac{1}{2g_0^2} \int d^4x \mathrm{Tr}\,(F_{02}^2 - F_{12}^2 + \lambda^{-2} F_{01}^2).$$

The longitudinally rescaled (2+1)-dimensional Hamiltonian, in temporal gauge ($A_0 = 0$), is

$$H = H_0 + \lambda^2 H_1, \tag{2.1.1}$$

where

$$H_0 = \int d^2x \left(\frac{g_0^2}{2} E_2^2 + \frac{1}{2g_0^2} B^2 \right),$$

and

$$H_1 = \int d^2x \frac{g_0^2}{2} E_1^2,$$

where $E_i = -i\delta/\delta A_i$, $B = \epsilon^{jk}(\partial_j A_k + A_j \times A_k)$, where $i, j, k = 1, 2$ and $(A_j \times A_k)^a = f^a_{bc} A^b_j A^c_k$. Physical states Ψ satisfy Gauss's law:

$$(D_1 E_1 + D_2 E_2)\Psi = 0. \tag{2.1.2}$$

The Hamiltonian (2.1.1) is simplified in the highly rescaled limit, $\lambda \to 0$. In this case we can treat the term $\lambda^2 H_1$ as a small perturbation. We will show in the next sections that the unperturbed Hamiltonian H_0 is integrable. Our goal is to calculate exact quantities in the unperturbed theory, then calculate corrections in powers of the small parameter λ^2.

A similar perturbation theory is possible in 3+1 dimensions [14]. The longitudinally rescaled Hamiltonian in temporal gauge is

$$H = H_0 + \lambda^2 H_1 + \lambda^2 H_2,$$

where

$$H_0 = \int d^3x \left(\frac{g_0^2}{2} E_1^2 + \frac{g_0^2}{2} E_2^2 + \frac{1}{2g_0^2} B_1^2 + \frac{1}{2g_0^2} B_2^2 \right),$$

$$H_1 = \int d^3x \frac{g_0^2}{2} E_3^2,$$

and

$$H_2 = \int d^3x \frac{1}{2g_0^2} B_3^2,$$

where $B_i = \epsilon^{ijk}(\partial_j A_k + A_j \times A_k)$, with $i, j, k = 1, 2, 3$.

In the next section we examine the (2+1)-dimensional Hamiltonian (2.1.1) in the axial gauge, $A_1 = 0$. Then we consider this Hamiltonian on the lattice. As $\lambda \to 0$, the lattice model is an array of integrable PCSM's.

2.2 The Yang-Mills Hamiltonian in the axial gauge in 2+1 dimensions

The Yang-Mills Hamiltonian in 2+1 dimensions, in $A_0 = 0$ gauge is

$$H = \int d^2x \left(\frac{g_0^2}{2} E_1^2 + \frac{g_0^2}{2} E_2^2 + \frac{1}{2g_0^2} B^2 \right).$$

Gauss's law (Eq. (2.1.2)) may be written more explicitly in terms of the gauge fields as

$$\begin{aligned} &\left\{ \left[\delta_{ac}\partial_1 + g_o f_{abc} A_1(x^1, x^2)^b \right] E_1(x^1, x^2)^c \right. \\ &\quad \left. + \left[\delta_{ac}\partial_2 + g_o f_{abc} A_2(x^1, x^2)^b \right] E_2(x^1, x^2)^c \right\} \Psi = 0. \end{aligned} \tag{2.2.1}$$

We can solve (2.2.1) to find E_1, in terms of A_1, A_2, and E_2. With the boundary conditions, $E_1 \to 0$ as $x^1 \to \pm\infty$:

$$\begin{aligned} E_1(x^1, x^2)^a \quad &= \quad \int_{-\infty}^{x^1} dy^1 \left\{ \mathcal{P} \exp \left[ie \int_{-\infty}^{y^1} dz^1 \mathcal{A}_1(x^1, x^2) \right] \right\}_a^{\;\;b} \\ &\quad \times \left\{ \left[\delta_{bd}\partial_2 + g_0 f_{bcd} A_2(y^1, x^2)^c \right] E_2(y^1, x^2)^d \right\} \end{aligned}$$

where $\mathcal{A}_1(x^1, x^2)_{ab} = i f_{abc} A_1(x^1, x^2)^c$ is the gauge field in the adjoint representation. There remains a global invariance given by

$$\left(\int_{-\infty}^{\infty} dy^1 \left\{ \mathcal{P} \exp \left[ie \int_{-\infty}^{y^1} dz^1 \mathcal{A}_1(x^1, x^2) \right] \right\}_a^{\;\;b} \left[\left(\delta_{bd}\partial_2 + g_0 f_{bcd} A_2(y^1, x^2)^c \right) E_2(y^1, x^2)^d \right] \right) \Psi = 0.$$

The temporal gauge condition, $A_0 = 0$ does not completely fix the gauge. Time-independent gauge transformations are consistent with this condition. We may now impose the axial gauge, $A_1(x^1, x^2) = 0$.

In the axial gauge, the electric field E_1 is

$$\begin{aligned} E_1(x^1, x^2)^a &= \int_{-\infty}^{x^1} dy^1 \left[\delta_{ac}\partial_2 + g_0 f_{abc} A_2(y^1, x^2)^b) E_2(y^1, x^2)^c\right] \\ &= \int_{-\infty}^{x^1} dy^1 D_2(y^1, x^2)_{ac} E_2(y^1, x^2)^c. \end{aligned} \tag{2.2.2}$$

After axial gauge fixing, there remains an invariance:

$$\left\{\int_{-\infty}^{\infty} dy^1 \left[(\delta_{ac}\partial_2 + g_0 f_{abc} A_2(y^1, x^2)^b) E_2(y^1, x^2)^c\right]\right\} \Psi = 0. \tag{2.2.3}$$

The condition (2.2.3) is imposed on the wave functional for each value of x^2.

Substituting Eq. (2.2.2) into the Hamiltonian:

$$\begin{aligned} H = &\int d^2x \left\{\frac{g_0^2}{2} E_2(x^1, x^2)^2 + \frac{1}{2}[\partial_1 A_2(x^1, x^1)]^2\right\} \\ &- \int dx^1 \int dy^1 \int dx^2 |x^1 - y^1| [D_2(x^1, x^2) E_2(x^1, x^2)] \\ &\times [D_2(y^1, x^2) E_2(y^1, x^2)]. \end{aligned} \tag{2.2.4}$$

The Hamiltonian (2.2.4) depends only on the transverse degrees of freedom A_2 and E_2. This Hamiltonian is nonlocal in the x^1-direction, which is an artifact of the axial gauge fixing. Eq. (2.2.4) can be made local, by reintroducing the A_0 component of the gauge field.

2.3 The Kogut-Susskind lattice Hamiltonian in the axial gauge

The Kogut-Susskind Hamiltonian in 2+1 dimensions, in temporal gauge ($A_0 = 0, U_0 = \mathbf{1}$), is

$$\begin{aligned} H &= \sum_{x^1=-\frac{L_1}{2}}^{\frac{L_1}{2}} \sum_{x^2=-\frac{L_2}{2}}^{\frac{L_2}{2}} \sum_{j=1}^{2} \sum_{b=1}^{N^2-1} \frac{g_0^2}{2a}[l_j(x)_b]^2 \\ &- \sum_{x^1=-\frac{L_1}{2}}^{\frac{L_1}{2}} \sum_{x^2=-\frac{L_2}{2}}^{\frac{L_2}{2}} \frac{1}{4g_0^2 a}[\mathrm{Tr}\, U_{12}^{\square}(x) + \mathrm{Tr}\, U_{21}^{\square}(x)], \end{aligned} \tag{2.3.1}$$

where

$$U_{jk}^{\square}(x) = U_j(x)U_k(x + \hat{j}a)U_j(x + \hat{k}a)^{\dagger}U_k(x)^{\dagger}.$$

We will impose axial gauge for the lattice model, just as we did in the previous section for the continuum gauge theory. We find the electric field component l_1 by solving Gauss's law (1.3.2):

$$l_1(x^1, x^2)_b = \sum_{y^1=-\frac{L_1}{2}}^{x^1} [\mathcal{D}_2 l_2(y^1, x^2)]_b. \tag{2.3.2}$$

There is a global invariance left after the axial gauge fixing $U_1(x) = \mathbf{1}$:

$$\sum_{x^1=-\frac{L_1}{2}}^{\frac{L_1}{2}} [\mathcal{D}_2 l_2(x^1, x^2)]_b \Psi = 0. \tag{2.3.3}$$

The lattice Hamiltonian in axial gauge is found by substituting the new nonlocal expression for the electric field (2.3.2) into (2.3.1):

$$
\begin{aligned}
H &= \sum_{x^1=-\frac{L_1}{2}}^{\frac{L_1}{2}} \sum_{x^2=-\frac{L_2}{2}}^{\frac{L_2}{2}} \frac{g_0^2}{2a}[l_2(x)]^2 \\
&- \sum_{x^1=-\frac{L_1}{2}}^{\frac{L_1}{2}} \sum_{x^2=-\frac{L_2}{2}}^{\frac{L_2}{2}} \frac{1}{2g_0^2 a}[\mathrm{Tr}\, U_2(x^1,x^2)^\dagger U_2(x^1+a,x^2) + c.c.] \\
&- \frac{(g_0')^2}{2a} \sum_{x^1=-\frac{L_1}{2}}^{\frac{L_1}{2}} \sum_{y^1=-\frac{L_1}{2}}^{\frac{L_1}{2}} \sum_{x^2=-\frac{L_2}{2}}^{\frac{L_2}{2}} |x^1-y^1| \\
&\times[l_2(x^1,x^2) - \mathcal{R}_2(x^1,x^2-a)l_2(x^1,x^2-a)] \\
&\times[l_2(y^1,x^2) - \mathcal{R}_2(y^1,x^2-a)l_2(y^1,x^2-a)]. \qquad (2.3.4)
\end{aligned}
$$

The Hamiltonian (2.3.4) is the discretized version of (2.2.4). Like (2.2.4), (2.3.4) is nonlocal in x^1, and depends only on the transverse degrees of freedom U_2, l_2. In the following section we explore the longitudinal rescaling of coordinates on the Hamiltonian (2.3.4).

2.4 Anisotropic Yang-Mills as an array of sigma models

Next we find the effect of longitudinal rescaling, $x^{0,1} \to \lambda x^{0,1}$, $x^2 \to x^2$ to a lattice gauge theory. The longitudinally-rescaled lattice has spacing λa in the $x^{0,1}$ directions and spacing a in the x^2 direction. In the $\lambda \to 0$ limit, it is sensible to treat x^0 and x^1 as continuous directions, and x^2 discrete.

Longitudinally rescaling the lattice Hamiltonian (2.3.4), gives $H = H_0 + \lambda^2 H_1$, where

$$
\begin{aligned}
H_0 &= \sum_{x^1=-\frac{L_1}{2}}^{\frac{L_1}{2}} \sum_{x^2=-\frac{L_2}{2}}^{\frac{L_2}{2}} \frac{g_0^2}{2a}[l_2(x)]^2 \\
&\quad - \sum_{x^1=-\frac{L_1}{2}}^{\frac{L_1}{2}} \sum_{x^2=-\frac{L_2}{2}}^{\frac{L_2}{2}} \frac{1}{2g_0^2 a}[\mathrm{Tr}\, U_2(x^1,x^2)^\dagger U_2(x^1+a,x^2) + c.c.], \\
H_1 &= -\frac{(g_0')^2}{2a} \sum_{x^1=-\frac{L_1}{2}}^{\frac{L_1}{2}} \sum_{y^1=-\frac{L_1}{2}}^{\frac{L_1}{2}} \sum_{x^2=-\frac{L_2}{2}}^{\frac{L_2}{2}} |x^1-y^1| \\
&\quad \times[l_2(x^1,x^2) - \mathcal{R}_2(x^1,x^2-a) l_2(x^1,x^2-a)] \\
&\quad \times[l_2(y^1,x^2) - \mathcal{R}_2(y^1,x^2-a) l_2(y^1,x^2-a)].
\end{aligned}
$$

Henceforth we drop the Lorentz index 2 from U_2, l_2.

We treat H_1 as a perturbation. In the interaction representation, U satisfies the Heisenberg equation of motion, $\partial_0 U = i[H_0, U]$. The solution of this equation of motion is

$$
\begin{aligned}
l(x^1,x^2)_b &= \frac{ia}{g_0^2}\mathrm{Tr}\, t_b \partial_0 U(x^1,x^2) U(x^1,x^2)^\dagger, \\
\mathcal{R}(x^1,x^2)_b{}^c l(x^1,x^2)_c &= \frac{ia}{g_0^2}\mathrm{Tr}\, t_b U(x^1,x^2)^\dagger \partial_0 U(x^1,x^2).
\end{aligned} \tag{2.4.1}
$$

Substituting (2.4.1) into H_0, and taking the continuum limit in the x^1 direction, we find

$$
\begin{aligned}
H_0 &= \sum_{x^2} H_0(x^2) = \sum_{x^2} \int dx^1 \frac{1}{2g_0^2}\left\{ \left[j_0^L(x^1,x^2)_b\right]^2 + \left[j_1^L(x^1,x^2)_b\right]^2 \right\} \\
&= \sum_{x^2} \int dx^1 \frac{1}{2g_0^2}\left\{ \left[j_0^R(x^1,x^2)_b\right]^2 + \left[j_1^R(x^1,x^2)_b\right]^2 \right\},
\end{aligned}
$$

where

$$j^L_\mu(x)_b = i\mathrm{Tr}\, t_b \partial_\mu U(x) U(x)^\dagger,\ j^R_\mu(x)_b = i\mathrm{Tr}\, t_b U(x)^\dagger \partial_\mu U(x), \tag{2.4.2}$$

where $\mu = 0, 1$.

We now note that $H_0(x^2)$ is the Hamiltonian of a (1+1)-dimensional principal chiral sigma model located at x^2. The principal chiral sigma model has the action

$$\mathcal{L}_{\mathrm{PCSM}} = \int d^2x \frac{1}{2g_0^2} \eta^{\mu\nu} \mathrm{Tr}\, \partial_\mu U^\dagger \partial_\nu U. \tag{2.4.3}$$

This model has a global $SU(N) \times SU(N)$ symmetry given by the transformation $U(x) \to V^L U(x) V^R$, where $V^{L,R} \in SU(N)$. The Noether currents corresponding to these global symmetries are $j^{L,R}$ given in (2.4.2). The Hamiltonian corresponding to the action (2.4.3) of a single principal chiral sigma model at fixed x^2 is $H_0(x^2)$. The unperturbed Hamiltonian, H_0, is an array of principal chiral sigma models, one at each value of x^2,

$$H_0 = \sum_{x^2} H_0(x^2) = \sum_{x^2} H_{\mathrm{PCSM}}(x^2).$$

The residual Gauss's law, (2.3.3) becomes

$$\int dx^1 \left[j^L_0(x^1, x^2)_b - j^R_0(x^1, x^2 - a)_b \right] \Psi = 0,$$

for each value of x^2, when x^1 is continuous.

Using (2.4.1), we write the interaction Hamiltonian H_1 in the continuous x^1 limit:

$$\begin{aligned} H_1 &= \sum_{x^2} \int dx^1 \int dy^1 \frac{1}{4g_0^2 a} |x^1 - y^1| \\ &\quad \times \left[j_0^L(x^1, x^2) - j_0^R(x^1, x^2 - a) \right] \\ &\quad \times \left[j_0^L(y^1, x^2) - j_0^R(y^1, x^2 - a) \right]. \end{aligned} \tag{2.4.4}$$

The Hamiltonian (2.4.4) couples adjacent sigma models, which allows particles to propagate in the x^2 direction. The coupling is suppressed in the $\lambda \to 0$ limit.

To summarize, longitudinally-rescaled Yang-Mills theory in $2+1$ dimensions consists of an array of principal chiral sigma models. The principal chiral sigma model is integrable and many exact results can be found.

The next chapter is a brief review of integrable quantum field theories in $1+1$ dimensions. In Chapter 4, we find exact form factors of the principal chiral sigma model, guided by its integrability. In Chapters 5 and 6 we apply these results to Yang-Mills theories.

Chapter 3

Integrable quantum field theory and form factor axioms

3.1 Integrability in quantum field theories

An integrable field theory has an infinite set of nontrivial local conserved charges, which are Lorentz tensors of increasing rank. We will call Q_n a conserved charge of rank n (with n Lorentz indices).

The existence of this set conserved charges has dramatic consequences in quantum field theory. In an integrable quantum field theory, all scattering is elastic. This means that in any scattering event the total number of particles is conserved, and furthermore, the set of their momenta is conserved. All scattering in an integrable quantum field theory is factorizable. A many-particle S-matrix can be broken down into a product of two-particle scatterings. The different ways one can factorize an S-matrix must be equivalent. From this follows that the two-particle S-matrix satisfies the Yang-Baxter equation (which we will discuss in more detail in the next section).

These restrictions, along with other physical considerations (like unitarity of the S-matrix, crossing symmetry, and analyticity), allow us to find the two-particle S-matrix exactly in many integrable field theories. In the rest of this section we briefly explain how elasticity and factorization follow from the existence of the set of charges Q_n.

We now examine the scattering of l particles into k particles in a generic integrable quantum field theory. A state with one particle in 1+1 dimensions is characterized by its two-momentum, p, and some set of quantum numbers c (these could include color, isospin, flavor, etc.). We define a state with l incoming particles as

$$|l, \{p\}, \{c\}\rangle_{\rm in} = |p_1, c_1; p_2, c_2; \dots; p_l, c_l\rangle_{\rm in},$$

where we assume that the incoming particles are widely separated in the infinite past. We can similarly define the k-particle outgoing state

$$|k, \{p'\}, \{c'\}\rangle_{\rm out} = |p'_1, c'_1; p'_2, c'_2; \dots; p'_l, c'_l\rangle_{\rm out}.$$

Acting with the conserved charge Q_n in the incoming and outgoing states yields

$$\begin{aligned} Q_n|l, \{p\}, \{c\}\rangle_{\rm in} &= \sum_{i=1}^{l} q_n(p_i)^{c'_i}_{c_i}|l, \{p\}, \{c'\}\rangle_{\rm in}, \\ Q_n|k, \{p'\}, \{c\}\rangle_{\rm out} &= \sum_{i=1}^{k} q_n(p'_i)^{c'_i}_{c_i}|k, \{p'\}, \{c'\}\rangle_{\rm out}, \end{aligned}$$

where $q_n(p_i)$ is some polynomial of rank n of the energy and momentum of the i-th particle. The charges Q_n are locally conserved, which means that we can add the individual contributions from each particle q_n if the particles are widely separated. The fact that Q_n is conserved implies

$$\sum_{i=1}^{l} q_n(p_i)_{c_i}^{c'_i} = \sum_{i=1}^{k} q_n(p'_i)_{c_i}^{c'_i}. \tag{3.1.1}$$

If there is an infinite number of charges Q_n, there is an infinite number of polynomial equations (3.1.1) of increasing rank that need to be satisfied by only $l + k$ variables. The only way way these infinite set of conditions can be satisfied is if $l = k$, and the set of momenta satisfies $\{p\} = \{p'\}$. Thus all scattering events are elastic.

To verify the factorizability of the S-matrix, we examine the Fourier transform of the incoming state:

$$|l, \{x\}, \{c\}\rangle_{\text{in}} = \left(\prod_{i=1}^{l} dp_i e^{ip_i(x_i - x_i^0)}\right) |l, \{p\}, \{c\}\rangle_{\text{in}},$$

where the center of mass of the i-th particle is located at x_i^0. We can now operate on the state $|l, \{x\}, \{c\}\rangle_{\text{in}}$ with the operator $e^{i\epsilon Q_n}$:

$$e^{i\epsilon Q_n}|l, \{x\}, \{c\}\rangle_{\text{in}} = \left(\prod_{i=1}^{l} e^{i\epsilon q_n(p_i)}\right) |l, \{x\}, \{c\}\rangle_{\text{in}}. \tag{3.1.2}$$

The operator, $e^{i\epsilon Q_n}$, commutes with the Hamiltonian, so it does not alter the S-matrix. In the incoming state from Eq. (3.1.2), the i-th particle is now centered at $x_i^0 - \epsilon(dq_n(p_i)/dp_i)$. The shift in the position of each particle depends on its momentum. By adjusting the parameter ϵ, we can change the position of each particle independently. By applying the operator $e^{i\epsilon Q_n}$, we change the order in which particles interact. We can break any scattering event into multiple two-particle scatterings, and we can exchange the order of the two-particle scatterings without affecting the total S-matrix. The S-matrix is therefore factorizable.

In the next section, we show how the elasticity and factorizability, along with unitarity, crossing symmetry and analyticity can be used to find the exact S-matrix in integrable quantum field theories.

3.2 Exact S-matrices

As we discussed in the previous section, all scattering in an integrable field theory is elastic. The S-matrix is nonzero only if there is an equal number of incoming and outgoing particles, and the set of incoming and outgoing momenta is the same. The S-matrix with l incoming and l outgoing particles is called the l-particle S-matrix.

S-matrices are factorizable. For the three-particle S-matrix, this means

$$\begin{aligned} S_{PPP}(\theta_1,\theta_2,\theta_3)^{c_1' \, c_2' \, c_3'}_{c_1 \, c_2 \, c_3} &= S_{PP}(\theta_1,\theta_2)^{c_1'' \, c_2''}_{c_1 \, c_2} S_{PP}(\theta_1,\theta_3)^{c_1' \, c_3''}_{c_1'' \, c_3} S_{PP}(\theta_2,\theta_3)^{c_2' \, c_3'}_{c_2'' \, c_3''} \\ &= S_{PP}(\theta_2,\theta_3)^{c_2'' \, c_3''}_{c_2 \, c_3} S_{PP}(\theta_1,\theta_3)^{c_1'' \, c_3'}_{c_1 \, c_3''} S_{PP}(\theta_1,\theta_2)^{c_1' \, c_2'}_{c_1'' \, c_2''}, \end{aligned} \tag{3.2.1}$$

Where c_i is the set of quantum numbers of the i-th particle, and θ_i is the rapidity of the i-th particle, defined by its energy and momentum: $E_i = m\cosh\theta_i$, $p_i = m\sinh\theta_i$. Equation (3.2.1) is called the Yang-Baxter equation.

The S-matrix (and time evolution) is unitary. For the two-particle S-matrix, this implies

$$S_{PP}(\theta_2,\theta_1)^{c_2'' \, c_1''}_{c_2 \, c_1} S_{PP}(\theta_1,\theta_2)^{c_1' \, c_2'}_{c_1'' \, c_2''} = \delta^{c_1'}_{c_1}\delta^{c_2'}_{c_2}. \tag{3.2.2}$$

The S-matrix should also be invariant under crossing symmetry. We can turn an incoming particle into an outgoing antiparticle by shifting is rapidity as $\theta \to \theta + \pi i$ (and an outgoing particle can be turned into an incoming antiparticle). The antiparticle-particle S-matrix, S_{AP}, can be found from the particle-particle S-matrix by crossing. This is

$$S_{AP}(\theta)_{c_1\,c_2}^{c_1'\,c_2'} = S_{PP}(\pi i - \theta)_{c_1\,c_2}^{c_1'\,c_2'}, \tag{3.2.3}$$

where $\theta = \theta_1 - \theta_2$.

The spectrum of a quantum field theory might have bound states of elementary particles. The bound state mass, m_B, of a particle composed of two particles of mass m_1 and m_2 is given by

$$m_B = \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos\eta}, \quad (0 < \eta < \pi), \tag{3.2.4}$$

where η is the fusion angle. One can define the S-matrix for a bound state with an elementary particle, S_{BP} from the two-particle S-matrix by

$$S_{BP}(\theta_B, \theta_1)_{d'\,c_3}^{d\,c_3'}\,\Gamma_{c_1\,c_2}^{d'} = \Gamma_{c_1'\,c_2'}^{d}\,S_{PP}(\theta_1, \theta_3)_{c_1\,c_3''}^{c_1'\,c_3'}S_{PP}(\theta_2, \theta_3)_{c_2\,c_3}^{c_2' c_3''}, \tag{3.2.5}$$

where d are the quantum numbers of the bound state, and $\Gamma_{c_1\,c_2}^{d}$ is defined by

$$i\mathrm{Res}|_{\theta=i\eta}\,S_{PP}(\theta)_{c_1\,c_2}^{c_1'\,c_2'} = \Gamma_{d}^{c_1'\,c_2'}\,\Gamma_{c_1\,c_2}^{d}. \tag{3.2.6}$$

The two-particle S-matrix for can be found up to a CDD ambiguity [21] by solving equations (3.2.1), (3.2.2), (3.2.3) and (3.2.5), for particles described by the quantum numbers $\{c\}$ and bound states described by the numbers d. The CDD ambiguity means that the two-particle S-matrix still satisfies the conditions (3.2.1), (3.2.2), (3.2.3) and (3.2.5) if we multiply with it with a factor

$$\prod_k \frac{\sinh\theta + i\sin\alpha_k}{\sinh\theta - i\sin\alpha_k}, \tag{3.2.7}$$

for some set of numbers $\{\alpha\}$. This ambiguity can be fixed by requiring that the S-matrix be maximally analytic. The S-matrix should not have any poles in the physical strip, $0 < \mathrm{Im}\theta < \pi$, that do not correspond to bound state fusion angles. One can multiply the S-matrix by factors of the form (3.2.7) to push any unnecessary poles outside the physical strip, obtaining the maximally analytic S-matrix.

This approach for finding S-matrices for integrable theories was first developed in References [15], [16], [17], [20], [18]. The exact two-particle S-matrix of the principal chiral sigma model was found in Reference [19]. In the next section we show how the exact S-matrix can be used to find form factors of local operators, and correlation functions.

3.3 The form factor bootstrap program

The l-excitation form factor of a local operator $\mathcal{O}(x)_{c_0}$ is defined as

$$F^{\mathcal{O}}(\theta_1,\dots,\theta_l)_{\{c\}} = \langle 0|\mathcal{O}(0)_{c_0}|\theta_1, c_1;\theta_2, c_2;\dots;\theta_l, c_l\rangle_{\mathrm{in}}, \tag{3.3.1}$$

where c_0 is the set of Lorentz, color, or flavor indices of the operator, and $\theta_1 > \theta_2 > \cdots > \theta_l$. Other orderings of the rapidities can be defined by analytic continuation of the form factor.

The two-particle form factors were first studied in References [23], [24]. Generalizations for form factors with many particles were discussed in [25]. Form factors of integrable theories satisfy a set of conditions called the "Smirnov axioms", [26], which require knowledge of the exact two-excitation S-matrix. These axioms provide such strong restrictions, that they yield the exact form factors. Once all the form factors of a local operator are known, one can find correlation functions by inserting complete sets of intermediate states between these operators. This approach of finding the S-matrix, using it to calculate form factors, and using these form factors to calculate correlation functions is called the integrable bootstrap program.

In the rest of these section we list and explain briefly the Smirnov axioms. These are the scattering axiom, periodicity axiom, Lorentz invariance axiom, annihilation pole axiom, bound state axiom, and minimality axiom.

We will first discuss the so-called scattering axiom (also known as Watson's theorem). The scattering axiom states that exchanging two of the incoming excitations in (3.3.1) is equivalent to scattering them, explicitly,

$$
\begin{aligned}
&F^{\mathcal{O}}(\theta_1, \ldots, \theta_j, \theta_i, \ldots, \theta_l)_{c_0 c_1 \ldots c_j c_i \ldots c_l} \\
&\qquad = S(\theta_i - \theta_j)_{c_j c_i}^{c_i' c_j'} F^{\mathcal{O}}(\theta_1, \ldots, \theta_i, \theta_j, \ldots, \theta_l)_{c_0 c_1 \ldots c_i c_j \ldots c_l}.
\end{aligned} \tag{3.3.2}
$$

One can keep applying the scattering axiom on the form factor to exchange the order of any two particles. The form factor with any order of rapidities can be found by multiplying many times with the S-matrix.

Next we look at the periodicity axiom. This axiom follows from crossing symmetry in two dimensions. By crossing symmetry, one can turn the $l-th$ incoming particle (or antiparticle) in (3.3.1) into an outgoing antiparticle (or particle), by shifting $\theta_l \to \theta_l - \pi i$. This way, if we know the form factor (3.3.1), we can find the form factors of an operator with outgoing excitations. The periodicity axiom states that the outgoing antiparticle (or particle) can be turned into the first incoming excitation, by shifting its rapidity again by $-\pi i$. Explicitly, this axiom states:

$$\begin{aligned} F^{\mathcal{O}}(\theta_1,\ldots,\theta_l)_{c_0 c_1\ldots c_l} &= F^{\mathcal{O}}(\theta_l - 2\pi i, \theta_1,\ldots,\theta_{l-1})_{c_l c_1\, c_{l-1}} \\ &= F^{\mathcal{O}}(\theta_{l-1} - 2\pi i, \theta_l - 2\pi i, \theta_1,\ldots,\theta_{l-2})_{c_0 c_{l-1} c_l c_1\ldots c_{l-2}} \\ &= \ldots\ . \end{aligned} \tag{3.3.3}$$

The Lorentz invariance axiom restricts how the form factor (3.3.1) transforms under a Lorentz boost. A Lorentz boost in (3.3.1) means shifting all the rapidities by $\theta_i \to \theta_i + \alpha$, for some constant α. If the Operator $\mathcal{O}$ has spin s, the form factor transforms as

$$F^{\mathcal{O}}(\theta_1+\alpha,\ldots,\theta_l+\alpha)_{\{c\}} = e^{s\alpha} F^{\mathcal{O}}(\theta_1,\ldots,\theta_l)_{\{c\}}. \tag{3.3.4}$$

If the operator $\mathcal{O}$ is a Lorentz scalar, then the form factor is invariant under a boost. By Lorentz invariance, we can also write the x-dependent form factor:

$$\langle 0|\mathcal{O}(x)_{c_0}|\theta_1, c_1; \theta_2, c_2; \ldots; \theta_l, c_l\rangle_{\text{in}} = e^{ix\cdot(p_1+\cdots+p_l)} F^{\mathcal{O}}(\theta_1,\ldots,\theta_l)_{\{c\}}.$$

There exists a possibility that a particle with rapidity θ_i and a neighboring antiparticle with rapidity θ_i+1 in the incoming state in (3.3.1) annihilate before reaching the operator $\mathcal{O}$. The form factor must then have an kinematic pole at $\theta_i - \theta_j = -\pi i$. The annihilation-pole axiom states that the residue of the l-particle form factor at this pole is proportional to the

$(l-2)$-particle form factor. Precisely, this axiom states:

$$\begin{aligned}
&\mathrm{Res}|_{\theta_{l-1}-\theta_l=-\pi i} F^{\mathcal{O}}(\theta_1,\ldots,\theta_l)_{\{c\}} = 2iF^{\mathcal{O}}(\theta_1,\ldots,\theta_{l-2})_{c_0 c'_1 \ldots c'_{l-2}}\delta_{c_l c'_{l-1}} \\
&\times \Big(\delta_{c_1 c'_1}\ldots\delta_{c_{l-2}c'_{l-2}}\delta_{c_{l-1}c'_{l-1}} - S(\theta_1-\theta_{l-1})^{c'_{l-1}c'_1}_{c''_1 c_1} S(\theta_2-\theta_{l-1})^{c''_1 c'_2}_{c''_2 c_2}\times\cdots \\
&\qquad \times S(\theta_i-\theta_{l-1})^{c''_{i-1}c'_i}_{c''_i c_i}\times\cdots\times S(\theta_{l-2}-\theta_{l-1})^{c''_{l-3}c'_{l-2}}_{c_{l-1}c_{l-2}}\Big), \qquad (3.3.5)
\end{aligned}$$

where the right hand of (3.3.5) side includes the $(l-2)$-particle form factor. The product of S-matrices in the right hand side of (3.3.5) accounts for the possibility that before the $(l-1)$-st particle (or antiparticle), annihilates with the l-th antiparticle (or particle), it can scatter with the rest of the $(l-2)$ excitations. The annihilation-pole axiom can be used as a recursion relation. Once the l-particle form factor is known, this axiom can be used to find information about the $(l+2)$-particle form factor, and so on.

As we discussed in the previous section, an integrable theory, may have bound states of elementary particles, with masses given by (3.2.4). If the i-th and $(i+1)$-st excitations in Eq. (3.3.1) can form a bound state, the form factor must have a pole at $\theta_i - \theta_{i+1} = i\eta$, where η is the fusion angle defined in (3.2.4). One can use this to find information about form factors with incoming bound-state particles. The bound-state axiom states that

$$\begin{aligned}
&\mathrm{Res}|_{\theta_1-\theta_2=i\eta} F^{\mathcal{O}}(\theta_1,\ldots,\theta_l)_{c_0 c_1\ldots c_l} \\
&\qquad = \sqrt{2}\Gamma^d_{c_1 c_2} F^{\mathcal{O}}(\theta^B_{(12)},\theta_3,\ldots,\theta_l)_{c_0 d\, c_3\ldots c_l}, \qquad (3.3.6)
\end{aligned}$$

where the right-hand side includes the form factor with an incoming bound state, $\theta^B_{(12)}$ is the rapidity of the bound state, and $\Gamma^d_{c_1 c_2}$ is defined in Eq. (3.2.6).

The axioms we have discussed in Equations (3.3.2), (3.3.3), (3.3.4), (3.3.5), and (3.3.6), are enough to determine the form factor up to an ambiguity, similar to the CDD ambiguity in the S-matrix. If we find a form factor, $F^{\mathcal{O}}_{\text{minimal}}(\theta_1,\ldots,\theta_l)_{\{c\}}$, which satisfies all the axioms we have mentioned, these axioms are also satisfied by

$$F^{\mathcal{O}}(\theta_1,\ldots,\theta_l)_{\{c\}} = \frac{P_l[\{\cosh(\theta_i-\theta_j)\}]}{Q_l[\{\cosh(\theta_i-\theta_j)\}]}F^{\mathcal{O}}_{\text{minimal}}(\theta_1,\ldots,\theta_l)_{\{c\}}, \tag{3.3.7}$$

where P_l, and Q_l are symmetric polynomials. The minimality axiom states that the form factor must be maximally analytic. Form factors should have no poles that are not either kinematic annihilation poles, or bound state poles in the physical strip. If a form factor is found which has additional, unphysical poles, one can use the ambiguity (3.3.7) to push these outside the physical strip. The physical form factor is then the one with the minimum number of poles.

Once all the exact form factors of an operator are known, they can be use to calculate non-time-ordered correlation functions. The simplest case is the two-point function of the operator $\mathcal{O}$, defined as

$$W^{\mathcal{O}}(x)_{c_0d_0} = \langle 0|\mathcal{O}(x)_{c_0}\mathcal{O}(0)_{d_0}|0\rangle.$$

The function $W^{\mathcal{O}}(x)_{c_0d_0}$ can be computed by inserting a complete set of intermediate states between the two operators. The correlation function is then expressed as a sum over all the form factors:

$$\begin{aligned} W^{\mathcal{O}}(x)_{c_0 d_0} &= \sum_{l=1}^{\infty} \frac{1}{l!} \int \prod_{j=1}^{l} \frac{d\theta_j}{4\pi} e^{-ix\cdot(p_1+\cdots+p_l)} \\ &\quad \times \langle 0|\mathcal{O}(0)_{c_0}|\theta_1, c_1, \ldots, \theta_l, c_l\rangle_{\text{in}} \langle 0|\mathcal{O}(0)_{d_0}|\theta_1, c_1, \ldots, \theta_l, c_l\rangle_{\text{in}}^{*} \\ &= \sum_{l=1}^{\infty} \frac{1}{l!} \int \prod_{j=1}^{l} \frac{d\theta_j}{4\pi} e^{-ix\cdot(p_1+\cdots+p_l)} \\ &\quad \times F^{\mathcal{O}}(\theta_1, \ldots, \theta_l)_{c_0 c_1 \ldots c_l} \times \left[F^{\mathcal{O}}(\theta_1, \ldots, \theta_l)_{d_0 c_1 \ldots c_l}\right]^{*}. \end{aligned}$$

Correlation functions of more than two operators can still be found by inserting a complete set of states between each pair of operators. One would then need the form factors with both incoming and outgoing particles, which can be found using crossing symmetry.

In the next chapter we use the integrable bootstrap program to find form factors and correlation functions of the principal chiral sigma model. We show how to apply these new exact results to Yang-Mills theories.

Chapter 4

Exact form factors of Noether current and energy-momentum tensor of the principal chiral sigma model

4.1 The principal chiral sigma model

This chapter contains material previously published in [27] and [28].

The principal chiral sigma model has the action

$$S_{\mathrm{PCSM}} = \int d^2x \frac{1}{2g_0^2} \eta^{\mu\nu} \operatorname{Tr} \partial_\mu U^\dagger(x) \partial_\nu U(x), \tag{4.1.1}$$

where the field $U(x)$ is in the fundamental representation of $SU(N)$, $\mu, \nu = 0, 1$, and $\eta^{00} = -\eta^{11} = 1$, $\eta^{01} = \eta^{10} = 0$. The action (4.1.1) has a global $SU(N) \times SU(N)$ symmetry, given by the transformation $U(x) \to V_L U(x) V_R$, where $V_{L,R} \in SU(N)$. The Noether currents associated with these symmetries are

$$\begin{aligned} j^L_\mu(x)^c_a &= \frac{-iN}{2g^2}\partial_\mu U_{ab}(x)U(x)^{\dagger\, bc}(x), \\ j^R_\mu(x)^d_b &= \frac{-iN}{2g^2}U^{\dagger\, da}(x)\partial_\mu U_{ab}(x), \end{aligned} \tag{4.1.2}$$

respectively, where we have included the color indices, $a, b, c, d = 1, \dots, N$, explicitly.

This model is asymptotically free, and has a mass gap, which we call m. It has been argued that this mass gap is generated by non-perturbative saddle points of the path integral [29]. The quantum integrability of this model was shown in References [30] and [31].

The sigma model has elementary particles of mass m which carry both left and right colors. These elementary particles form bound states that obey a sine formula [34]

$$m_r = m\frac{\sin\left(\frac{\pi r}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}, \quad r = 1, \dots, N-1, \tag{4.1.3}$$

where m_r is the mass of an r-particle bound state. In the large-N limit, the mass of an r-particle bound state is $m_r = m\, r$, for finite r. This means that there are no bound states of a finite number of elementary particles in the planar limit, since the binding energy vanishes.This is the 't Hooft large-N limit, where the mass gap m is fixed as N goes to infinity. An alternative large-N limit has been examined in References [35] where the mass m_{N-1} is kept fixed and the mass gap goes to zero as N goes to infinity.'t Hooft

We introduce particle and antiparticle creation operators $\mathfrak{A}^\dagger_P(\theta)_{ab}$ and $\mathfrak{A}^\dagger_A(\theta)_{ba}$, respectively, where θ is the particle rapidity, defined in terms of the momentum vector by $p_0 = m\cosh\theta, p_1 = m\sinh\theta$, and a, $b = 1, \dots, N$ are left and right color indices, respectively. A state with many particles is created by acting on the vacuum with a product of creation operators in order of increasing rapidity, from left to right,

$$|P,\theta_1,a_1,b_1;A,\theta_2,b_2,a_2;\dots\rangle_{\rm in} = \mathfrak{A}^\dagger_P(\theta_1)_{a_1b_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\dots|\,0\rangle, \tag{4.1.4}$$

where $\theta_1 > \theta_2 > \dots$.

The S-matrix, $S_{PP}(\theta)^{c_2d_2;c_1d_1}_{a_1b_1;a_2b_2}$ of two particles with incoming rapidities θ_1 and θ_2, and outgoing rapidities θ_1' and θ_2' is defined by

$$\begin{aligned} &{}_{\rm out}\langle P,\theta_1',c_1,d_1;A,\theta_2',d_2,c_2\mid P,\theta_1,a_1,b_1;A,\theta_2,b_2,a_2\rangle_{\rm in} \\ &\qquad = S_{PP}(\theta)^{c_2d_2;c_1d_1}_{a_1b_1;a_2b_2}\,4\pi\,\delta(\theta_1'-\theta_1)\,4\pi\,\delta(\theta_2'-\theta_2), \end{aligned}$$

where $\theta=\theta_1-\theta_2$. This S-matrix was found exactly in Reference [19] to be

$$S_{PP}(\theta)^{c_2d_2;c_1d_1}_{a_1b_1;a_2b_2} = \chi(\theta)\,S_{\rm CGN}(\theta)^{c_2;c_1}_{a_1;a_2}\,S_{\rm CGN}(\theta)^{d_2d_1}_{b_1;b_2}, \tag{4.1.5}$$

where $S_{\rm CGN}$ is the S-matrix of two elementary excitations of the $SU(N)$ chiral Gross-Neveu model [32], [33]:

$$S_{\rm CGN}(\theta)^{c_2c_1}_{a_1a;a_2} = \frac{\Gamma(i\theta/2\pi+1)\Gamma(-i\theta/2\pi-1/N)}{\Gamma(i\theta/2\pi+1-1/N)\Gamma(-i\theta/2\pi)}\left(\delta^{c_1}_{a_1}\delta^{c_2}_{a_2}-\frac{2\pi i}{N\theta}\delta^{c_1}_{a_2}\delta^{c_2}_{a_1}\right),$$

and $\chi(\theta)$ is the CDD factor [21]:

$$\chi(\theta) = \frac{\sinh\left(\frac{\theta}{2}-\frac{\pi i}{N}\right)}{\sinh\left(\frac{\theta}{2}+\frac{\pi i}{N}\right)}. \tag{4.1.6}$$

The particle-antiparticle S-matrix is related to the particle-particle S-matrix by crossing symmetry, i.e. $\theta\to\hat\theta=\pi i-\theta$. The S-matrix for a particle with incoming rapidity θ_1 and outgoing rapidity θ_1' and an antiparticle with incoming rapidity θ_2 and outgoing rapidity θ_2' is

$$\begin{aligned}
S_{AP}(\theta)^{d_2c_2;c_1d_1}_{a_1b_1;b_2a_2} &= S(\hat\theta,N)\left[\delta^{c_1}_{a_1}\delta^{c_2}_{a_2}\delta^{d_1}_{b_1}\delta^{d_2}_{b_2}\right.\\
&\quad -\frac{2\pi i}{N\hat\theta}\left(\delta_{a_1a_2}\delta^{c_1c_2}\delta^{d_2}_{b_2}\delta^{d_1}_{b_1}+\delta^{c_2}_{a_2}\delta^{c_1}_{a_1}\delta_{b_1b_2}\delta^{d_1d_2}\right)\\
&\quad \left.-\frac{4\pi^2}{N^2\hat\theta^2}\delta_{a_1a_2}\delta^{c_1c_2}\delta_{b_1b_2}\delta^{d_1d_2}\right],
\end{aligned}$$

where

$$\begin{aligned}
S(\theta,N) &= \frac{\sinh\left(\frac{\theta}{2}-\frac{\pi i}{N}\right)}{\sinh\left(\frac{\theta}{2}+\frac{\pi i}{N}\right)}\left[\frac{\Gamma(i\theta/2\pi+1)\Gamma(-i\theta/2\pi-1/N)}{\Gamma(i\theta/2\pi+1-1/N)\Gamma(-i\theta/2\pi)}\right]^2\\
&= 1+\mathcal{O}\left(\frac{1}{N^2}\right).
\end{aligned} \tag{4.1.7}$$

The creation operators satisfy the Zamolodchikov algebra:

$$\begin{aligned}
\mathfrak{A}^\dagger_P(\theta_1)_{a_1b_1}\mathfrak{A}^\dagger_P(\theta_2)_{a_2b_2} &= S_{PP}(\theta)^{c_2d_2;c_1d_1}_{a_1b_1;a_2b_2}\mathfrak{A}^\dagger_P(\theta_2)_{c_2d_2}\mathfrak{A}^\dagger_P(\theta_1)_{c_1d_1},\\
\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2} &= S_{AA}(\theta)^{d_2c_2;d_1c_1}_{b_1a_1;b_2a_2}\mathfrak{A}^\dagger_A(\theta_2)_{d_2c_2}\mathfrak{A}^\dagger_A(\theta_1)_{d_1c_1},\\
\mathfrak{A}^\dagger_P(\theta_1)_{a_1b_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2} &= S_{AP}(\theta)^{d_2c_2;c_1d_1}_{a_1b_1;b_2a_2}\mathfrak{A}^\dagger_A(\theta_2)_{d_2c_2}\mathfrak{A}^\dagger_P(\theta_1)_{c_1d_1}.
\end{aligned} \tag{4.1.8}$$

In the next section we will use the Smirnov form factor axioms to calculate the two-particle form factors of the Noether current operators (4.1.2) at finite and infinite N. We calculate explicitly the four-particle form factors at large N. We later find a general expression for the form factor with any number of particles. The two-point correlation function of two Noether currents is found by summing over all the form factors. The same procedure is later repeated for the energy-momentum tensor operator. This combination of the integrable bootstrap and the large-N limit was also used by P. Orland in References [36] to calculate all the form factors of the renormalized field operator.

4.2 The two-particle form factor

In this section, we calculate the first non vanishing form factor of the current operators. We will discuss only the left-handed current $j^L_\mu(x)^c_a$ in detail, since the same method yields the right-handed-current form factor.

Under a global $SU(N) \times SU(N)$ transformation, the current and the particle and antiparticle creation operators transform as

$$j^L_\mu(x) \to V_L j^L_\mu(x) V_L^\dagger, \qquad \mathfrak{A}^\dagger_P(\theta) \to V_R^\dagger \mathfrak{A}^\dagger_P(\theta) V_L^\dagger,$$
$$\mathfrak{A}^\dagger_A(\theta) \;\; \to \;\; V_R \mathfrak{A}^\dagger_A(\theta) V_L^{;}.$$

Only form factors with an equal number of particles and antiparticles are invariant under such global transformations.

The first nontrivial form factor is

$$\begin{aligned}\langle 0| j^L_\mu(x)_{a_0 c_0} \mid A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2\rangle_{\rm in} &= \langle 0| j^L_\mu(x)_{a_0 c_0} \mathfrak{A}^\dagger_A(\theta_1)_{b_1 a_1} \mathfrak{A}^\dagger_P(\theta_2)_{a_2 b_2}|0\rangle \\ &= e^{-ix\cdot(p_1+p_2)} F_\mu(\theta)_{a_0 a_1 a_2 c_0; b_1 b_2}, \end{aligned} \tag{4.2.1}$$

for $\theta_1 > \theta_2$. The function $F_\mu(\theta)_{a_0 a_1 a_2 c_0; b_1 b_2}$ is restricted by the conservation and the tracelessness of the Noether current, such that

$$(p_1 + p_2)^\mu F_\mu(\theta)_{a_0 a_1 a_2 c_0; b_1 b_2} = 0, \tag{4.2.2}$$

and

$$\delta^{a_0 c_0} F_\mu(\theta)_{a_0 a_1 a_2 c_0; b_1 b_2} = 0. \tag{4.2.3}$$

The condition (4.2.2) means that we can write

$$\begin{aligned} F_\mu(\theta)_{a_0 a_1 a_2 c_0; b_1 b_2} &= (p_1 - p_2)_\mu F(\theta)_{a_0 a_1 a_2 c_0; b_1 b_2} \\ &= -\epsilon_{\mu\nu}(p_1 + p_2)^\nu \tanh\left(\frac{\theta}{2}\right) F(\theta)_{a_0 a_1 a_2 c_0; b_1 b_2}. \end{aligned} \tag{4.2.4}$$

The condition (4.2.2) is satisfied by choosing

$$F(\theta)_{a_0 a_1 a_2 c_0; b_1 b_2} = F(\theta)\left(\delta_{a_0 a_2}\delta_{c_0 a_1}\delta_{b_1 b_2} - \frac{1}{N}\delta_{a_0 c_0}\delta_{a_1 a_2}\delta_{b_1 b_2}\right).$$

For $\theta_2 > \theta_1$, we define the form factor

$$\begin{aligned} &\langle 0 | j^L_\mu(x)_{a_0 c_0} \mid P, \theta_2, a_2, b_2; A, \theta_1, b_1, a_1\rangle_{\text{in}} \\ &\quad = (p_1 - p_2)_\mu e^{-ix\cdot(p_1+p_2)} F'(\theta)\left(\delta_{a_0 a_2}\delta_{c_0 a_1}\delta_{b_1 b_2} - \frac{1}{N}\delta_{a_0 c_0}\delta_{a_1 a_2}\delta_{b_1 b_2}\right). \end{aligned}$$

Next we further restrict the functions $F(\theta)$, $F'(\theta)$ by applying the Smirnov form factor axioms. First we apply the scattering axiom. The scattering axiom follows from the Zamolodchikov algebra (4.1.8). This axiom implies

$$\begin{aligned} &\langle 0 | j^L_\mu(0)_{a_0 c_0} \mathfrak{A}^\dagger_P(\theta_2)_{a_2 b_2} \mathfrak{A}^\dagger_A(\theta_1)_{b_1 a_1} | 0\rangle \\ &\quad = S_{AP}(\theta)^{d_1 c_1; c_2 d_2}_{a_2 b_2; b_1 a_1} \langle 0 | j^L_\mu(0)_{a_0 c_0} \mathfrak{A}^\dagger_A(\theta_1)_{d_1 c_1} \mathfrak{A}^\dagger_P(\theta_2)_{c_2 d_2} | 0\rangle, \end{aligned} \tag{4.2.5}$$

or

$$F'(\theta) = S(\hat{\theta}, N)\left(1 - \frac{2\pi i}{\hat{\theta}}\right) F(\theta).$$

We next consider the Smirnov periodicity axiom. For the two-particle form factor, this axiom implies

$$\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_2)_{a_2b_2}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}|0\rangle \\ = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1 - 2\pi i)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_2)_{a_2b_2}|0\rangle,$$

or

$$F'(\theta) = F(\theta - 2\pi i). \tag{4.2.6}$$

Combining (4.2.5) and (4.2.6) gives

$$F(\theta - 2\pi i) = \hat{S}(\theta, N)\left(\frac{\theta + \pi i}{\theta - \pi i}\right) F(\theta), \tag{4.2.7}$$

where we have defined $\hat{S}(\theta, N) = S(\hat{\theta}, N)$.

Equation (4.2.7) can be easily solved at large N. For large N we expand $\hat{S}(\theta, N) = 1 + \mathcal{O}\left(\frac{1}{N^2}\right)$, and $F(\theta) = F^0(\theta) + \frac{1}{N}F^1(\theta) + \frac{1}{N^2}F^2(\theta) + \ldots$, and keep only terms up to order $\frac{1}{N^0}$, so that

$$F^0(\theta - 2\pi i) = \left(\frac{\theta + \pi i}{\theta - \pi i}\right) F^0(\theta). \tag{4.2.8}$$

The general solution to (4.2.8) is

$$F^0(\theta) = \frac{g(\theta)}{\theta + \pi i},$$

where $g(\theta)$ satisfies the periodicity condition, $g(\theta - 2\pi i) = g(\theta)$. The minimal choice is taking $g(\theta) = g$ to be a constant.

Next, we determine the value of g. There is a conserved charge, $Q^L_{a_0c_0}$, associated with the current operator. This charge is

$$Q^L_{a_0c_0} = \int dx^1\, j^L_0(x)_{a_0c_0}.$$

We fix the value of g by requiring that the charge is a generator of $SU(N)$, so it satisfies the Lie algebra

$$[Q^{L\,c_1}_{a_1}, Q^{L\,c_2}_{a_2}] = -(\delta^{c_2}_{a_1}\delta^{a_3}_{a_2}\delta^{c_1}_{c_3} - \delta^{c_1}_{a_2}\delta^{a_3}_{a_1}\delta^{c_2}_{c_3})Q^{L\,c_3}_{a_3}. \tag{4.2.9}$$

We cross the incoming particle in Eq. (4.2.1) to an outgoing antiparticle, by shifting its rapidity, $\theta_2 \to \theta_2 - \pi i$:

$$\begin{aligned}
&\langle A, \theta_2, b_2, a_2 \mid j^L_0(x)_{a_0c_0} | A, \theta_1, b_1, a_1\rangle \\
&\quad = \; m\,(\cosh\,\theta_1 + \cosh\,\theta_2)\exp\big\{-im[x^0(\cosh\,\theta_1 - \cosh\,\theta_2) \\
&\qquad -x^1(\sinh\,\theta_1 - \sinh\,\theta_2)]\big\}\,F(\theta + \pi i) \\
&\qquad \times \left(\delta_{a_0a_2}\delta_{b_1b_2}\delta_{c_0a_1} - \frac{1}{N}\delta_{a_0c_0}\delta_{b_1b_2}\delta_{a_1a_2}\right).
\end{aligned}$$

Integrating over x^1 gives the matrix element of the charge operator:

$$\begin{aligned}\langle A,\theta_2,b_2,a_2|Q^L_{a_0c_0}|A,\theta_1,b_1,a_1\rangle &= (2\pi)^2\,2(p_1)_0\delta(\theta_1-\theta_2)\\ &\times \left(\delta_{a_0a_2}\delta_{b_1b_2}\delta_{c_0a_1}-\frac{1}{N}\delta_{a_0c_0}\delta_{b_1b_2}\delta_{a_1a_2}\right)F(\pi i).\end{aligned}$$

The matrix element of the commutator of two charges is found by inserting a complete set of intermediate states:

$$\begin{aligned}&\langle A,\theta_2,b_2,a_2|[Q^L_{a_0c_0},\,Q^L_{a_4c_4}]|A,\theta_1,b_1,a_1\rangle\\ &= \int\frac{d\theta_3}{4\pi}\langle A,\theta_2,b_2,a_2|Q^L_{a_0c_0}|A,\theta_3,b_3,a_3\rangle\langle A,\theta_3,b_3,a_3|Q^L_{a_4c_4}|A,\theta_1,b_1,a_1\rangle\\ &\quad-\int\frac{d\theta_3}{4\pi}\langle A,\theta_2,b_2,a_2|Q^L_{a_4c_4}|A,\theta_3,b_3,a_3\rangle\\ &\quad\times\langle A,\theta_3,b_3,a_3|Q^L_{a_0c_0}|A,\theta_1,b_1,a_1\rangle.\end{aligned}\tag{4.2.10}$$

With the choice $F(\pi i)=1$, Eq. (4.2.10) becomes

$$\begin{aligned}&\langle A,\theta_2,b_2,a_2|[Q^L_{a_0c_0},\,Q^L_{a_4c_4}]|A,\theta_1,b_1,a_1\rangle\\ &= -(\delta^{c_2}_{a_1}\delta^{a_3}_{a_2}\delta^{c_1}_{c_3}-\delta^{c_1}_{a_2}\delta^{a_3}_{a_1}\delta^{c_2}_{c_3})\langle A,\theta_2,b_2,a_2|Q^{L\,c_5}_{a_5}|A,\theta_1,b_1,a_1\rangle,\end{aligned}$$

which is equivalent to Eq. (4.2.9). This fixes the constant $g=2\pi i$.

To find the form factor for general, finite N, we need to solve the full Equation (4.2.7) instead of the simpler (4.2.8). We start with the ansatz

$$F(\theta)=\frac{g(\theta)}{\theta+\pi i}.\tag{4.2.11}$$

In terms of $g(\theta)$, Equation (4.2.7) becomes

$$g(\theta - 2\pi i) = \hat{S}(\theta, N) g(\theta). \tag{4.2.12}$$

We solve Equation (4.2.12) by a contour-integration method first used in Ref. [24]. We define a contour C to be that from $-\infty$ to ∞ and from $\infty + 2\pi i$ to $-\infty + 2\pi i$, bounding the strip in which the form factor is holomorphic. Then

$$\ln g(\theta) = \int_C \frac{dz}{4\pi i} \coth \frac{z-\theta}{2} \ln g(z) = \int_{-\infty}^{\infty} \frac{dz}{4\pi i} \coth \frac{z-\theta}{2} \ln \frac{g(z)}{g(z+2\pi i)}.$$

We differentiate both sides with repeat to θ, and use Eq. (4.2.12) to write

$$\frac{d}{d\theta} [\ln g(\theta)] = \frac{1}{8\pi i} \int_{-\infty}^{\infty} \frac{dz}{\sinh^2 \frac{1}{2}(z-\theta)} \ln \hat{S}(z, N). \tag{4.2.13}$$

The solution to (4.2.13) is

$$g(\theta) = g \exp \int_0^{\infty} dx\, A(x, N) \frac{\sin^2[x(\pi i - \theta)/2\pi]}{\sinh x}, \tag{4.2.14}$$

where the function $A(x, N)$ is defined by

$$\hat{S}(\theta, N) = \exp \int_0^{\infty} dx\, A(x, N) \sinh\left(\frac{x\theta}{\pi i}\right), \tag{4.2.15}$$

and g is a constant. Note that expanding the S matrix in powers of $1/N$ yields $A(x, N) = \frac{1}{N^2} B(x) + \mathcal{O}(\frac{1}{N^3})$.

To express the function $\hat{S}(\theta, N)$, presented in (4.1.7), in the form (4.2.15), we use the integral formula of the gamma function [24], [37],

$$\Gamma(z) = \exp \int_0^\infty \frac{dx}{x} \left[\frac{e^{-xz} - e^{-x}}{1 - e^{-x}} + (z-1)e^{-x} \right], \quad \text{for} \operatorname{Re} z > 0.$$

Then

$$\begin{aligned} &\left[\frac{\Gamma\left(\frac{i\hat{\theta}}{2\pi} + 1\right) \Gamma\left(\frac{-i\hat{\theta}}{2\pi} - \frac{1}{N}\right)}{\Gamma\left(\frac{i\hat{\theta}}{2\pi} + 1 - \frac{1}{N}\right) \Gamma\left(\frac{-i\hat{\theta}}{2\pi}\right)} \right]^2 \\ &\qquad = \exp \int_0^\infty \frac{dx}{x} \frac{4e^{-x}\left(e^{2x/N} - 1\right)}{1 - e^{-2x}} \sinh\left(\frac{x\theta}{\pi i}\right), \end{aligned} \tag{4.2.16}$$

for $N > 2$. We use the formula [22]

$$\frac{\sin \frac{\pi}{2}(z+a)}{\sin \frac{\pi}{2}(z-a)} = \exp 2 \int_0^\infty \frac{dx}{x} \frac{\sinh x(1-z)}{\sinh x} \sinh(xa), \quad \text{for } 0 < z < 1,$$

to write the CDD factor as

$$\begin{aligned} \frac{\sinh\left(\frac{\hat{\theta}}{2} - \frac{\pi i}{N}\right)}{\sinh\left(\frac{\hat{\theta}}{2} + \frac{\pi i}{N}\right)} &= \frac{\sin \frac{\pi}{2}\left(\left(1 - \frac{2}{N}\right) - \frac{\theta}{\pi i}\right)}{\sin \frac{\pi}{2}\left(\left(1 - \frac{2}{N}\right) + \frac{\theta}{\pi i}\right)} \\ &= \exp \int_0^\infty \frac{dx}{x} \frac{-2\sinh(2x/N)}{\sinh x} \sinh\left(\frac{x\theta}{\pi i}\right), \end{aligned} \tag{4.2.17}$$

for $N > 2$. Combining (4.2.16) and (4.2.17) gives

$$\begin{aligned} \hat{S}(\theta, N) \;\; = \;\; \exp \int_0^\infty \frac{dx}{x} &\left[\frac{-2\sinh(2x/N)}{\sinh x} \right. \\ &\left. + \frac{4e^{-x}\left(e^{2x/N} - 1\right)}{1 - e^{-2x}} \right] \sinh\left(\frac{x\theta}{\pi i}\right). \end{aligned} \tag{4.2.18}$$

From (4.2.11) and (4.2.14), the form factor is

$$\begin{aligned} F_1(\theta) &= \frac{g}{(\theta+\pi i)} \exp \int_0^\infty \frac{dx}{x} \left[\frac{-2\sinh\left(\frac{2x}{N}\right)}{\sinh x} \right. \\ & \left. + \frac{4e^{-x}\left(e^{2x/N}-1\right)}{1-e^{-2x}} \right] \frac{\sin^2[x(\pi i - \theta)/2\pi]}{\sinh x}. \end{aligned} \tag{4.2.19}$$

The condition $F_1(\pi i) = 1$ implies $g = 2\pi i$.

4.3 Four-particle form factors

In this section we find the four-excitation form factor of the Noether current operator, in the large-N limit. Only the form factor with two particles and two antiparticles is nonzero, because of the global color symmetry. The most general Lorentz- and $SU(N)\times SU(N)$-invariant four-particle form factor, respecting the tracelessness of the current operator is

$$\begin{aligned} &\langle 0 | j_\mu^L(0)_{a_0c_0} \mid A,\theta_1,b_1,a_1;A,\theta_2,b_2,a_2;P,\theta_3,a_3,b_3;P,\theta_4,a_4,b_4\rangle \\ &\quad = \langle 0 | j_\mu^L(0)_{a_0c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2a_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4b_4} |0\rangle \\ &\quad = -\epsilon_{\mu\nu}(p_1+p_2+p_3+p_4)^\nu \frac{1}{N} \\ &\qquad \times \vec{F}(\theta_1,\theta_2,\theta_3,\theta_4)\cdot \vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4}, \end{aligned} \tag{4.3.1}$$

for $\theta_1 > \theta_2 > \theta_3 > \theta_4$,

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0} \mid A,\theta_1,b_1,a_1;P,\theta_2,a_2,b_2;A,\theta_3,b_3,a_3;P,\theta_4,a_4,b_4\rangle \\
&\quad= \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle \\
&\quad= -\epsilon_{\mu\nu}(p_1+p_2+p_3+p_4)^\nu\frac{1}{N} \\
&\qquad\times\vec{G}(\theta_1,\theta_2,\theta_3,\theta_4)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},
\end{aligned} \tag{4.3.2}$$

for $\theta_1 > \theta_3 > \theta_2 > \theta_4$,

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0} \mid A,\theta_1,b_1,a_1;P,\theta_2,a_2,b_2;P,\theta_3,a_3,b_3;A,\theta_4,b_4,a_4\rangle \\
&\quad= \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle \\
&\quad= -\epsilon_{\mu\nu}(p_1+p_2+p_3+p_4)^\nu\frac{1}{N} \\
&\qquad\times\vec{H}(\theta_1,\theta_2,\theta_3,\theta_4)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},
\end{aligned} \tag{4.3.3}$$

for $\theta_1 > \theta_3 > \theta_4 > \theta_2$,

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0} \mid P,\theta_1,a_1,b_1;A,\theta_2,b_2,a_2;P,\theta_3,a_3,b_3;A,\theta_4,b_4,a_4\rangle \\
&\quad= \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle \\
&\quad= -\epsilon_{\mu\nu}(p_1+p_2+p_3+p_4)^\nu\frac{1}{N} \\
&\qquad\times\vec{K}(\theta_1,\theta_2,\theta_3,\theta_4)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},
\end{aligned} \tag{4.3.4}$$

for $\theta_3 > \theta_1 > \theta_4 > \theta_2$,

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0} \mid P,\theta_1,a_1,b_1;P,\theta_2,a_2,b_2;A,\theta_3,b_3,a_3;A,\theta_4,b_4,a_4\rangle \\
&\quad = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle \\
&\quad = -\epsilon_{\mu\nu}(p_1+p_2+p_3+p_4)^\nu\frac{1}{N} \\
&\qquad \times\vec{L}(\theta_1,\theta_2,\theta_3,\theta_4)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},
\end{aligned} \tag{4.3.5}$$

for $\theta_3 > \theta_4 > \theta_1 > \theta_2$,

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0} \mid P,\theta_1,a_1,b_1;A,\theta_2,b_2,a_2;A,\theta_3,b_3,a_3;P,\theta_4,a_4,b_4\rangle \\
&\quad = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3a_3}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle \\
&\quad = -\epsilon_{\mu\nu}(p_1+p_2+p_3+p_4)^\nu\frac{1}{N} \\
&\qquad \times\vec{Q}(\theta_1,\theta_2,\theta_3,\theta_4)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},
\end{aligned} \tag{4.3.6}$$

for $\theta_3 > \theta_1 > \theta_2 > \theta_4$,

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0} \mid A,\theta_2,b_2,a_2;A,\theta_1,b_1,a_1;P,\theta_3,a_3,b_3;P,\theta_4,a_4,b_4\rangle \\
&\quad = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle \\
&\quad = -\epsilon_{\mu\nu}(p_1+p_2+p_3+p_4)^\nu\frac{1}{N} \\
&\qquad \times\vec{F}(\theta_2,\theta_1,\theta_3,\theta_4)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},
\end{aligned} \tag{4.3.7}$$

for $\theta_2 > \theta_1 > \theta_3, > \theta_4$, and

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0} \mid A,\theta_1,b_1,a_1;A,\theta_2,b_2,a_2;P,\theta_4,a_4,b_4;P,\theta_3,a_3,b_3\rangle \\
&\quad= \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}|0\rangle \\
&\quad= -\epsilon_{\mu\nu}(p_1+p_2+p_3+p_4)^\nu\frac{1}{N} \\
&\qquad\times\vec{F}(\theta_1,\theta_2,\theta_4,\theta_3)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},
\end{aligned} \tag{4.3.8}$$

for $\theta_1 > \theta_2 > \theta_4 > \theta_3$, where we define the eight-component vectors

$$[\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4}] = \begin{pmatrix}
\delta_{a_0a_3}\delta_{a_1c_0}\delta_{a_2a_4}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_3}\delta_{a_2a_4}\delta_{b_1b_3}\delta_{b_2b_4} \\
\delta_{a_0a_3}\delta_{a_1c_0}\delta_{a_2a_4}\delta_{b_1b_4}\delta_{b_2b_3} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_3}\delta_{a_2a_4}\delta_{b_1b_4}\delta_{b_2b_3} \\
\delta_{a_0a_4}\delta_{a_1c_0}\delta_{a_2a_3}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_4}\delta_{a_2a_3}\delta_{b_1b_3}\delta_{b_2b_4} \\
\delta_{a_0a_4}\delta_{a_1c_0}\delta_{a_2a_3}\delta_{b_1b_4}\delta_{b_2b_3} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_4}\delta_{a_2a_3}\delta_{b_1b_4}\delta_{b_2b_3} \\
\delta_{a_0a_3}\delta_{a_1a_4}\delta_{a_2c_0}\delta_{b_1b_4}\delta_{b_2b_3} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_3}\delta_{a_1a_4}\delta_{b_1b_4}\delta_{b_2b_3} \\
\delta_{a_0a_3}\delta_{a_1a_4}\delta_{a_2c_0}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_3}\delta_{a_1a_4}\delta_{b_1b_3}\delta_{b_2b_4} \\
\delta_{a_0a_4}\delta_{a_1a_3}\delta_{a_2c_0}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_4}\delta_{a_1a_3}\delta_{b_1b_3}\delta_{b_2b_4} \\
\delta_{a_0a_4}\delta_{a_1a_3}\delta_{a_2a_0}\delta_{b_1b_4}\delta_{b_2b_3} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_4}\delta_{a_1a_3}\delta_{b_1b_4}\delta_{b_2b_3}
\end{pmatrix},$$

$$[\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4)] = \begin{pmatrix} F_1(\theta_1,\theta_2,\theta_3,\theta_4) \\ \\ F_2(\theta_1,\theta_2,\theta_3,\theta_4) \\ \\ F_3(\theta_1,\theta_2,\theta_3,\theta_4) \\ \\ F_4(\theta_1,\theta_2,\theta_3,\theta_4) \\ \\ F_5(\theta_1,\theta_2,\theta_3,\theta_4) \\ \\ F_6(\theta_1,\theta_2,\theta_3,\theta_4) \\ \\ F_7(\theta_1,\theta_2,\theta_3,\theta_4) \\ \\ F_8(\theta_1,\theta_2,\theta_3,\theta_4) \end{pmatrix},$$

and similarly for $\vec{G}$, $\vec{H}$, $\vec{K}$, $\vec{L}$ and $\vec{Q}$.

The scattering axiom relates the form factors with different ordering of rapidities, yielding

$$\begin{aligned}
&\langle 0|j^L_\mu(0\,)_{\,a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle \\
&\quad = \quad S_{AP}(\theta_{23})^{d_2c_2;c_3d_3}_{a_3b_3;b_2a_2} \\
&\qquad\qquad \times\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{d_2c_2}\mathfrak{A}^\dagger_P(\theta_3)_{c_3d_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle,
\end{aligned}$$

$$
\begin{aligned}
&\langle 0|j^L_\mu(0\,)_{\,a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle \\
&\quad= \; S_{AP}(\theta_{24})^{d_2c_2;c_4d_4}_{a_4b_4;b_2a_2} \\
&\qquad \times\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_2)_{d_2c_2}\mathfrak{A}^\dagger_P(\theta_4)_{c_4d_4}|0\rangle,
\end{aligned}
$$

$$
\begin{aligned}
&\langle 0|j^L_\mu(0\,)_{\,a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle \\
&\quad= \; S_{AP}(\theta_{13})^{d_1c_1;c_3d_3}_{a_3b_3;b_1a_1} \\
&\qquad \times\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{d_1c_1}\mathfrak{A}^\dagger_P(\theta_3)_{c_3d_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle,
\end{aligned}
$$

$$
\begin{aligned}
&\langle 0|j^L_\mu(0\,)_{\,a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_1}|0\rangle \\
&\quad= \; S_{AP}(\theta_{14})^{d_1c_1;c_4d_4}_{a_4b_4;b_1a_1} \\
&\qquad \times\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_1)_{d_1c_1}\mathfrak{A}^\dagger_P(\theta_4)_{c_4d_4}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle,
\end{aligned}
$$

$$
\begin{aligned}
&\langle 0|j^L_\mu(0\,)_{\,a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle \\
&\quad= \; S_{AP}(\theta_{13})^{d_1c_1;c_3d_3}_{a_3b_3;b_1a_1} \\
&\qquad \times\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{d_1c_1}\mathfrak{A}^\dagger_P(\theta_3)_{c_3d_3}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle,
\end{aligned}
$$

$$
\begin{aligned}
&\langle 0|j^L_\mu(0\,)_{\,a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}_P(\theta_4)_{a_4b_4}|0\rangle \\
&\quad= \; S_{AA}(\theta_{12})^{d_2c_2;d_1c_1}_{b_1a_1;b_2a_2} \\
&\qquad \times\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_2)_{d_2c_2}\mathfrak{A}^\dagger_A(\theta_1)_{d_1c_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle,
\end{aligned}
$$

$$\begin{aligned}
&\langle 0|j^L_\mu(0\,)_{\,a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}_P(\theta_4)_{a_4b_4}|0\rangle \\
&\quad = \quad S_{PP}(\theta_{34})^{c_4d_4;c_3d_3}_{a_3b_3;a_4b_4} \\
&\qquad\qquad \times\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{c_4d_4}\mathfrak{A}^\dagger_P(\theta_3)_{c_3d_3}|0\rangle,
\end{aligned}$$

where $\theta_{jk} = \theta_j - \theta_k$. These imply, respectively,

$$
\begin{aligned}
&\vec{G}(\theta_1, \theta_2,\, \theta_3\ , \theta_4) \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-2\pi i}{N\hat{\theta}_{23}} & \left(1 - \frac{2\pi i}{\hat{\theta}_{23}}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-2\pi i}{N\hat{\theta}_{23}} & 0 & \left(1 - \frac{2\pi i}{\hat{\theta}_{23}}\right) & 0 & 0 & 0 & \frac{-2\pi i}{N\hat{\theta}_{23}} & 0 \\
0 & \frac{-1}{N}\left(\frac{2\pi i}{\hat{\theta}_{23}} + \frac{4\pi^2}{\hat{\theta}_{23}^2}\right) & \frac{-1}{N}\left(\frac{2\pi i}{\hat{\theta}_{23}} + \frac{4\pi^2}{\hat{\theta}_{23}^2}\right) & \left(1 - \frac{4\pi i}{\hat{\theta}_{23}} - \frac{4\pi^2}{\hat{\theta}_{23}^2}\right) & 0 & 0 & 0 & \frac{-1}{N}\left(\frac{2\pi i}{\hat{\theta}_{23}} + \frac{4\pi^2}{\hat{\theta}_{23}^2}\right) \\
0 & 0 & 0 & 0 & \left(1 - \frac{2\pi i}{\hat{\theta}_{23}}\right) & \frac{-2\pi i}{N\hat{\theta}_{23}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\hat{\theta}_{23}} & \left(1 - \frac{2\pi i}{\hat{\theta}_{23}}\right)
\end{pmatrix} \\
&\quad \times \vec{F}(\theta_1, \theta_2, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right) \\
&\equiv \overleftrightarrow{M}_1(\theta_2, \theta_3)\vec{F}(\theta_1\theta_2, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right),
\end{aligned}
\tag{4.3.9}
$$

$$
\begin{aligned}
&\vec{H}(\theta_1, \theta_2, \theta_3, \theta_4) \\
= &\begin{pmatrix}
\left(1-\frac{4\pi i}{\hat\theta_{24}}-\frac{4\pi^2}{\hat\theta_{24}^2}\right) & \frac{-1}{N}\left(\frac{2\pi i}{\hat\theta_{24}}+\frac{4\pi^2}{\hat\theta_{24}^2}\right) & \frac{-1}{N}\left(\frac{2\pi i}{\hat\theta_{24}}+\frac{4\pi^2}{\hat\theta_{24}^2}\right) & 0 & 0 & \frac{-1}{N}\left(\frac{2\pi i}{\hat\theta_{24}}-\frac{4\pi^2}{\hat\theta_{24}^2}\right) & 0 & 0 \\
0 & \left(1-\frac{2\pi i}{\hat\theta_{24}}\right) & 0 & \frac{-2\pi i}{N\hat\theta_{24}} & \frac{-2\pi i}{N\hat\theta_{24}^2} & 0 & 0 & 0 \\
0 & 0 & \left(1-\frac{2\pi i}{\hat\theta_{24}}\right) & \frac{-2\pi i}{N\hat\theta_{24}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-2\pi i}{N\hat\theta_{24}} & \left(1-\frac{2\pi i}{\hat\theta_{24}}\right) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \left(1-\frac{2\pi i}{\hat\theta_{24}}\right) & \frac{-2\pi i}{N\hat\theta_{24}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \\
&\times \vec{G}(\theta_1, \theta_2, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right) \\
\equiv\ &\overleftrightarrow{M}_2(\theta_2, \theta_4)\vec{G}(\theta_1, \theta_2, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right),
\end{aligned}
\tag{4.3.10}
$$

$$
\begin{aligned}
&\vec{K}(\theta_1,\theta_2,\ \theta_3\ ,\theta_4) \\
&= \begin{pmatrix}
\left(1-\frac{2\pi i}{\hat{\theta}_{13}}\right) & \frac{-2\pi i}{N\hat{\theta}_{13}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \left(1-\frac{2\pi i}{\hat{\theta}_{13}}\right) & \frac{-2\pi i}{N\hat{\theta}_{13}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-2\pi i}{N\hat{\theta}_{13}} & \left(1-\frac{2\pi i}{\hat{\theta}_{13}}\right) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{-1}{N}\left(\frac{2\pi i}{\hat{\theta}_{13}}+\frac{4\pi^2}{\hat{\theta}_{13}^2}\right) & \left(1-\frac{4\pi i}{\hat{\theta}_{13}}-\frac{4\pi^2}{\hat{\theta}_{13}^2}\right) & \frac{-1}{N}\left(\frac{2\pi i}{\hat{\theta}_{13}}+\frac{4\pi^2}{\hat{\theta}_{13}^2}\right) \\
0 & 0 & 0 & \frac{-2\pi i}{N\hat{\theta}_{13}} & \frac{-2\pi i}{N\hat{\theta}_{13}} & 0 & 0 & \left(1-\frac{2\pi i}{\hat{\theta}_{13}}\right)
\end{pmatrix} \\
&\quad \times \vec{H}(\theta_1,\theta_2,\theta_3,\theta_4)+\mathcal{O}\left(\frac{1}{N^2}\right) \\
&\equiv \overleftrightarrow{M}_3(\theta_1,\theta_3)\vec{H}(\theta_1,\theta_2,\theta_3,\theta_4)+\mathcal{O}\left(\frac{1}{N^2}\right),
\end{aligned}
\tag{4.3.11}
$$

$$
\begin{aligned}
\vec{L}(\theta_1,\theta_2,\ \theta_3\ ,\theta_4)
&= \left(\begin{array}{ccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-2\pi i}{N\hat{\theta}_{14}} & \left(1-\frac{2\pi i}{\hat{\theta}_{14}}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-2\pi i}{N\hat{\theta}_{14}} & \left(1-\frac{2\pi i}{\hat{\theta}_{14}}\right) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \left(1-\frac{4\pi i}{\hat{\theta}_{14}}-\frac{4\pi^2}{\hat{\theta}_{14}^2}\right) & \frac{-1}{N}\left(\frac{2\pi i}{\hat{\theta}_{14}}+\frac{4\pi^2}{\hat{\theta}_{14}^2}\right) & 0 & \frac{-1}{N}\left(\frac{2\pi i}{\hat{\theta}_{14}}+\frac{4\pi^2}{\hat{\theta}_{14}^2}\right) \\
\frac{-2\pi i}{N\hat{\theta}_{14}} & 0 & 0 & 0 & 0 & \left(1-\frac{2\pi i}{\hat{\theta}_{14}}\right) & \frac{-2\pi i}{N\hat{\theta}_{14}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\hat{\theta}_{14}} & \left(1-\frac{2\pi i}{\hat{\theta}_{14}}\right)
\end{array}\right) \\
&\quad \times\vec{K}(\theta_1,\theta_2,\theta_3,\theta_4)+\mathcal{O}\left(\frac{1}{N^2}\right) \\
&\equiv \overleftrightarrow{M}_4(\theta_1,\theta_4)\vec{K}(\theta_1,\theta_2,\theta_3,\theta_4)+\mathcal{O}\left(\frac{1}{N^2}\right),
\end{aligned}
\tag{4.3.12}
$$

$$
\vec{Q}(\theta_1,\theta_2,\theta_3,\theta_4)=\overleftrightarrow{M}_3(\theta_1,\theta_3)\vec{G}(\theta_1,\theta_2,\theta_3,\theta_4)+\mathcal{O}\left(\frac{1}{N^2}\right), \tag{4.3.13}
$$

$$\begin{aligned}\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4) &= \begin{pmatrix} 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 1 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 \\ \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 1 \\ 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 0 & 1 & \frac{-2\pi i}{N\theta_{12}} \\ 1 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 \\ \frac{-2\pi i}{N\theta_{12}} & 1 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 0 \\ 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 1 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} \\ 0 & 0 & 1 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 \end{pmatrix} \\ &\quad \times\vec{F}(\theta_2,\theta_1,\theta_3,\theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right) \\ &\equiv \overleftrightarrow{I}_1(\theta_1,\theta_2)\vec{F}(\theta_2,\theta_1,\theta_3,\theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right), \end{aligned} \tag{4.3.14}$$

$$\begin{aligned}\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4) &= \begin{pmatrix} 0 & \frac{-2\pi i}{N\theta_{34}} & \frac{-2\pi i}{N\theta_{34}} & 1 & 0 & 0 & 0 & 0 \\ \frac{-2\pi i}{N\theta_{34}} & 0 & 1 & \frac{-2\pi i}{N\theta_{34}} & 0 & 0 & 0 & 0 \\ \frac{-2\pi i}{N\theta_{34}} & 1 & 0 & \frac{-2\pi i}{N\theta_{34}} & 0 & 0 & 0 & 0 \\ 1 & \frac{-2\pi i}{N\theta_{34}} & \frac{-2\pi i}{N\theta_{34}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{34}} & 1 & \frac{-2\pi i}{N\theta_{34}} \\ 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{34}} & 0 & \frac{-2\pi i}{N\theta_{34}} & 1 \\ 0 & 0 & 0 & 0 & 1 & \frac{-2\pi i}{N\theta_{34}} & 0 & \frac{-2\pi i}{N\theta_{34}} \\ 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{34}} & 1 & \frac{-2\pi i}{N\theta_{34}} & 0 \end{pmatrix} \\ &\quad \times\vec{F}(\theta_1,\theta_2,\theta_4,\theta_3) + \mathcal{O}\left(\frac{1}{N^2}\right) \\ &\equiv \overleftrightarrow{I}_2(\theta_3,\theta_4)\vec{F}(\theta_1,\theta_2,\theta_4,\theta_3) + \mathcal{O}\left(\frac{1}{N^2}\right). \end{aligned} \tag{4.3.15}$$

Next we apply the Smirnov periodicity axiom (4.4.4):

$$\begin{aligned}&\langle 0|j^L_\mu(0\,)_{\,a_0c_0}\mathfrak{A}^\dagger_A(\theta_1-2\pi i)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle\\ &\qquad = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}|0\rangle,\end{aligned}$$

$$\begin{aligned}&\langle 0|j^L_\mu(0\,)_{\,a_0c_0}\mathfrak{A}^\dagger_A(\theta_2-2\pi i)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}|0\rangle\\ &\qquad = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle,\end{aligned}$$

$$\begin{aligned}&\langle 0|j^L_\mu(0\,)_{\,a_0c_0}\mathfrak{A}^\dagger_P(\theta_3-2\pi i)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle\\ &\qquad = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}|0\rangle,\end{aligned}$$

$$\begin{aligned}&\langle 0|j^L_\mu(0\,)_{\,a_0c_0}\mathfrak{A}^\dagger_P(\theta_4-2\pi i)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}|0\rangle\\ &\qquad = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle,\end{aligned}$$

which imply, respectively,

$$\vec{F}(\theta_1-2\pi i,\theta_2,\theta_3,\theta_4) = \vec{H}(\theta_2,\theta_1,\theta_3,\theta_4), \tag{4.3.16}$$

$$\vec{H}(\theta_2-2\pi i,\theta_1,\theta_3,\theta_4) = \vec{L}(\theta_1,\theta_2,\theta_3,\theta_4), \tag{4.3.17}$$

$$\vec{L}(\theta_1,\theta_2,\theta_3-2\pi i,\theta_4) = \vec{Q}(\theta_1,\theta_2,\theta_4,\theta_3), \tag{4.3.18}$$

$$\vec{Q}(\theta_1,\theta_2,\theta_4-2\pi i,\theta_3) = \vec{F}(\theta_1,\theta_2,\theta_3,\theta_4). \tag{4.3.19}$$

We combine Watson's theorem with the periodicity axiom, to express Equations (4.3.16), (4.3.17), (4.3.18) and (4.3.19) in terms of only $\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4)$. We combine (4.3.16) with

(4.3.12), (4.3.11) and (4.3.14), and find

$$\begin{aligned}&\vec{F}(\theta_1 - 2\pi i, \theta_2, \theta_3, \theta_4)\\ &\quad= \overleftrightarrow{M}_4(\theta_1, \theta_4)\overleftrightarrow{M}_3(\theta_1, \theta_3)\left[\overleftrightarrow{I}_1(\theta_1, \theta_2)\right]^{-1}\vec{F}(\theta_1, \theta_2, \theta_3, \theta_4).\end{aligned} \tag{4.3.20}$$

Combining (4.3.17) with (4.3.10), (4.3.9) and (4.3.14) gives

$$\begin{aligned}&\left[\overleftrightarrow{I}_1(\theta_1, \theta_2 - 2\pi i)\right]^{-1}\vec{F}(\theta_1, \theta_2 - 2\pi i, \theta_3, \theta_4)\\ &\qquad=\overleftrightarrow{M}_2(\theta_2, \theta_4)\overleftrightarrow{M}_1(\theta_2, \theta_4)\vec{F}(\theta_1, \theta_2, \theta_3, \theta_4).\end{aligned} \tag{4.3.21}$$

Combining (4.3.18) with (4.3.11), (4.3.9) and (4.3.15) gives

$$\begin{aligned}&\overleftrightarrow{M}_3(\theta_1, \theta_3 - 2\pi i)\overleftrightarrow{M}_1(\theta_2, \theta_3 - 2\pi i)\vec{F}(\theta_1, \theta_2, \theta_3, \theta_4)\\ &\qquad=\left[\overleftrightarrow{I}_2(\theta_3, \theta_4)\right]^{-1}\vec{F}(\theta_1, \theta_2, \theta_3, \theta_4).\end{aligned} \tag{4.3.22}$$

Finally, we combine (4.3.19) with (4.3.12), (4.3.10) and (4.3.15) to find

$$\begin{aligned}&\overleftrightarrow{M}_4(\theta_1, \theta_4 - 2\pi i)\overleftrightarrow{M}_2(\theta_2, \theta_4 - 2\pi i)\left[\overleftrightarrow{I}_2(\theta_3, \theta_4 - 2\pi i)\right]^{-1}\vec{F}(\theta_1, \theta_2, \theta_3, \theta_4 - 2\pi i)\\ &\qquad=\vec{F}(\theta_1, \theta_2, \theta_3, \theta_4).\end{aligned} \tag{4.3.23}$$

The set of equations (4.3.20), (4.3.21), (4.3.22) and (4.3.23) are difficult to solve, for finite N. In the large-N limit, the matrices $\overleftrightarrow{M}_{1,2,3,4}$ become diagonal and mutually commute, and the matrices $\overleftrightarrow{I}_{1,2}$ become their own inverses. This greatly simplifies the problem, allowing us to find the form factors. We expand the form factors in powers of $1/N$ as $\vec{F}(\theta_1, \theta_2, \theta_3, \theta_4) = \vec{F}^0(\theta_1, \theta_2, \theta_3, \theta_4) + \frac{1}{N}\vec{F}^1(\theta_1, \theta_2, \theta_3, \theta_4) + \dots$, simplifying the periodicity conditions for $\vec{F}^0(\theta_1,\theta_2,\theta_3,\theta_4)$. We combine (4.3.20) and (4.3.21) to get

$$\begin{aligned}&\vec{F}^0(\theta_1-2\pi i,\theta_2-2\pi i,\theta_3,\theta_4)\\&=\overleftrightarrow{M}_4(\theta_1,\theta_4)\overleftrightarrow{M}_3(\theta_1,\theta_3)\overleftrightarrow{M}_2(\theta_2,\theta_4)\overleftrightarrow{M}_1(\theta_2,\theta_3)\vec{F}^0(\theta_1,\theta_2,\theta_3,\theta_4),\end{aligned}\tag{4.3.24}$$

or explicitly, in terms of the components of $\vec{F}^0(\theta_1,\theta_2,\theta_3,\theta_4)$,

$$\begin{aligned}F_1^0(\theta_1-2\pi i,\theta_2-2\pi i,\theta_3,\theta_4)&=\left(\frac{\theta_{13}+\pi i}{\theta_{13}-\pi i}\right)\left(\frac{\theta_{24}+\pi i}{\theta_{24}-\pi i}\right)^2F_1^0(\theta_1,\theta_2,\theta_3,\theta_4),\\F_2^0(\theta_1-2\pi i,\theta_2-2\pi i,\theta_3,\theta_4)&=\left(\frac{\theta_{14}+\pi i}{\theta_{14}-\pi i}\right)\left(\frac{\theta_{23}+\pi i}{\theta_{23}-\pi i}\right)\\&\quad\times\left(\frac{\theta_{24}+\pi i}{\theta_{24}-\pi i}\right)F_2^0(\theta_1,\theta_2,\theta_3,\theta_4),\\F_3^0(\theta_1-2\pi i,\theta_2-2\pi i,\theta_3,\theta_4)&=\left(\frac{\theta_{13}+\pi i}{\theta_{13}-\pi i}\right)\left(\frac{\theta_{23}+\pi i}{\theta_{23}-\pi i}\right)\\&\quad\times\left(\frac{\theta_{24}+\pi i}{\theta_{24}-\pi i}\right)F_3^0(\theta_1,\theta_2,\theta_3,\theta_4),\\F_4^0(\theta_1-2\pi i,\theta_2-2\pi i,\theta_3,\theta_4)&=\left(\frac{\theta_{14}+\pi i}{\theta_{14}-\pi i}\right)\left(\frac{\theta_{23}+\pi i}{\theta_{23}-\pi i}\right)^2F_4^0(\theta_1,\theta_2,\theta_3,\theta_4),\end{aligned}$$

$$
\begin{aligned}
F_5^0(\theta_1-2\pi i,\theta_2-2\pi i,\theta_3,\theta_4) &= \left(\frac{\theta_{14}+\pi i}{\theta_{14}-\pi i}\right)^2\left(\frac{\theta_{23}+\pi i}{\theta_{23}-\pi i}\right)F_5^0(\theta_1,\theta_2,\theta_3,\theta_4),\\
F_6^0(\theta_1-2\pi i,\theta_2-2\pi i,\theta_3,\theta_4) &= \left(\frac{\theta_{14}+\pi i}{\theta_{14}-\pi i}\right)\left(\frac{\theta_{13}+\pi i}{\theta_{13}-\pi i}\right)\\
&\quad\times\left(\frac{\theta_{24}+\pi i}{\theta_{24}-\pi i}\right)F_6^0(\theta_1,\theta_2,\theta_3,\theta_4),\\
F_7^0(\theta_1-2\pi i,\theta_2-2\pi i,\theta_3,\theta_4) &= \left(\frac{\theta_{13}+\pi i}{\theta_{13}-\pi i}\right)^2\left(\frac{\theta_{24}+\pi i}{\theta_{24}-\pi i}\right)F_7^0(\theta_1,\theta_2,\theta_3,\theta_4),\\
F_8^0(\theta_1-2\pi i,\theta_2-2\pi i,\theta_3,\theta_4) &= \left(\frac{\theta_{14}+\pi i}{\theta_{14}-\pi i}\right)\left(\frac{\theta_{13}+\pi i}{\theta_{13}-\pi i}\right)\\
&\quad\times\left(\frac{\theta_{23}+\pi i}{\theta_{23}-\pi i}\right)F_8^0(\theta_1,\theta_2,\theta_3,\theta_4).
\end{aligned}
$$

The solution that satisfies (4.3.24), (4.3.14) and (4.3.15) is

$$
\begin{aligned}
F_1^0(\theta_1,\theta_2,\theta_3,\theta_4) &= \frac{g_1(\theta_1,\theta_2,\theta_3,\theta_4)}{(\theta_{13}+\pi i)(\theta_{24}+\pi i)^2},\\
F_2^0(\theta_1,\theta_2,\theta_3,\theta_4) &= \frac{g_2(\theta_1,\theta_2,\theta_3,\theta_4)}{(\theta_{14}+\pi i)(\theta_{23}+\pi i)(\theta_{24}+\pi i)},\\
F_3^0(\theta_1,\theta_2,\theta_3,\theta_4) &= \frac{g_2(\theta_1,\theta_2,\theta_4,\theta_3)}{(\theta_{13}+\pi i)(\theta_{23}+\pi i)(\theta_{24}+\pi i)},\\
F_4^0(\theta_1,\theta_2,\theta_3,\theta_4) &= \frac{g_1(\theta_1,\theta_2,\theta_4,\theta_3)}{(\theta_{14}+\pi i)(\theta_{23}+\pi i)^2},\\
F_5^0(\theta_1,\theta_2,\theta_3,\theta_4) &= \frac{g_1(\theta_2,\theta_1,\theta_3,\theta_4)}{(\theta_{14}+\pi i)^2(\theta_{23}+\pi i)},\\
F_6^0(\theta_1,\theta_2,\theta_3,\theta_4) &= \frac{g_2(\theta_2,\theta_1,\theta_3,\theta_4)}{(\theta_{14}+\pi i)(\theta_{13}+\pi i)(\theta_{24}+\pi i)},\\
F_7^0(\theta_1,\theta_2,\theta_3,\theta_4) &= \frac{g_1(\theta_2,\theta_1,\theta_4,\theta_3)}{(\theta_{13}+\pi i)^2(\theta_{24}+\pi i)},\\
F_8^0(\theta_1,\theta_2,\theta_3,\theta_4) &= \frac{g_2(\theta_2,\theta_1,\theta_4,\theta_3)}{(\theta_{14}+\pi i)(\theta_{13}+\pi i)(\theta_{23}+\pi i)},
\end{aligned}
\tag{4.3.25}
$$

where the functions $g_{1,\ldots,8}(\theta_1,\theta_2,\theta_3,\theta_4)$ are periodic under $\theta_{1,2}\to\theta_{1,2}-2\pi i$.

Instead of the analyis of the previous paragraph, we could have combined (4.3.22) and

(4.3.23) to obtain

$$\begin{aligned}
&\overleftrightarrow{M}_4(\theta_1,\theta_4-2\pi i)\overleftrightarrow{M}_3(\theta_1,\theta_3-2\pi i)\overleftrightarrow{M}_2(\theta_2,\theta_4-2\pi i)\\
&\times\overleftrightarrow{M}_1(\theta_2,\theta_3-2\pi i)\vec{F}^0(\theta_1,\theta_2,\theta_3-2\pi i,\theta_4-2\pi i)\\
&=\vec{F}^0(\theta_1,\theta_2,\theta_3,\theta_4).
\end{aligned} \tag{4.3.26}$$

The condition (4.3.26) is equivalent to (4.3.24). The solution of (4.3.26) is (4.3.25)

The functions $g_{1,\ldots 8}(\theta_1,\theta_2,\theta_3,\theta_4)$ are fixed by the annihilation pole axiom. Any of the two particles of the four-excitation form factor can annihilate with any of the two antiparticles. This means there must be annihilation poles at $\theta_{24},\theta_{14},\theta_{23},\theta_{13}=-\pi i$. For the $\theta_{24}=-\pi i$ pole, the annihilation-pole axiom implies

$$\begin{aligned}
\mathrm{Res}|_{\theta_{24}=-\pi i}&\langle 0\mid j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle\\
&= 2i\Big\{\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}|0\rangle\delta_{a_2a_4}\delta_{b_2b_4}\\
&\quad-\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1'a_1'}\mathfrak{A}_P(\theta_3)_{a_3'b_3'}|0\rangle\\
&\quad\times\delta_{a_2'a_4}\delta_{b_2'b_4}S_{AA}(\theta_{12})^{b_2'a_2';b_1'a_1'}_{d_1c_1;b_1a_1}S_{AP}(\theta_{23})^{d_1c_1;a_3'b_3'}_{a_3b_3;b_2a_2}\Big\}.
\end{aligned} \tag{4.3.27}$$

We substitute the two-particle form factor into the right-hand side of (4.3.27) to find

$$
\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}|0\rangle\delta_{a_2a_4}\delta_{b_2b_4} \\
&-\langle 0|\mathcal{O}_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b'_1a'_1}\mathfrak{A}_P(\theta_3)_{a'_3b'_3}|0\rangle\delta_{a'_2a_4}\delta_{b'_2b_4}S_{AA}(\theta_{12})^{b'_2a'_2;b'_1a'_1}_{d_1c_1;b_1a_1}S_{AP}(\theta_{23})^{d_1c_1;a'_3b'_3}_{a_3b_3;b_2a_2} \\
&\quad = -\epsilon_{\mu\nu}(p_1+p_3)^\nu\tanh\left(\frac{\theta_{13}}{2}\right) \\
&\quad \times\frac{2\pi}{(\theta_{13}+\pi i)}\left\{\frac{2\pi i}{N\hat\theta_{23}}\left(\delta_{a_0a_4}\delta_{a_2a_3}\delta_{c_0a_1}\delta_{b_1b_3}\delta_{b_2b_4}-\frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_4}\delta_{a_2a_3}\delta_{b_1b_3}\delta_{b_2b_4}\right)\right. \\
&\quad +\frac{1}{N}\left(\frac{-2\pi i}{\hat\theta_{23}}+\frac{2\pi i}{\theta_{12}}-\frac{4\pi^2}{\theta_{12}\hat\theta_{23}}\right) \\
&\quad \times\left(\delta_{a_0a_3}\delta_{a_2a_4}\delta_{a_1c_0}\delta_{b_2b_3}\delta_{b_1b_4}-\frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_3}\delta_{a_2a_4}\delta_{b_2b_3}\delta_{b_1b_4}\right) \\
&\quad \left.\frac{-2\pi i}{N\theta_{12}}\left(\delta_{a_0a_3}\delta_{a_1a_4}\delta_{a_2c_0}\delta_{b_1b_3}\delta_{b_2b_4}-\frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_3}\delta_{a_1a_4}\delta_{b_1b_3}\delta_{b_2b_4}\right)\right\}.
\end{aligned}
$$

We can repeat this for every other annihilation pole, which fixes the functions:

$$
\begin{aligned}
g_1(\theta_1,\theta_2,\theta_3,\theta_4) &= 0 \\
g_2(\theta_1,\theta_2,\theta_3,\theta_4) &= 8\pi^2 i\tanh\left(\frac{\theta_{13}}{2}\right) \\
g_3(\theta_1,\theta_2,\theta_3,\theta_4) &= 8\pi^2 i\tanh\left(\frac{\theta_{14}}{2}\right) \\
g_4(\theta_1,\theta_2,\theta_3,\theta_4) &= 0 \\
g_5(\theta_1,\theta_2,\theta_3,\theta_4) &= 0 \\
g_6(\theta_1,\theta_2,\theta_3,\theta_4) &= 8\pi^2 i\tanh\left(\frac{\theta_{23}}{2}\right) \\
g_7(\theta_1,\theta_2,\theta_3,\theta_4) &= 0 \\
g_8(\theta_1,\theta_2,\theta_3,\theta_4) &= 8\pi^2 i\tanh\left(\frac{\theta_{24}}{2}\right).
\end{aligned}
$$

The minimal four-particle form factor satisfying all of Smirnov's axioms for large N is

$$\begin{aligned}
&\langle 0 \quad | j^L_\mu(0)_{a_0c_0} | A,\theta_1,b_1,a_1; A,\theta_2,b_2,a_2; P,\theta_3,a_3,b_3; P,\theta_4,a_4,b_4\rangle \\
&= \quad -\epsilon_{\mu\nu}(p_1+p_2+p_3+p_4)^\nu \frac{8\pi^2 i}{N} \\
&\quad \times \Bigg\{ \frac{\tanh\left(\frac{\theta_{13}}{2}\right)}{(\theta_{14}+\pi i)(\theta_{23}+\pi i)(\theta_{24}+\pi i)} \\
&\quad \times \left(\delta_{a_0a_3}\delta_{a_1c_0}\delta_{a_2a_4}\delta_{b_1b_4}\delta_{b_2b_3} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_3}\delta_{a_2a_4}\delta_{b_1b_4}\delta_{b_2b_3} \right) \\
&\quad + \frac{\tanh\left(\frac{\theta_{14}}{2}\right)}{(\theta_{13}+\pi i)(\theta_{23}+\pi i)(\theta_{24}+\pi i)} \\
&\quad \times \left(\delta_{a_0a_4}\delta_{a_1c_0}\delta_{a_2a_3}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_4}\delta_{a_2a_3}\delta_{b_1b_3}\delta_{b_2b_4} \right) \\
&\quad + \frac{\tanh\left(\frac{\theta_{23}}{2}\right)}{(\theta_{14}+\pi i)(\theta_{13}+\pi i)(\theta_{24}+\pi i)} \\
&\quad \left(\delta_{a_0a_3}\delta_{a_1a_4}\delta_{a_2c_0}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_3}\delta_{a_1a_4}\delta_{b_1b_3}\delta_{b_2b_4} \right) \\
&\quad + \frac{\tanh\left(\frac{\theta_{24}}{2}\right)}{(\theta_{14}+\pi i)(\theta_{13}+\pi i)(\theta_{23}+\pi i)} \\
&\quad \left(\delta_{a_0a_4}\delta_{a_1a_3}\delta_{a_2c_0}\delta_{b_1b_4}\delta_{b_2b_3} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_4}\delta_{a_1a_3}\delta_{b_1b_4}\delta_{b_2b_3} \right) \Bigg\}.
\end{aligned} \tag{4.3.28}$$

4.4 Form factors of an arbitrary number of particles

In this section we generalize our results to find the form factor with M particles and M antiparticles. We introduce the permutation $\sigma \in S_{M+1}$ which takes the set of numbers $0, 1, \ldots, M$ to $\sigma(0), \sigma(1), \ldots, \sigma(M)$, respectively, and the permutation $\tau \in S_M$ which takes the set of numbers $1, 2, \ldots, M$ to $\tau(1), \tau(2), \ldots, \tau(M)$, respectively.

The form factor of the current operator with $2M$ excitations is

$$\begin{aligned}
&\langle 0 \mid j^L_\mu(x)_{a_0\, a_{2M+1}} \mid A, \theta_1, b_1, a_1; \ldots \\
&\qquad ; A, \theta_M, b_M, a_M; P, \theta_{M+1}, a_{M+1}, b_{M+1}; \ldots ; P, \theta_{2M}, a_{2M}, b_{2M}\rangle_{\text{in}} \\
&= \langle 0| j^L_\mu(x)_{a_0 a_{2M+1}} \mathfrak{A}^\dagger_A(\theta_1)_{b_1 a_1} \ldots \mathfrak{A}^\dagger_A(\theta_M)_{b_M a_M} \\
&\qquad \times \mathfrak{A}^\dagger_P(\theta_{M+1})_{a_{M+1} b_{M+1}} \ldots \mathfrak{A}^\dagger_P(\theta_{2M})_{a_{2M} b_{2M}} |0\rangle \\
&= \frac{-\epsilon_{\mu\nu}}{N^{M-1}} (p_1 + \cdots + p_{2M})^\nu \sum_{\sigma,\tau} F_{\sigma\tau}(\theta_1, \ldots, \theta_{2M})\, e^{-ix\cdot\sum_{j=1}^{2M} p_j} \\
&\times \Bigg[\prod_{j=0}^{M} \delta_{a_j a_{\sigma(j)+M}} \prod_{k=1}^{M} \delta_{b_k b_{\tau(k)+M}} \\
&\qquad - \frac{1}{N} \delta_{a_0 a_{2M+1}} \delta_{a_{l_\sigma} a_{\sigma(0)+M}} \prod_{j=1,\, j\neq l_\sigma}^{M} \delta_{a_j a_{\sigma(j)+M}} \prod_{k=1}^{M} \delta_{b_k b_{\tau(k)+M}} \Bigg], \qquad (4.4.1)
\end{aligned}$$

where l_σ is defined by $\sigma(l_\sigma) + M = 2M + 1$. This is the most general expression consistent with Lorentz invariance, a conserved traceless current (guaranteed by the second term in square brackets) and crossing.

To simplify our terminology, we say that excitation h is the particle or antiparticle with rapidity θ_h and left and right indices a_h, b_h, respectively.

We expand the functions $F_{\sigma\tau}(\theta_1, \ldots, \theta_{2M})$ in powers of $1/N$:

$$\begin{aligned}
F_{\sigma\tau}(\theta_1, \ldots, \theta_{2M}) &= F^0_{\sigma\tau}(\theta_1, \ldots, \theta_{2M}) \\
&\quad + \frac{1}{N} F^1_{\sigma\tau}(\theta_1, \ldots, \theta_{2M}) + \frac{1}{N^2} F^2_{\sigma\tau}(\theta_1, \ldots, \theta_{2M}) + \cdots
\end{aligned}$$

keeping only the first term.

We now apply the scattering axiom on Eq.(4.4.1):

$$\begin{aligned}
&\langle 0 \mid j^L_\mu(x)_{a_0 a_{2M+1}} \mathfrak{A}^\dagger_{I_1}(\theta_1)_{C_1} \dots \mathfrak{A}^\dagger_{I_i}(\theta_i)_{C_i} \mathfrak{A}^\dagger_{I_{i+1}}(\theta_{i+1})_{C_{i+1}} \dots \mathfrak{A}^\dagger_{I_{2M}}(\theta_{2M})_{C_{2M}} |0\rangle \\
&= \quad S_{I_{i+1} I_i}(\theta_i - \theta_{i+1})^{C'_{i+1}; C'_i}_{C_i; C_{i+1}} \\
&\quad \times \langle 0 | j^L_\mu(x)_{a_0 a_{2M+1}} \mathfrak{A}^\dagger_{I_1}(\theta_1)_{C_1} \dots \\
&\quad \times \mathfrak{A}^\dagger_{I_{i+1}}(\theta_{i+1})_{C'_{i+1}} \mathfrak{A}^\dagger_{I_i}(\theta_i)_{C'_i} \dots \mathfrak{A}^\dagger_{I_{2M}}(\theta_{2M})_{C_{2M}} |0\rangle,
\end{aligned} \tag{4.4.2}$$

where, for each k, $I_k = P$ for a particle or $I_k = A$ for an antiparticle, and C_k is the ordered set of indices $C_k = (a_k, b_k)$ for $I_k = P$, or $C_k = (b_k, a_k)$ for $I_k = A$. We use (4.4.2) to interchange the creation operator of the excitation h with the creation operator of the excitation i in (4.4.1). There are four different ways the function $F^0_{\sigma\tau}(\theta_1, \dots, \theta_{2M})$ can be affected by interchanging the excitations h and i, for a given σ and τ. If excitation h and excitation i are both particles or both antiparticles, then the rapidities θ_h and θ_i are interchanged in the function $F^0_{\sigma\tau}(\theta_1, \dots, \theta_{2M})$. If excitation h is a particle, excitation i is an antiparticle, and $\sigma(i) + M \neq h, \tau(i) + M \neq h$, then the function $F^0_{\sigma\tau}(\theta_1, \dots, \theta_{2M})$ is unchanged. If excitation h is a particle, excitation i an antiparticle, and either $\sigma(i) + M = h$, $\tau(i) + M \neq h$, or $\sigma(i) + M \neq h$, $\tau(i) + M = h$, then we multiply $F^0_{\sigma\tau}(\theta_1, \dots, \theta_{2M})$ by the pure phase $\frac{\theta_{ih} + \pi \mathrm{i}}{\theta_{ih} - \pi \mathrm{i}}$. If excitation h is a particle, excitation i is an antiparticle and $\sigma(i) + M = h$, $\tau(i) + M = h$, then we multiply the function $F^0_{\sigma\tau}(\theta_1, \dots, \theta_{2M})$ by the pure phase $\left(\frac{\theta_{ih} + \pi \mathrm{i}}{\theta_{ih} - \pi \mathrm{i}}\right)^2$. The rules for interchanging creation operators described in the previous paragraph suggest an underlying Abelian structure for the large-N limit. The pure phase we use in the scattering axiom, namely 1, $\frac{\theta_{ih} + \pi \mathrm{i}}{\theta_{ih} - \pi \mathrm{i}}$ or $\left(\frac{\theta_{ih} + \pi \mathrm{i}}{\theta_{ih} - \pi \mathrm{i}}\right)^2$ is similar to the S-matrix element of a theory of colorless particles.

Smirnov's periodicity axiom states

$$\begin{aligned}&\langle 0 \mid j^L_\mu(x)_{a_0 a_{2M+1}} \mathfrak{A}^\dagger_{I_1}(\theta_1)_{C_1} \mathfrak{A}^\dagger_{I_1}(\theta_2)_{C_2} \ldots \mathfrak{A}^\dagger_{I_M}(\theta_M)_{C_M} |0\rangle \\ &= \langle 0| j^L_\mu(x)_{a_0 a_{2M+1}} \mathfrak{A}^\dagger_{I_M}(\theta_M - 2\pi\mathrm{i})_{C_M} \\ &\quad \times \mathfrak{A}^\dagger_{I_1}(\theta_1)_{C_1} \ldots \mathfrak{A}^\dagger_{I_{M-1}}(\theta_{M-1})_{C_{M-1}} |0\rangle . \end{aligned} \tag{4.4.3}$$

In terms of the function $F^0_{\sigma\tau}(\theta_1, \ldots, \theta_{2M})$, (4.4.3) is

$$\begin{aligned} F^0_{\sigma\tau} &(\theta_1, \ldots, \theta_{2M}) \\ &= F^0_{\sigma\tau}(\theta_{2M} - 2\pi\mathrm{i}, \theta_1, \ldots, \theta_{2M-1}) \\ &= F^0_{\sigma\tau}(\theta_{2M-1} - 2\pi\mathrm{i}, \theta_{2M} - 2\pi\mathrm{i}, \theta_1, \ldots, \theta_{2M-2}) \\ &= \cdots . \end{aligned} \tag{4.4.4}$$

The general solution of (4.4.2) and (4.4.4) is

$$\begin{aligned} F^0_{\sigma\tau} &(\theta_1, \ldots, \theta_{2M}) \\ &= \frac{H_{\sigma\tau}(\theta_1, \ldots, \theta_{2M})}{\prod_{j=1; j\neq l_\sigma}^{M} \left(\theta_j - \theta_{\sigma(j)+M} + \pi\mathrm{i}\right) \prod_{k=1}^{M} \left(\theta_k - \theta_{\tau(k)+M} + \pi\mathrm{i}\right)} , \end{aligned} \tag{4.4.5}$$

where $\sigma(l_\sigma) + M = 2M + 1$, and the functions $H_{\sigma\tau}(\theta_1, \ldots, \theta_{2M})$ are holomorphic and periodic in θ_j, with period $2\pi\mathrm{i}$, for each $j = 1, \ldots, 2M$.

The annihilation-pole axiom states that the $2M+2$-excitation form factor:

$$\begin{aligned}
&\langle 0 \mid j^L_\mu(0)_{a_0 a_{2M+3}} \left[\prod_{j=1}^{M} \mathfrak{A}^\dagger_A(\theta_j)_{b_j a_j}\right] \left[\prod_{k=M+1}^{2M} \mathfrak{A}^\dagger_P(\theta_k)_{a_k b_k}\right] \\
&\quad \times \mathfrak{A}^\dagger_A(\theta_{2M+1})_{b_{2M+1} a_{2M+1}} \mathfrak{A}^\dagger_P(\theta_{2M+2})_{a_{2M+2} b_{2M+2}} |0\rangle \\
&= \quad -\epsilon_{\mu\nu}(p_1+\cdots+p_{2M+2})^\nu \\
&\quad \times \mathcal{F}(\theta_1,\ldots,\theta_{2M+2})_{a_0 a_2 \ldots a_{2M+3}; b_1 \ldots b_{2M+2}},
\end{aligned}$$

has a pole at $\theta_{2M+1}-\theta_{2M+2}=-\pi i$, with a residue proportional to the form factor of $2M$ excitations, such that

$$\begin{aligned}
\mathrm{Res}|_{\theta_{2M+1}-\theta_{2M+2}=-\pi i} \quad & \mathcal{F}(\theta_1,\ldots,\theta_{2M+2})_{a_0 \ldots a_{2M+3};\, b_1 \ldots b_{2M+2}} \\
= \quad & 2i\mathcal{F}(\theta_1,\ldots,\theta_{2M})_{a_0 a'_1 \ldots a'_{2M} a_{2M+3};\, b'_1 \cdots b'_{2M}} \delta_{a'_{2M+1} a_{2M+2}} \delta_{b'_{2M+1} b_{2M+2}} \\
& \times \Big[\delta_{a'_1 a_1} \delta_{b'_1 b_1} \cdots \delta_{a'_{2M+1} a_{2M+1}} \delta_{b'_{2M+1} b_{2M+1}} \\
& - S_{AA}(\theta_{1\,2M+1})^{b'_{2M+1} a'_{2M+1}; b'_1 a'_1}_{d_1 c_1; b_1 a_1} \cdots S_{AA}(\theta_{M\,2M+1})^{d_{M-1} c_{M-1}; b'_M a'_M}_{d_M c_M; b_M a_M} \\
& \times S_{AP}(\theta_{2M+1\,M+1})^{d_M c_M; a'_{M+1} b'_{M+1}}_{c_{M+1} d_{M+1}; a_{M+1} b_{M+1}} \cdots \\
& \times S_{AP}(\theta_{2M+1\,2M})^{c_{2M-1} d_{2M-1}; a'_{2M} b'_{2M}}_{c_{2M} d_{2M}; a_{2M} b_{2M}} \Big] . \qquad (4.4.6)
\end{aligned}$$

Equation (4.4.6) fixes the functions:

$$H_{\sigma\tau} = \begin{cases} 2\pi \mathrm{i} (4\pi)^{M-1} \tanh\left(\frac{\theta_{l_\sigma} - \theta_{\sigma(0)+M}}{2}\right) & , \quad \text{if } \sigma(j) \neq \tau(j), \text{ for all } j \\ 0 & , \quad \text{otherwise} \end{cases} . \qquad (4.4.7)$$

This concludes our derivation of all the form factors of the current operator. They are completely specified in (4.4.1), (4.4.5) and (4.4.7).

4.5 Vacuum expectation values of products of current operators

Once we have calculated the general form factor for the Noether current operator, we can calculate its two-point correlation function. The current-current correlation function is

$$W^{j}_{\mu\nu}(x)_{a_0c_0;e_0f_0} = \langle 0|j^L_\mu(x)_{a_0c_0}\; j^L_\nu(0)_{e_0f_0}|0\rangle = \sum_{M=1}^{\infty} W^{2M}_{\mu\nu}(x)_{a_0c_0e_0f_0}, \tag{4.5.1}$$

where the contribution from the $2M$-excitation form factor is given by

$$\begin{aligned}
W^{2M}_{\mu\nu}(x)_{a_0c_0e_0f_0} &= \frac{1}{N(M!)^2}\int \frac{d\theta_1\dots d\theta_{2M}}{(2\pi)^{2M}}\; e^{-ix\cdot\sum_{j=1}^{2M}p_j} \\
&\times\langle 0|j^L_\mu(0)_{a_0c_0}|A,\theta_1,b_1,a_1;\dots; \\
&\qquad A,\theta_M,b_Ma_M;P,\theta_{M+1},a_{M+1},b_{M+1};\dots;P,\theta_{2M},a_{2M},b_{2M}\rangle_{\mathrm{in}} \\
&\times\langle 0|j^L_\nu(0)_{e_0f_0}|A,\theta_1,b_1,a_1;\dots; \\
&\qquad A,\theta_M,b_Ma_M;P,\theta_{M+1},a_{M+1},b_{M+1};\dots;P,\theta_{2M},a_{2M},b_{2M}\rangle^*_{\mathrm{in}}.
\end{aligned}$$

Substituting the form factors (4.4.1), (4.4.5), (4.4.7), we find

$$\begin{aligned}
W^{2M}_{\mu\nu}(x)_{a_0c_0e_0f_0} &= \frac{1}{(M!)^2}\int \prod_{j=1}^{2M}\frac{d\theta_j}{4\pi}\;\; e^{-ix\cdot\sum_{j=1}^{2M}p_j} \\
&\times\epsilon_{\mu\alpha}\epsilon_{\nu\beta}(p_1+\dots+p_{2M})^\alpha(p_1+\dots+p_{2M})^\beta \\
&\times\left[\sum_{\sigma,\tau\in S_M}\frac{|H_{\sigma\tau}(\theta_1,\dots,\theta_{2M})|^2(\delta_{a_0e_0}\delta_{c_0f_0}-\frac{1}{N}\delta_{a_0c_0}\delta_{e_0f_0})}{\prod_{j=1;\,j\neq l_\sigma}^{M}|\theta_j-\theta_{\sigma(j)+M}+\pi\mathrm{i}|^2\prod_{k=1}^{M}|\theta_k-\theta_{\tau(k)+M}+\pi\mathrm{i}|^2}\right. \\
&\left.+\mathcal{O}\left(\frac{1}{N^2}\right)\right],
\end{aligned} \tag{4.5.2}$$

where we have used

$$\begin{aligned}
\sum_{a_1,\dots,a_{2M},b_1,\dots,b_{2M}} & \Bigg[\prod_{j=0}^{M}\delta_{a_j a_{\sigma(j)+M}}\prod_{k=1}^{M}\delta_{b_k b_{\tau(k)+M}} \\
& \qquad -\frac{1}{N}\delta_{a_0 a_{2M+1}}\delta_{a_{l_\sigma} a_{\sigma(0)+M}}\prod_{j=1;\, j\neq l_\sigma}^{M}\delta_{a_j a_{\sigma(j)+M}}\prod_{k=1}^{M}\delta_{b_k b_{\tau(k)+M}}\Bigg] \\
\times & \Bigg[\prod_{j=0}^{M}\delta_{a'_j a'_{\omega(j)+M}}\prod_{k=1}^{M}\delta_{b_k b_{\varphi(k)+M}} \\
& \qquad -\frac{1}{N}\delta_{a'_0 a'_{2M+1}}\delta_{a'_{l_\omega} a'_{\omega(0)}}\prod_{j=1\, j\neq l_\omega}^{M}\delta_{a'_j a'_{\omega(j)+M}}\prod_{k=1}^{M}\delta_{b_k b_{\varphi(k)+M}}\Bigg] \\
= \quad & N^{2M-1}(\,\delta_{a_0 e_0}\delta_{c_0 f_0}-\delta_{a_0 c_0}\delta_{e_0 f_0}/N\,)\left[\delta_{\sigma\omega}\delta_{\tau\varphi}+\mathcal{O}\left(\frac{1}{N^2}\right)\right],
\end{aligned}$$

where $\{a_j\} = a_0, a_1, a_2, \dots, a_{2M}, c_0$ and $\{a'_j\} = e_0, a_1, a_2, \dots, a_{2M}, f_0$.

The contribution to (4.5.2) from each pair σ, τ is the same (because there is no change if the integration variables are interchanged). There are $(M!)^2$ pairs σ, τ that satisfy $H_{\sigma\tau} \neq 0$, by (4.4.7). We choose the contribution from one pair σ, τ in (4.5.2) and multiply it by $(M!)^2$. We choose $\tau(j) = j$, for $j = 1, \dots, M$, and $\sigma(1) = 2M+1, \sigma(j) = j-1$, for $j = 2, \dots, M$, such that

$$\begin{aligned}
W^{2M}_{\mu\nu}(x)_{a_0c_0e_0f_0} &= \int \prod_{j=1}^{2M} \frac{d\theta_j}{4\pi} \;\; e^{-ix\cdot\sum_{j=1}^{2M} p_j} \; 4\pi^2(4\pi)^{2M-2} \\
&\quad \times \left(\delta_{a_0e_0}\delta_{c_0f_0} - \frac{1}{N}\delta_{a_0c_0}\delta_{e_0f_0}\right) \\
&\times \;\; \epsilon_{\mu\alpha}\,\epsilon_{\nu\beta}(p_1+\cdots+p_{2M})^\alpha(p_1+\cdots+p_{2M})^\beta \\
&\quad \times \frac{1}{(\theta_1-\theta_{M+1})^2+\pi^2}\frac{1}{(\theta_2-\theta_{M+2})^2+\pi^2}\cdots \\
&\quad \times \frac{1}{(\theta_M-\theta_{2M})^2+\pi^2} \\
&\quad \times \frac{1}{(\theta_2-\theta_{M+1})^2+\pi^2}\frac{1}{(\theta_3-\theta_{M+2})^2+\pi^2}\cdots \\
&\quad \times \frac{1}{(\theta_M-\theta_{2M-1})^2+\pi^2}\tanh^2\left(\frac{\theta_1-\theta_{2M}}{2}\right) + O\left(\frac{1}{N^2}\right)
\end{aligned}$$

We further relabel the integration variables as $\theta_1 \to \theta_1, \theta_2 \to \theta_3, \theta_3 \to \theta_5, \ldots, \theta_M \to \theta_{2M-1}; \theta_{M+1} \to \theta_2, \theta_{M+2} \to \theta_4, \ldots, \theta_{2M} \to \theta_{2M}$. This yields the expression for the non-time-ordered correlation function of two current operators:

$$\begin{aligned}
W^j_{\mu\nu}(x)_{a_0c_0e_0f_0} &= \left(\delta_{a_0e_0}\delta_{c_0f_0} - \frac{1}{N}\delta_{a_0c_0}\delta_{e_0f_0}\right) \\
&\times \;\; \sum_{M=1}^{\infty}\frac{1}{4}\int\prod_{j=1}^{2M} d\theta_j \;\; e^{-ix\cdot\sum_{j=1}^{2M} p_j} \\
&\quad \times\epsilon_{\mu\alpha}\,\epsilon_{\nu\beta}(p_1+\cdots+p_{2M})^\alpha(p_1+\cdots+p_{2M})^\beta \\
&\quad \times \prod_{j=1}^{2M-1}\frac{\tanh^2\left(\frac{\theta_1-\theta_{2M}}{2}\right)}{(\theta_j-\theta_{j+1})^2+\pi^2} \; + \; O\left(\frac{1}{N^2}\right).
\end{aligned} \tag{4.5.3}$$

4.6 Form factors of the stress-energy-momentum tensor

The stess- energy momentum tensor of the principal chiral sigma model is given by

$$T_{\mu\nu}(x) = \frac{1}{2\pi}(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\beta_\mu \delta^\alpha_\nu - \eta_{\mu\nu}\eta^{\alpha\beta}) \operatorname{Tr} \partial_\alpha U(x)^\dagger \partial_\beta U(x) + \lambda \eta_{\mu\nu},$$

where λ is chosen to normal order this operator, so that the vacuum energy is zero. This operator is explicitly conserved ($\partial^\mu T_{\mu\nu} = 0$), symmetric ($T_{\mu\nu} = T_{\nu\mu}$) and it is an $SU(N) \times SU(N)$-color singlet. The fact that the bare field is unitary means that there is less freedom when defining an stress-energy-momentum tensor, quadratic in derivatives, than in ordinary scalar field theories. There is no color-singlet total divergence term of dimension two we can add to the $T_{\mu\nu}$.

This stress-energy-momentum tensor operator is invariant under $\mathrm{SU}(N)\times\mathrm{SU}(N)$ transformations. Thus the only non-vanishing form factors have equal number of particles and antiparticles in the in-state ket. The general form factor with M particles and M antiparticles, which respects energy-momentum conservation, is

$$\begin{aligned}
\langle 0|\, T_{\mu\nu}(0) \;|\; & A, \theta_1, b_1, a_1; \ldots; A, \theta_M, b_M, a_M; P, \theta_{M+1}, a_{M+1}, b_{M+1}; \\
& \ldots; P, \theta_{2M}, a_{2M}, b_{2M}\rangle_{\rm in} \\
= \; & \left[(p_1 + \cdots + p_{2M})_\mu (p_1 + \cdots + p_{2M})_\nu - \eta_{\mu\nu}(p_1 + \cdots + p_{2M})^2\right] \\
& \times \frac{1}{N^{M-1}} \sum_{\sigma,\tau \in S_M} F_{\sigma\tau}(\theta_1, \ldots, \theta_{2M}) \\
& \times \prod_{j=1}^{M} \delta_{a_j a_{\sigma(j)+M}} \prod_{k=1}^{M} \delta_{b_k b_{\tau(k)+M}},
\end{aligned} \tag{4.6.1}$$

where $\sigma, \tau \in S_M$ are permutations of the integers $1, 2, \ldots, M$ (this is different from the convention in Section III. Recall that there the permutation σ was defined as an element of S_{M+1}).

We expand the function $F_{\sigma,\tau}(\theta_1, \ldots, \theta_{2M})$ in powers of $1/N$, *i.e.*, as $F^0_{\sigma,\tau}(\theta_1, \ldots, \theta_{2M}) + \frac{1}{N}F^1_{\sigma,\tau}(\theta_1, \ldots, \theta_{2M}) + \cdots$, keeping only the first term.

The form factors in (4.6.1) behave the same way as the current-operator form factors under Watson's theorem and the periodicity axiom. These two axioms give us the solution

$$\begin{aligned} F^0_{\sigma\tau}&(\theta_1, \ldots, \theta_{2M}) \\ &= \frac{H_{\sigma\tau}(\theta_1, \ldots, \theta_{2M})}{\prod_{j=1}^{M}(\theta_j - \theta_{\sigma(j)+M} + \pi \mathrm{i}) \prod_{k=1}^{M}(\theta_k - \theta_{\tau(k)+M} + \pi \mathrm{i})}. \end{aligned} \tag{4.6.2}$$

The minimal choice is to make $H_{\sigma,\tau}(\theta_1, \ldots, \theta_{2M}) = H_{\sigma\tau}$ constants. These constants can be fixed by the annihilation pole axiom, once we fix the constant for the two-particle form factor.

For $M = 1$, Equation (4.6.1) becomes

$$\begin{aligned} &\langle 0 \mid T_{\mu\nu}(x) | A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2\rangle_{\mathrm{in}} \\ &= [(p_1+p_2)_\mu (p_1+p_2)_\nu - \eta_{\mu\nu}(p_1+p_2)^2] \frac{g}{(\theta_{12}+\pi \mathrm{i})^2} e^{-ix(p_1+p_2)} \delta_{a_1 a_2}\delta_{b_1 b_2} \\ &\quad + O\left(\frac{1}{N}\right). \end{aligned} \tag{4.6.3}$$

We fix the constant g by requiring that

$$\int dx^1\, T_{00}(x) | A, \theta_1, b_1, a_1\rangle_{\mathrm{in}} = m \cosh\theta_1 | A, \theta_1, b_1, a_1\rangle_{\mathrm{in}}. \tag{4.6.4}$$

Notice that the pole in (4.6.3) has vanishing residue. Therefore, by the annihilation-pole axiom, the vacuum energy is zero.

We next apply crossing, changing one of the incoming particles in (4.6.3) to an outgoing antiparticle:

$$\begin{aligned}
&\langle A, \theta_2, b_2, a_2 \mid T_{\mu\nu}(x) | A, \theta_1, b_1, a_1 \rangle \\
&\quad = \quad [(p_1 - p_2)_\mu (p_1 - p_2)_\nu - \eta_{\mu\nu}(p_1 - p_2)^2] \frac{g}{(\theta_{12} + 2\pi i)^2} \\
&\quad\quad\quad \times e^{-ix\cdot(p_1 - p_2)} \delta_{a_1 a_2} \delta_{b_1 b_2}.
\end{aligned}$$

The matrix element of the energy density is then

$$\begin{aligned}
&\langle A, \theta_2, b_2, a_2 \mid T_{00}(x) | A, \theta_1, b_1, a_1 \rangle \\
&\quad = \quad [(p_{1\,0} p_{1\,0} + p_{2\,0} p_{2\,0} - 2 p_{1\,0} p_{2\,0} - 2m^2 + 2 p_1 \cdot p_2] \\
&\quad\quad\quad \times \frac{g}{(\theta_{12} + 2\pi i)^2} e^{-ix\cdot(p_1 - p_2)} \delta_{a_1 a_2} \delta_{b_1 b_2}. \\
&\quad = \quad [(p_{1\,0} + p_{2\,0})^2 - 2m^2 + 2 p_1 \cdot p_2] \\
&\quad\quad\quad \times \frac{g}{(\theta_{12} + 2\pi i)^2} e^{-ix\cdot(p_1 - p_2)} \delta_{a_1 a_2} \delta_{b_1 b_2}.
\end{aligned}$$

We integrate over the spatial coordinate x^1, yielding

$$\begin{aligned}
&\int dx^1 \, \langle A, \theta_2, b_2, a_2 \mid T_{00}(x) | A, \theta_1, b_1, a_1 \rangle \\
&\quad = \quad [(m \cosh \theta_1 + m \cosh \theta_2)^2 - 2m^2 + 2 p_1 \cdot p_2] \\
&\quad\quad\quad \times \frac{g}{(\theta_{12} + 2\pi i)^2} \frac{2\pi}{m \cosh \theta_2} \\
&\quad\quad\quad \times \delta(\theta_1 - \theta_2) \delta_{a_1 a_2} \delta_{b_1 b_2}. \\
&\quad = \quad m \cosh \theta_1 \frac{g}{2\pi^2} 4\pi \delta(\theta_1 - \theta_2) \delta_{a_1 a_2} \delta_{b_1 b_2}.
\end{aligned}$$

The condition (4.6.4) imples $g = -2\pi^2$.

The constants $H_{\sigma\tau}$ for the $2M$-particle form factor are fixed by the annihilation pole axiom (Equation (4.4.6)), which gives the values

$$H_{\sigma\tau} = \begin{cases} (-2\pi^2)(4\pi)^{M-1} \ , & \text{for}\, \sigma(j) \neq \tau(j), \,\text{for all}\, j \\ 0 \ , & \text{otherwise} \end{cases} . \tag{4.6.5}$$

The $2M$-particle form factor has a total of $(M!)^2/2$ non-vanishing terms.

To summarize the results of this section, (4.6.1), (4.6.2), (4.6.5) determine all the form factors of the stress-energy-momentum tensor.

4.7 Correlation function of the stress-energy-momentum tensor

In this section we obtain the vacuum expectation value of the product of two stress-energy-momentum-tensor operators. In other words, we find

$$W^T_{\mu\nu\alpha\beta}(x) = \frac{1}{N^2}\langle 0|T_{\mu\nu}(x)T_{\alpha\beta}(0)|0\rangle = \sum_{M=1}^{\infty} W^{2M}_{\mu\nu\alpha\beta}(x)\,, \tag{4.7.1}$$

where the terms in the sum over M are defined as

$$\begin{aligned} W^{2M}_{\mu\nu\alpha\beta}(x) &= \frac{1}{N^2}\frac{1}{(M!)^2}\int\prod_{j=1}^{2M}\frac{d\theta_j}{4\pi}\, e^{-ix\cdot\sum_{j=1}^{2M}p_j} \\ &\quad\times\langle 0|T_{\mu\nu}(0)|A,\theta_1,b_1,a_1;\dots;A,\theta_M,b_M,a_M; \\ &\qquad\quad P,\theta_{M+1},a_{M+1},b_{M+1};\dots;P,\theta_{2M},a_{2M},b_{2M}\rangle_{\rm in}. \\ &\quad\times\langle 0|T_{\alpha\beta}(0)|A,\theta_1,b_1,a_1;\dots;A,\theta_M,b_M,a_M; \\ &\qquad\quad P,\theta_{M+1},a_{M+1},b_{M+1};\dots;P,\theta_{2M},a_{2M},b_{2M}\rangle^*_{\rm in}. \end{aligned}$$

Substituting our form factors (4.6.1), (4.6.2), (4.6.5) gives

$$\begin{aligned} W^{2M}_{\mu\nu\alpha\beta}(x) &= \frac{1}{(M!)^2}\int\prod_{j=1}^{2M}\frac{d\theta_j}{4\pi}\, e^{-ix\cdot\sum_{j=1}^{2M}p_j} \\ &\quad\times\left[(p_1+\dots+p_{2M})_\mu(p_1+\dots+p_{2M})_\nu-\eta_{\mu\nu}(p_1+\dots+p_{2M})^2\right] \\ &\quad\times\left[(p_1+\dots+p_{2M})_\alpha(p_1+\dots+p_{2M})_\beta-\eta_{\alpha\beta}(p_1+\dots+p_{2M})^2\right] \\ &\quad\times\sum_{\sigma,\tau\in S_M}\frac{|H_{\sigma\tau}|^2}{\prod_{j=1}^M\left(\theta_j-\theta_{\sigma(j)+M}+\pi\mathrm{i}\right)^2\prod_{k=1}^M\left(\theta_k-\theta_{\tau(k)+M}+\pi\mathrm{i}\right)^2} \\ &\quad+\mathcal{O}\left(\frac{1}{N}\right), \end{aligned}$$

where we have used

$$\begin{aligned} &\sum_{a_1,\dots,a_{2M},b_1,\dots,b_{2M}}\left[\prod_{j=1}^M\prod_{k=1}^M\delta_{a_ja_{\sigma(j)+M}}\delta_{b_kb_{\tau(k)+M}}\right]\left[\prod_{j=1}^M\prod_{k=1}^M\delta_{a_ja_{\omega(j)+M}}\delta_{b_kb_{\varphi(k)+M}}\right] \\ &\qquad= N^{2M}\left[\delta_{\sigma\omega}\delta_{\tau\varphi}+\mathcal{O}\left(\frac{1}{N}\right)\right]. \end{aligned}$$

The contribution to $W^{2M}_{\mu\nu\alpha\beta}(x)$ from each pair σ,τ is the same. There are $\frac{(M!)^2}{2}$ possible pairs σ,τ. We write the contribution from just one of these pairs, and multiply by the factor

$\frac{(M!)^2}{2}$. We choose $\sigma(j) = j$ for $j = 1, \dots, M$, and $\tau(1) = M$, $\tau(j) = j - 1$ for $j = 2, \dots, M$. Then we have

$$
\begin{aligned}
W^{2M}_{\mu\nu\alpha\beta}(x) \;=\; & \frac{1}{2} \int \prod_{j=1}^{2M} \frac{d\theta_j}{4\pi}\, e^{-ix\cdot\sum_{j=1}^{2M} p_j} 4\pi^4 (4\pi)^{2M-2} \\
& \times \left[(p_1 + \cdots + p_{2M})_\mu (p_1 + \cdots + p_{2M})_\nu - \eta_{\mu\nu}(p_1 + \cdots + p_{2M})^2\right] \\
& \times \left[(p_1 + \cdots + p_{2M})_\alpha (p_1 + \cdots + p_{2M})_\beta - \eta_{\alpha\beta}(p_1 + \cdots + p_{2M})^2\right] \\
& \times \frac{1}{(\theta_1 - \theta_{M+1})^2 + \pi^2} \frac{1}{(\theta_2 - \theta_{M+2})^2 + \pi^2} \cdots \frac{1}{(\theta_M - \theta_{2M})^2 + \pi^2} \\
& \times \frac{1}{(\theta_1 - \theta_{2M})^2 + \pi^2} \frac{1}{(\theta_2 - \theta_{M+1})^2 + \pi^2} \cdots \frac{1}{(\theta_M - \theta_{2M-1})^2 + \pi^2} \\
& + O\left(\frac{1}{N}\right). \qquad (4.7.2)
\end{aligned}
$$

Finally, we relabel the integration variables by $\theta_1 \to \theta_1,\ \theta_2 \to \theta_3,\ \theta_3 \to \theta_5,\ \dots,\ \theta_M \to \theta_{2M-1},\ \theta_{M+1} \to \theta_2,\ \theta_{M+2} \to \theta_4,\ \dots,\ \theta_{2M} \to \theta_{2M}$. This gives the expression for the non-time-ordered correlation function

$$
\begin{aligned}
W^{T}_{\mu\nu\alpha\beta}(x) \;=\; & \frac{\pi^2}{8} \sum_{M=1}^{\infty} \int \prod_{j=1}^{2M} d\theta_j\, e^{-ix\cdot\sum_{j=1}^{2M} p_j} \\
& \times \left[(p_1 + \cdots + p_{2M})_\mu (p_1 + \cdots + p_{2M})_\nu - \eta_{\mu\nu}(p_1 + \cdots + p_{2M})^2\right] \\
& \times \left[(p_1 + \cdots + p_{2M})_\alpha (p_1 + \cdots + p_{2M})_\beta - \eta_{\alpha\beta}(p_1 + \cdots + p_{2M})^2\right] \\
& \times \frac{1}{(\theta_1 - \theta_{2M})^2 + \pi^2} \prod_{j=1}^{2M-1} \frac{1}{(\theta_j - \theta_{j+1})^2 + \pi^2} + O\left(\frac{1}{N}\right). \qquad (4.7.3)
\end{aligned}
$$

Chapter 5

The spectrum of (1+1)-dimensional massive Yang-Mills theory

5.1 Massive Yang-Mills in 1+1 dimensions

This chapter contains material previously published in [38].

Yang-Mills theory in $1+1$ dimensions has no local degrees of freedom. Introducing an explicit mass $\mathcal{M}$ gives a theory of longitudinally-polarized gluons at tree level. It may seem intuitively obvious, for small gauge coupling, that a particle is either a vector Boson, with a mass roughly equal to $\mathcal{M}$, or a bound state of such vector Bosons. This intuition, however, is wrong.We show in this chapter that the massive Yang-Mills theory describes an infinite number of particles, with masses that are much less than $\mathcal{M}$. We call this, dynamical mass reduction.

The massive Yang-Mills model can be thought of as a gauge field, coupled to a principal chiral nonlinear sigma model. The equivalence is seen by choosing the unitary gauge condition. The sigma model behaves like a Higgs field, which gives mass to the vector gluons. It was shown by Bardeen and Shizuya that this model is renormalizable [39]

The tree-level description fails because the excitations of the sigma model (without the gauge field) are not Goldstone Bosons. These excitations are massive. The gauge field produces a confining force between these excitations. There is no Higgs or Coulomb phase. There is only a confined phase. In this respect, the model is similar to a non-Abelian gauge theory with massive adjoint fermions [43].

A quantum field theory of an $SU(N)$ gauge field, coupled minimally to an adjoint matter field, can have distinct Higgs and confinement phases [45], separated by a phase boundary, for space-time dimension greater than two. If this dimension is two, however, there is only the confined phase. In the confined phase, the excitations are bound states of the massive particles of the sigma model. These massive particles are color multiplets of degeneracy N^2 [19].

The action of the massive $SU(N)$ Yang-Mills field in $1+1$ dimensions is

$$S = \int d^2x \left(-\frac{1}{4} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} + \frac{e^2}{2g_0^2} \mathrm{Tr}\, A_\mu A^\mu \right), \tag{5.1.1}$$

where A_μ is Hermitian and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}e[A_\mu, A_\nu]$ with $\mu, \nu = 0, 1$ and indices are raised by $\eta^{\mu\nu}$, where $\eta^{00} = -\eta^{11} = 1$, $\eta^{01} = \eta^{10} = 0$. If we drop the cubic and quartic terms from (5.1.1), the particles are gluons with mass $\mathcal{M} = e/g_0$.

The action (5.1.1) is equivalent to that of a principal chiral sigma model with one of its $SU(N)$ symmetries gauged by a Yang-Mills field. We promote the left-handed $SU(N)$ global symmetry of the sigma model to a local symmetry, by introducing the covariant derivative $D_\mu = \partial_\mu - \mathrm{i}eA_\mu$, where A_μ is a new Hermitian vector field that transforms as

$A_\mu \to V_L^\dagger(x) A_\mu V_L(x) - \frac{i}{e} V_L^\dagger(x) \partial_\mu V_L(x)$. We do not gauge the right-handed symmetry. The action is now

$$S = \int d^2x \left[-\frac{1}{4} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} + \frac{1}{2g_0^2} \mathrm{Tr}\, (D_\mu U)^\dagger D^\mu U \right]. \qquad (5.1.2)$$

In the unitary gauge, with $U(x) = 1$ everywhere, this action (5.1.2) reduces to (5.1.1). In the remainder of this chapter, however, we will study (5.1.2) in the axial gauge.

It is best to think of the left-handed symmetry as (confined) color-SU(N) and the right-handed symmetry as flavor-SU(N). Confinement of left-handed color means that only singlets of the left-handed color group exist in the spectrum. There are "mesonic" bound states, as well as "baryonic" bound states. The mesonic bound states have one elementary particle of the sigma model and one elementary antiparticle. The simplest baryonic bound states consist of N of these elementary particles, with no antiparticles. There are also more complicated bound states, which exist because there are excitations in the sigma model (with no gauge field) transforming as higher representations of the color group [31], [30]. In this thesis, we only discuss the mesonic states in detail.

A mesonic bound state, in the axial gauge, is a sigma-model particle-antiparticle pair, confined by a linear potential. The string tension is

$$\sigma = e^2 C_N, \qquad (5.1.3)$$

where C_N is the smallest eigenvalue of the Casimir operator of SU(N). The mass gap is

$$M = 2m + E_0 \ll \mathcal{M},$$

where E_0 is the smallest (positive) binding energy, and m is the mass of a sigma-model elementary excitation. This mass M is finite, for fixed m, as the ultraviolet cutoff is removed. In contrast, the bare Yang-Mills mass $\mathcal{M}$, which is proportional to $1/g_0$, diverges.

Our approach is similar to that of Ref. [1]. We find the wave function of an unbound particle-antiparticle pair, taking into account scattering at the origin. Next, we generalize this to the wave function of the pair, confined by a linear potential. The method is inspired by the determination of the spectrum of the two-dimensional Ising model in an external magnetic field [47]. More sophisticated approaches to this and other two-dimensional models of confinement [48], [49], [50], including fine structure (form factors) of the fundamental excitations, have been developed. We do not take into account decays or corrections to the spectrum from matrix elements with more fundamental excitations [51] in this chapter. For a general review, see Ref. [52].

5.2 The axial gauge Hamiltonian and confinement

In this section we derive the massive Yang-Mills Hamiltonian in the axial gauge. We show that this action describes sigma-model particles and antiparticles confined by a linear potential.

If the axial gauge $A_1 = 0$, is chosen, the action (5.1.2) is

$$S = \int d^2x \left[\frac{1}{2} \mathrm{Tr}\,(\partial_1 A_0)^2 + \frac{1}{2g_0^2} \mathrm{Tr}\,(\partial_0 U^\dagger + \mathrm{i}e U^\dagger A_0)(\partial_0 U - \mathrm{i}e A_0 U) \right.$$
$$\left. - \frac{1}{2g_0^2} \mathrm{Tr}\,\partial_1 U^\dagger \partial_1 U \right] .$$

If we naively eliminate A_0, by its equation of motion (or integrate A_0 from the functional integral), we obtain the effective action

$$S = \int d^2x \left(\frac{1}{2g_0^2} \mathrm{Tr}\, \partial_\mu U^\dagger \partial^\mu U + \frac{1}{2} j^L_{0\,a} \frac{1}{-\partial_1^2 + e^2/g_0^2} j^L_{0\,a} \right), \tag{5.2.1}$$

where $j^L_\mu(x)_b = -\mathrm{i}\mathrm{Tr}\, t_b \partial_\mu U(x) U^\dagger(x)$ is the Noether current of the left-handed SU(N) symmetry. The potential induced on the color-charge density, in the second term of (5.2.1), indicates that charges are screened, instead of confined. This conclusion, however, is based on the fact that $U^\dagger U = 1$. In the renormalized theory, U is not a physical field. The physical scaling field of the principal chiral nonlinear sigma model is not a unitary matrix. This fact is discussed more explicitly in Refs. [36], in the limit $N \to \infty$, with $g_0^2 N$ fixed. The actual excitations of the principal chiral model are massive, with a left and right color charge [19], so that no screening takes place.

A more careful approach is to first find the Hamiltonian in the temporal gauge $A_0 = 0$. Gauge invariance, or Gauss' law, must be imposed on physical states. The Hamiltonian is

$$\begin{aligned} H &= \int dx^1 \left\{ \frac{g_0^2}{2} [j^L_0(x^1)_b]^2 + \frac{1}{2g_0^2} [j^L_1(x^1)_b]^2 + \frac{1}{2} [E(x^1)_b]^2 \right. \\ & \qquad \left. + \frac{e}{g_0^2} j^L_1(x^1)_b A_1(x^1)_b \right\}, \end{aligned} \tag{5.2.2}$$

where $A_1(x^1)_b = \mathrm{Tr}\, t_b A$ and E_a is the electric field, obeying $[E(x^1)_a, A_1(y^1)_b] = -\mathrm{i}\delta_{ab}\delta(x^1 - y^1)$. The Hamiltonian (5.2.2) must be supplemented by Gauss' law $G(x^1)_a \Psi = 0$, for any physical state Ψ, where $G(x^1)_a$ is the generator of spatial gauge transformations:

$$G(x^1)_a = \partial_1 E(x^1)_a + e f_{abc} A_1(x^1)^b E(x^1)_c - \frac{e}{g_0^2} j^L_0(x^1)_a\,. \tag{5.2.3}$$

If we require that the electric field vanishes at the boundaries $x^1 = \pm l/2$, Gauss' law may be explicitly solved [46], to yield the expression for the electric field:

$$E(x^1)_a = \int_{-l/2}^{x^1} dy^1 \left\{ \mathcal{P} \exp \left[ie \int_{-l/2}^{y^1} dz^1 \mathcal{A}_1(z^1) \right] \right\}_a^{\ \ b} \frac{e}{g_0^2} j_0^L(y^1)_b, \tag{5.2.4}$$

where $\mathcal{A}_1(x^1)_a{}^b = \mathrm{i} f_{abc} A_1(x^1)_c$ is the gauge field in the adjoint representation. There remains a global gauge invariance, which must be satisfied by physical states, *i.e.*, $\Gamma_a \Psi = 0$, where

$$\Gamma_a = \int_{-l/2}^{l/2} dy^1 \left\{ \mathcal{P} \exp \left[ie \int_{-l/2}^{y^1} dz^1 \mathcal{A}_1(z^1) \right] \right\}_a^{\ \ b} \frac{e}{g_0^2} j_0^L(y^1)_b. \tag{5.2.5}$$

Now we are free to chose $A_1(x^1)_b = 0$, which simplifies (5.2.4) and (5.2.5). The solution for the electric field yields the Hamiltonian

$$\begin{aligned} H &= \int dx^1 \left\{ \frac{g_0^2}{2} [j_0^L(x^1)_b]^2 + \frac{1}{2g_0^2} [j_1^L(x^1)_b]^2 \right\} \\ &\quad - \frac{e^2}{2g_0^4} \int dx^1 \int dy^1 \, |x^1 - y^1| \, j_0^L(x^1)_b \, j_0^L(y^1)_b, \end{aligned} \tag{5.2.6}$$

where in the last step, we have taken the size l of the system to infinity. The last term is a linear potential which confines left-handed color. The Hamiltonian (5.2.6) is not bounded from below on the full Hilbert space. This is because of the last, nonlocal term; the energy can be lowered by adding pairs of colored particles (or antiparticles) and by separating them. The residual Gauss-law condition $\Gamma_a \Psi = 0$, forces the global left-handed color to be a singlet, thereby removing the instability,

5.3 The free particle-antiparticle wave function: $N > 2$

The wave function of a free antiparticle at x^1 and a free particle at x^2, with momenta p_1 and p_2, respectively, is

$$\Psi_{p_1,p_2}(x^1,y^1)_{a_1a_2;b_1b_2} = \begin{cases} e^{\mathrm{i}p_1x^1+\mathrm{i}p_2y^1}A_{a_1a_2;b_1b_2}, & \text{for } x^1 < y^1, \\ \\ e^{\mathrm{i}p_2x^1+\mathrm{i}p_1y^1}S_{AP}(\theta)^{d_2c_2;c_1d_1}_{a_1b_1;b_2a_2}A_{c_1c_2;d_1d_2}, & \text{for } x^1 > y^1. \end{cases} \tag{5.3.1}$$

where $A_{a_1a_2;b_1b_2}$ is set of arbitrary complex numbers, and $S_{AP}(\theta)^{d_2c_2;c_1d_1}_{a_1b_1;b_2a_2}$ is the principal chiral sigma model S-matrix.

The residual Gauss' law in the axial gauge, $\Gamma_a\Psi = 0$, restricts physical states to those which are invariant under global left-handed SU(N) color transformations. This means that the particle-antiparticle state of the form (5.3.1) must be projected to a global left-color singlet. A left-color-singlet wave function is

$$\Psi_{p_1p_2}(x^1,y^1)_{b_1b_2} = \delta^{a_1a_2}\Psi_{p_1,p_2}(x^1,y^1)_{a_1a_2b_1b_2}. \tag{5.3.2}$$

There are states of degeneracy N^2-1, which resemble massive gluons. These transform as the adjoint representation of the right-handed color symmetry. The wave function of such a state is traceless in the right-handed color indices:

$$\delta^{b_1b_2}\Psi_{p_1p_2}(x^1,y^1)_{b_1b_2} = 0. \tag{5.3.3}$$

We use a non-relativistic approximation $p_{1,2} \ll m$. The wave function in this limit becomes

$$\Psi_{p_1 p_2}(x^1, y^1)_{b_1 b_2} = \begin{cases} e^{\mathrm{i}p_1 x^1 + \mathrm{i}p_2 y^1} A_{b_1 b_2}, & \text{for } x^1 < y^1, \\ \\ e^{\mathrm{i}p_2 x^1 + \mathrm{i}p_1 y^1} \exp(\mathrm{i}\pi - \frac{ih_N}{\pi m}|p_1 - p_2|) A_{b_1 b_2}, & \text{for } x^1 > y^1. \end{cases} \tag{5.3.4}$$

where $\mathrm{Tr}\, A = 0$, and

$$\begin{aligned} h_N &= 2\int_0^\infty \frac{d\xi}{\sinh\xi}\left[2(e^{2\xi/N} - 1) - \sinh(2\xi/N)\right] \\ &= -4\gamma - \psi\left(\frac{1}{2} + \frac{1}{N}\right) - 3\psi\left(\frac{1}{2} - \frac{1}{N}\right) - 4\ln 4, \end{aligned} \tag{5.3.5}$$

where γ is the Euler-Mascheroni constant, and $\psi(x) = d\ln\Gamma(x)/dx$ is the digamma function. The expression in (5.3.4) must be equal to the wave function of two confined particles for sufficiently small $|x^1 - y^1|$. To compare the two expressions, it is convenient to use center-of-mass coordinates, X, x, and their respective momenta P, p. Explicitly, $X = x^1 + y^1$, $x = y^1 - x^1$, $P = p_1 + p_2$ and $p = p_2 - p_1$. In these coordinates, the wave function is

$$\Psi_p(x)_{b_1 b_2} = \begin{cases} \cos(px + \omega) A_{b_1 b_2}, & \text{for } x > 0, \\ \\ \cos[-px + \omega - \phi(p)] A_{b_1 b_2}, & \text{for } x < 0, \end{cases} \tag{5.3.6}$$

for some constant ω, with the phase shift $\phi(p) = \pi - \frac{h_N}{\pi m}|p|$.

Another type of mesonic state is the right-handed color singlet, with $A_{b_1 b_2} = \delta_{b_1 b_2}$. The non-relativistic limit of the wave function in this case is

$$\Psi_p(x)_{\text{singlet}} = \begin{cases} \cos(px+\omega), & \text{for } x>0, \\ \\ \cos[-px+\omega-\chi(p)], & \text{for } x<0, \end{cases} \tag{5.3.7}$$

where $\chi(p) = -\frac{h_N}{\pi m}|p|$.

5.4 Mesonic states of massive Yang-Mills theory: $N > 2$

The wave function of a particle-antiparticle pair, confined by string tension σ, satisfies the Schroedinger equation

$$-\frac{1}{m}\frac{d^2}{dx^2}\Psi(x)_{b_1b_2} + \sigma\,|x|\;\Psi(x)_{b_1b_2} = E\Psi(x)_{b_1b_2}, \tag{5.4.1}$$

where E is the binding energy [47]. The solution to Equation (5.4.1) is

$$\Psi(x)_{b_1b_2} = \begin{cases} C\mathrm{Ai}\left[(m\sigma)^{\frac{1}{3}}\left(x-\frac{E}{\sigma}\right)\right]A_{b_1b_2}, & \text{for } x>0 \\ \\ C'\mathrm{Ai}\left[(m\sigma)^{\frac{1}{3}}\left(-x-\frac{E}{\sigma}\right)\right]A_{b_1b_2}, & \text{for } x<0, \end{cases} \tag{5.4.2}$$

where $\mathrm{Ai}(x)$ is the Airy function of the first kind, and C, C' are constants.

For $|x| \ll (m\sigma)^{-1/3}$, the potential energy in (5.4.1) is sufficiently small that the wave function is (5.3.6), with $|p| = (mE)^{\frac{1}{2}}$. The wave function (5.4.2) is approximated in this region by

$$\Psi(x)_{b_1 b_2} = \begin{cases} C\frac{1}{\left(x+\frac{E}{\sigma}\right)^{\frac{1}{4}}}\cos\left[\frac{2}{3}(m\sigma)^{\frac{1}{2}}\left(-x+\frac{E}{\sigma}\right)^{\frac{3}{2}}-\frac{\pi}{4}\right]A_{b_1 b_2}, & \text{for } x>0, \\ \\ C'\frac{1}{\left(x+\frac{E}{\sigma}\right)^{\frac{1}{4}}}\cos\left[-\frac{2}{3}(m\sigma)^{\frac{1}{2}}\left(x+\frac{E}{\sigma}\right)^{\frac{3}{2}}+\frac{\pi}{4}\right]A_{b_1 b_2}, & \text{for } x<0. \end{cases}$$

Let us now consider the (N^2-1)-plet of mesonic states. The wave functions (5.3.6) and (5.4.2) should be the same for $x \downarrow 0$, yielding

$$\frac{C}{(\frac{E}{\sigma})^{\frac{1}{4}}}\cos\left[\frac{2}{3}(m\sigma)^{\frac{1}{2}}\left(\frac{E}{\sigma}\right)^{\frac{3}{2}}-\frac{\pi}{4}\right]=\cos(\omega). \tag{5.4.3}$$

Equation (5.4.3) implies

$$C=\left(\frac{E}{\sigma}\right)^{\frac{1}{4}}, \quad \omega=\frac{2}{3}(m\sigma)^{\frac{1}{2}}\left(\frac{E}{\sigma}\right)^{\frac{3}{2}}-\frac{\pi}{4}.$$

The wave functions (5.3.6) and (5.4.2) should also be the same for $x \uparrow 0$, yielding

$$\frac{C'}{\left(\frac{E}{\sigma}\right)^{\frac{1}{4}}}\cos\left[-\frac{2}{3}(m\sigma)^{\frac{1}{2}}\left(\frac{E}{\sigma}\right)^{\frac{3}{2}}+\frac{\pi}{4}\right]=\cos\left[\omega-\pi+\frac{h_N}{\pi m}(mE)^{\frac{1}{2}}\right], \tag{5.4.4}$$

hence $C' = C = \left(\frac{E}{\sigma}\right)^{\frac{1}{4}}$. The arguments of the cosine on each side of (5.4.4) must be the same, modulo 2π:

$$-\frac{2}{3}(m\sigma)^{\frac{1}{2}}\left(\frac{E}{\sigma}\right)^{\frac{3}{2}}+\frac{\pi}{4}+2\pi n=\frac{2}{3}(m\sigma)^{\frac{1}{2}}\left(\frac{E}{\sigma}\right)^{\frac{3}{2}}-\frac{5\pi}{4}+\frac{h_N}{\pi m}(mE)^{\frac{1}{2}},$$

for $n = 0, 1, 2, \ldots$. We simplify this to

$$\frac{4}{3}(m\sigma)^{\frac{1}{2}}\left(\frac{E}{\sigma}\right)^{\frac{3}{2}} + \frac{h_N}{\pi m}(mE)^{\frac{1}{2}} - \left(n + \frac{3}{4}\right)2\pi = 0. \tag{5.4.5}$$

An analysis which is similar to that of the previous paragraph yields the quantization condition for the right-handed singlet state (5.3.7). This is

$$\frac{4}{3}(m\sigma)^{\frac{1}{2}}\left(\frac{E}{\sigma}\right)^{\frac{3}{2}} + \frac{h_N}{\pi m}(mE)^{\frac{1}{2}} - \left(n + \frac{1}{4}\right)2\pi = 0. \tag{5.4.6}$$

Equations (5.4.5) and (5.4.6) are depressed cubic equations of the variable $Z_n = E_n^{\frac{1}{2}}$. These cubic equations have only one real solution for each value of n, because $h_N/(\pi m^{\frac{1}{2}}) > 0$. The solution of Equations (5.4.5) and (5.4.6) is

$$E_n = \left\{\left[\epsilon_n + \left(\epsilon_n^2 + \beta_N^3\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \left[\epsilon_n - \left(\epsilon_n^2 + \beta_N^3\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}\right\}^{\frac{1}{2}}, \tag{5.4.7}$$

where

$$\epsilon_n = \frac{3\pi}{4}\left(\frac{\sigma}{m}\right)^{\frac{1}{2}}\left(n + \frac{1}{2} \pm \frac{1}{4}\right), \qquad \beta_N = \frac{h_N \sigma^{\frac{1}{2}}}{4\pi m}, \tag{5.4.8}$$

where $\pm = +$ for the (N^2-1)-plet, and $\pm = -$ for the singlet.

We show in the next section that the expressions (5.4.7) and (5.4.8) remain valid for the SU(2) case, with $h_2 = -4\ln 2 + 2$ and, significantly, with a reversal of the sign in (5.4.8). For $N = 2$ *only* we must take $\pm = -$ for the (N^2-1)-plet (the triplet) and $\pm = +$ for the singlet.

Another interesting special case is the 't Hooft limit, $N \to \infty$. The mass gap of the sigma model should be fixed in this limit. The string tension σ will be fixed as well [53], provided e^2N is fixed. In this limit $h_N \to 0$, and we find

$$E_n = \left[\frac{3\pi}{2}\left(\frac{\sigma}{m}\right)^{\frac{1}{2}}\left(n+\frac{1}{2}\pm\frac{1}{4}\right)\right]^{1/3}. \tag{5.4.9}$$

5.5 The $N = 2$ case

The exponential expression for the S-matrix (4.2.18) is only correct for $N > 2$. The principal chiral model with $SU(2)\times SU(2)$ symmetry is equivalent to the $O(4)$-symmetric nonlinear sigma model. We will express the S-matrix, first found in Ref. [18], by an exponential expression [24].

A state with one excitation has a left-handed color index $a = 1, 2$ and a right-handed color index $b = 1, 2$. In the $O(4)$ formulation, excitations have a single species index $j = 1, 2, 3, 4$. The $SU(2)\times SU(2)$-symmetric states are related to the $O(4)$-symmetric states by

$$\begin{aligned} |P,\theta,a,b\rangle_{\rm in} &= \sum_j \frac{1}{\sqrt{2}}\left(\delta^{j4}\delta_{ab} - i\sigma^j_{ab}\right)|\theta,j\rangle_{\rm in}, \\ |A,\theta,a,b\rangle_{\rm in} &= \sum_j \frac{1}{\sqrt{2}}\left(\delta^{j4}\delta_{ab} - i\sigma^j_{ab}\right)^*|\theta,j\rangle_{\rm in}, \end{aligned}$$

where σ^j with $j = 1, 2, 3$ are the Pauli matrices. The $O(4)$ two-excitation S-matrix, $S(\theta)^{j_1 j_2}_{j_1' j_2'}$ is given by

$$_{\rm out}\langle\theta_1', j_1'; \theta_2', j_2'|\theta_1, j_1; \theta_2, j_2\rangle_{\rm in} = S(\theta)^{j_1 j_2}_{j_1' j_2'}\, 4\pi\delta(\theta_1-\theta_1')\, 4\pi\delta(\theta_2-\theta_2'),$$

where [24]

$$S(\theta)^{j_1 j_2}_{j_1' j_2'} = \left[\frac{\theta + \pi i}{\theta - \pi i}(P^0)^{j_1 j_2}_{j_1' j_2'} + \frac{\theta - \pi i}{\theta + \pi i}(P^+)^{j_1 j_2}_{j_1' j_2'} + (P^-)^{j_1 j_2}_{j_1' j_2'}\right] Q(\theta),$$

$$Q(\theta) = \exp 2 \int_0^\infty \frac{d\xi}{\xi}\frac{e^{-\xi} - 1}{e^{\xi} + 1} \sinh\left(\frac{\xi\theta}{\pi \mathrm{i}}\right),$$

and P^0, P^+, and P^- are the singlet, symmetric-traceless, and antisymmetric projectors, which are

$$\begin{aligned}(P^0)^{j_1 j_2}_{j_1' j_2'} = \frac{1}{4}\delta^{j_1 j_2}\delta_{j_1' j_2'} \quad &, \quad (P^+)^{j_1 j_2}_{j_1' j_2'} = \frac{1}{2}(\delta^{j_1}_{j_1'}\delta^{j_2}_{j_2'} + \delta^{j_1}_{j_2'}\delta^{j_2}_{j_1'}) - \frac{1}{4}\delta^{j_1 j_2}\delta_{j_1' j_2'}, \\ (P^-)^{j_1 j_2}_{j_1' j_2'} \quad &= \quad \frac{1}{2}(\delta^{j_1}_{j_1'}\delta^{j_2}_{j_2'} - \delta^{j_1}_{j_2'}\delta^{j_2}_{j_1'}),\end{aligned}$$

respectively.

We write the left-color-singlet wave function for a free particle and antiparticle:

$$\Psi_{p_1,p_2}(x^1, y^1)_{b_1 b_2} \quad = \quad D^{j_1 j_2}_{b_1 b_2} \begin{cases} e^{ip_1 x^1 + ip_2 y^1} A_{j_1 j_2}, & \text{for } x^1 > y^1 \\ \\ e^{ip_2 x^1 + ip_1 y^1} S(\theta)^{j_1' j_2'}_{j_1 j_2} A_{j_1' j_2'}, & \text{for } x^1 < y^1, \end{cases} \tag{5.5.1}$$

where

$$D^{j_1 j_2}_{b_1 b_2} = \frac{1}{2}\delta^{a_1 a_2}\left(\delta^{j_1 4}\delta_{a_1 b_1} - i\sigma^{j_1}_{a_1 b_1}\right)^* \left(\delta^{j_2 4}\delta_{a_2 b_2} - i\sigma^{j_2}_{a_2 b_2}\right).$$

There is a triplet of degenerate states and one singlet state. The triplet satisfies

$$\delta^{b_1 b_2}\Psi_{p_1,p_2}(x^1,y^1)_{b_1 b_2} = 0. \tag{5.5.2}$$

Substituting (5.5.1) into (5.5.2) gives the condition

$$\delta^{b_1 b_2}\, D^{j_1 j_2}_{b_1 b_2}\, A_{j_1 j_2} = \delta^{j_1 j_2} A_{j_1 j_2} = 0\,.$$

The traceless matrix $A_{j_1 j_2}$ can be split into a symmetric and an antisymmetric part, $A^{+}_{j_1 j_2} = (A_{j_1 j_2} + A_{j_2 j_1})/2$ and $A^{-}_{j_1 j_2} = (A_{j_1 j_2} - A_{j_2 j_1})/2$, respectively. The matrix $A^{+}_{j_1 j_2}$, however, does not contribute to the wave function (5.5.1), because

$$D^{j_1 j_2}_{b_1 b_2} A^{+}_{j_1 j_2} = \frac{1}{2}\delta_{b_1 b_2}\,\mathrm{Tr}\, A^{+} = 0.$$

The matrix $A^{-}_{j_1 j_2}$ satisfies [18], [24]:

$$S(\theta)^{j_1 j_2}_{j_1' j_2'} A^{-}_{j_1 j_2} = Q(\theta) A^{-}_{j_1' j_2'}. \tag{5.5.3}$$

Substituting (5.5.3) into (5.5.1), in center-of-mass coordinates and the non-relativistic limit, we find

$$\Psi_p(x)_{b_1 b_2} = D^{j_1 j_2}_{b_1 b_2} \begin{cases} \cos(px+\omega) A_{j_1 j_2}, & \text{for } x>0, \\ \\ \cos[-px+\omega-\phi(p)] A_{j_1 j_2}, & \text{for } x<0, \end{cases} \tag{5.5.4}$$

where $\phi(p) = -\frac{ih_2}{\pi m}|p|$, where

$$h_2 = 2\int_0^\infty d\xi \, \frac{e^{-\xi}-1}{e^{\xi}+1} = -4\ln 2 + 2. \tag{5.5.5}$$

The wave function of the right-color-singlet bound state is

$$\Psi^{\text{singlet}}_{p_1,p_2}(x^1,y^1) = \begin{cases} e^{ip_1x^1+ip_2y^1}, & \text{for } x^1 > y^1, \\ \\ e^{ip_2x^1+ip_1y^1}\frac{\theta+\pi \mathrm{i}}{\theta-\pi \mathrm{i}}Q(\theta), & \text{for } x^1 < y^1. \end{cases} \tag{5.5.6}$$

In center-of-mass coordinates, in the non-relativistic approximation, this becomes

$$\Psi^{\text{singlet}}_{p}(x) = \begin{cases} \cos(px+\omega), & \text{for } x > 0, \\ \\ \cos[-px+\omega-\chi(p)], & \text{for } x < 0, \end{cases} \tag{5.5.7}$$

where $\chi(p) = \pi - \frac{ih_2}{\pi m}|p|$.

From this point onward, the analysis is similar to what we've presented in the last two sections. We obtain (5.4.7), (5.4.8), except that h_N (defined in (5.3.5)) is replaced with h_2 (defined in (5.5.5)), with one important difference; we have $\pm = +$ for the singlet and $\pm = -$ for the triplet in Eq. (5.4.8).

5.6 Relativistic corrections

In the future, we would like to find relativistic corrections to the mass spectrum. This was done in Ref. [50] for the Ising model in an external magnetic field. The goal would be to find mesonic eigenstates of the Hamiltonian (5.2.6) of the form:

$$|\Psi_B\rangle_{b_1b_2} = |\Psi_B^{(2)}\rangle_{b_1b_2} + |\Psi_B^{(4)}\rangle_{b_1b_2} + |\Psi_B^{(6)}\rangle_{b_1b_2} + \dots,$$

where the state $|\Psi_B^{(2M)}\rangle_{b_1b_2}$ contains M particles and M antiparticles. The multi-particle contributions are included because an electric string may break [51], producing pairs of sigma-model excitations. Nonetheless, for small gauge coupling, the "two-quark" approximation is valid. In the this approximation, the bound state is treated as

$$\begin{aligned}
|\Psi_B\rangle_{b_1b_2} \approx |\Psi_B^{(2)}\rangle_{b_1b_2} &= \frac{1}{2}\int \frac{d\theta_1}{4\pi}\frac{d\theta_2}{4\pi}\Psi(p_1,p_2)_{a_2a_2}|A,\theta_1,b_1,a_1;P,\theta_2,a_2,b_2\rangle, \\
&\quad \text{where,} \\
\Psi(p_1,p_2)_{a_1a_2} &= S(\theta)\left[\delta_{a_1}^{c_1}\delta_{a_2}^{c_2} - \frac{2\pi\mathrm{i}}{N(\pi\mathrm{i}-\theta)}\delta_{a_1a_2}\delta^{c_1c_2}\right]\Psi(p_2,p_1)_{c_1c_2}.
\end{aligned} \tag{5.6.1}$$

The spectrum of masses Δ, of the states (5.6.1) is found from the Bethe-Salpeter equation $(H-\Delta)|\Psi_B^{(2)}\rangle_{b_1b_2} = 0$. Acting with the Hamiltonian (5.2.6) on this state, yields

$$\begin{aligned}
&(m\cosh\theta_1 + m\cosh\theta_2 - \Delta)\,\Psi(p_1',p_2')_{c_1c_2}\delta_{b_1d_1}\delta_{b_2d_2} \\
&\quad = \frac{e^2}{4g_0^4}\int\frac{d\theta_1}{4\pi}\frac{d\theta_2}{4\pi}\Psi(p_1,p_2)_{a_1a_2}\int dx^1dy^1|x^1-y^1| \\
&\quad\quad \times\langle A,\theta_1',d_1,c_1;P,\theta_2',c_2,d_2|\mathrm{Tr}\left[j_0^L(x^1)j_0^L(y^1)\right]|A,\theta_1,b_1,a_1;P,\theta_2,a_2,b_2\rangle,
\end{aligned} \tag{5.6.2}$$

where the operator Tr $\left[j_0^L(x^1)j_0^L(y^1)\right]$ is not time-ordered. The matrix element

$$\langle A,\theta_1',d_1,c_1;P,\theta_2',c_2,d_2|\mathrm{Tr}\,\left[j_0^L(x^1)j_0^L(y^1)\right]|A,\theta_1,b_1,a_1;P,\theta_2,a_2,b_2\rangle$$

is obtained by inserting a complete set of states between the current operators and using the exact form factors of the currents of the principal chiral sigma model. For finite N, only the leading two-particle form factors of currents are known from the previous chapter, and only a vacuum insertion can be made. The complete matrix element is known at large N, which should help in finding the relativistic corrections to the eigenvalues of Eq. (5.6.2).

Chapter 6

(2+1)-dimensional Yang-Mills theory and form factor perturbation theory

6.1 The low lying glueball mass spectrum

In this chapter we turn our attention to the anisotropic (2+1)-dimensional Yang-Mills theory described in Chapter 2. In the highly anisotropic limit, this theory becomes an array principal chiral sigma models, weakly coupled in the x^2 direction. We calculate physical quantities of the anisotropic theory using exact, non-perturbative information from the sigma model. The physical quantities we calculate are the low lying glueball masses, and the string tensions for quarks separated either in the x^1, or x^2 direction.

I should clarify that these calculations have been done before by P. Orland, in References [1], [2], [3], for the $SU(2)$ gauge group. The glueball masses and string tensions in these references were calculated using the $O(4)$-nonlinear sigma model S-matrix and form factors (similar to our calculation from Section 5.5). In this chapter we merely generalize these calculations for the $SU(N)$ group, using our new results from Chapter 4. The reason this has not been done before is that the form factors of the principal chiral sigma model were not known until now.

In this section we calculate the masses of the lightest particles in the anisotropic Yang-Mills theory (which we call glueballs). We recall the Hamiltonian of the anisotropic theory (in axial gauge):

$$H = H_0 + \lambda^2 H_1,$$

with

$$H_0 = \sum_{x^2} H_{\mathrm{PCSM}}(x),$$

and

$$\begin{aligned} H_1 \quad = \quad & \sum_{x^2} \int dx^1 \int dy^1 \frac{1}{4g_0^2 a} |x^1 - y^1| \\ & \times \left[j_0^L(x^1, x^2) - j_0^R(x^1, x^2 - a) \right] \\ & \times \left[j_0^L(y^1, x^2) - j_0^R(y^1, x^2 - a) \right] . \end{aligned} \tag{6.1.1}$$

After axial gauge fixing, there is the residual (global) Gauss's law constraint on physical states:

$$\int dx^1 \left[j_0^L(x^1, x^2)_b - j_0^R(x^1, x^2 - a)_b \right] \Psi = 0, \tag{6.1.2}$$

The constraint (6.1.2) requires that there be an equal number of particles and antiparticles in each x^2 layer. Furthermore, it requires that the excitations form left- and right-color singlets. If the sigma model at x^2 has a particle with a left color index, a_1, this index has to be contracted with either the left-color index of an antiparticle in the x^2 layer, or the right color index of a particle in the $(x^2 + a)$ layer. A glueball in this theory consists of several sigma-model excitations, forming a color-singlet bound state.

The simplest and lightest glueball is one composed of only one particle and one antiparticle, at the same value of x^2. The gauss law constraint requires that their left and right handed color indices be contracted. The interaction Hamiltonian (6.1.1) provides a confining linear potential, with string tension

$$\sigma = 2\sigma^H = 2\lambda^2 \frac{g_0^2}{a^2} C_N. \tag{6.1.3}$$

The factor of 2 comes the fact that both the left and right color charges are confined.

The problem is now essentially (1+1)-dimensional. We can calculate the massive gluon spectrum doing exactly the same calculation as in Chapter 5, but using the new string tension, σ, from Eq. (6.1.3).

The glueball masses are given by $M_n = 2m + E_n$, where m is the mass of the sigma model particles, and E_n is the binding energy. Because both the left and right handed colors are confined, The glueball masses correspond to the singlet mesons from last chapter. For $N > 2$, the binding energy is then given by

$$E_n = \left\{ \left[\epsilon_n + \left(\epsilon_n^2 + \beta_N^3 \right)^{\frac{1}{2}} \right]^{\frac{1}{3}} + \left[\epsilon_n - \left(\epsilon_n^2 + \beta_N^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{3}} \right\}^{\frac{1}{2}}, \tag{6.1.4}$$

where

$$\epsilon_n = \frac{3\pi}{4}\left(\frac{\sigma}{m}\right)^{\frac{1}{2}}\left(n+\frac{1}{4}\right), \quad \beta_N = \frac{h_N \sigma^{\frac{1}{2}}}{4\pi m},$$

for $n = 0, 1, 2, \ldots$, where σ is defined in Eq. (6.1.3) and h_N is defined in Eq. (5.3.5).

The spectrum (6.1.4) generalizes the result from Ref. [1] for general $N > 2$. The result found in Ref. [1] corresponds to our $N = 2$ spectrum from Section 5.5. The $N = 2$ low lying glueball spectrum is given by Eq. (6.1.4) but substituting

$$\epsilon_n = \frac{3\pi}{4}\left(\frac{\sigma}{m}\right)^{\frac{1}{2}}\left(n+\frac{5}{4}\right),$$

and $h_2 = -4\ln 2 + 2$.

6.2 The horizontal string tension

In this section we compute quantum corrections to the string tension σ^H. This calculation has been done before, in Reference [2], for $N = 2$ using the form factors of the $O(4)$ sigma model. In this section we generalize these results for $N > 2$, using the form factors from Chapter 4.

It is convenient to rewrite the Hamiltonian (6.1.1) by reintroducing the auxiliary field $\Phi = -A_0$, such that

$$\begin{aligned} H_1 \;=\; & \sum_{x^2}\int dx^1 \left\{ \frac{g_0^2\, a^2}{4}\partial_1\Phi(x^1,x^2)\partial_1\Phi(x^1,x^2) \right. \\ & \left. -j_0^L(x^1,x^2)\Phi(x^1,x^2) - j_0^R(x^1,x^2)\Phi(x^1,x^2+a)\right\}. \end{aligned} \tag{6.2.1}$$

By integrating out the auxiliary field, Φ, we see the Hamiltonians, (6.2.1) and (6.1.1) are equivalent.

We can easily introduce static quarks into the Hamiltonian (6.2.1) by coupling them to the auxiliary field, Φ. Our goal is to find the potential energy of a quark-antiquark pair separated only in the x^1 direction. By integrating out the sigma model degrees of freedom, we can find the quantum corrections to the string tension σ^H. The Hamiltonian with a quark of charge q at the space point (u^1, u^2), and an antiquark of charge q' at the space-time point (v^1, v^2), is

$$\begin{aligned} H_1 &= \sum_{x^2} \int dx^1 \left\{ \frac{g_0^2\, a^2}{4} \partial_1 \Phi(x^1, x^2) \partial_1 \Phi(x^1, x^2) \right. \\ & -j_0^L(x^1, x^2)\Phi(x^1, x^2) - j_0^R(x^1, x^2)\Phi(x^1, x^2 + a) \\ & \left. + g_0^2\, q\, \Phi(u^1, u^2) - g_0^2\, q' \Phi(v^1, v^2) \right\}. \end{aligned} \tag{6.2.2}$$

With these static quarks, the residual gauss law on physical states is modified to:

$$\begin{aligned} \int dx^1 & \left[j_0^L(x^1, x^2)_b - j_0^R(x^1, x^2 - a)_b \right. \\ & \left. + q_b \delta(x^1 - u^1)\delta_{x^2 u^2} - q'_b \delta(x^1 - v^1)\delta_{x^2 v^2} \right] \Psi = 0. \end{aligned} \tag{6.2.3}$$

To find the string tension, σ^H, we set $u^2 = v^2$, and integrate out the sigma model field, U. We obtain an effective action, $S_{\text{eff}}(\Phi)$, by

$$e^{iS_{\text{eff}}(\Phi)} = \langle 0 | \mathcal{T}\, e^{i \int dx^0 \lambda^2 H_1} | 0 \rangle, \tag{6.2.4}$$

where $\mathcal{T}$ stands for time ordering. The field Φ in (6.2.4) is treated as a background classical field. Expanding (6.2.4) in powers of λ, up to quartic order, we find

$$
\begin{aligned}
i\,S_{\rm eff}(A_0) &\approx -i\lambda^2 \sum_{x^2} \int d^2x\, \frac{g_0^2 a^2}{4} \Phi \partial_1^2 \Phi + i\lambda^4 S^{(2)}(\Phi) + \mathcal{O}(\lambda^6) \\
&- \quad \lambda^2 \sum_{x^2} \int d^2x \left[g_0^2\, q(x^0)\, \Phi(x^0,u^1,u^2) - g_0^2\, q'(x^0) \Phi(x^0,v^1,v^2) \right]
\end{aligned}
\tag{6.2.5}
$$

where

$$
iS^{(2)} \equiv -\frac{1}{2} \sum_{x^2} \int d^2x d^2y\, D(x^0,x^1,y^0,y^1,x^2)_{acef} \Phi(x^0,x^1,x^2)_{ac} \Phi(y^0,y^1,x^2)_{ef},
$$

where

$$
D(x^0,x^1,y^0,y^1,x^2)_{acef} \equiv \langle 0 | \mathcal{T} j_0^L(x^0,x^1,x^2)_{ac}\, j_0^L(y^0,y^1,x^2)_{ef} | 0 \rangle. \tag{6.2.6}
$$

We compute the correlation function (6.2.6) by introducing a complete set of intermediate states between the two operators. The non-time-ordered correlation function is given by

$$
\begin{aligned}
\langle 0 | j_0^L(x)_{ac}\, j_0^L(y)_{ef} | 0 \rangle &= \sum_{M=1}^{\infty} \frac{1}{N(M!)^2} \int \frac{d\theta_1 \dots d\theta_{2M}}{(2\pi)^{2M}} e^{-i(x-y)\cdot\left[\sum_{j=1}^{2M} p_j\right]} \\
&\times \langle 0 | j_\mu^L(0)_{a_0 c_0} | A, \theta_1, b_1, a_1; \dots; \\
&\quad A, \theta_M, b_M a_M; P, \theta_{M+1}, a_{M+1}, b_{M+1}; \dots; P, \theta_{2M}, a_{2M}, b_{2M} \rangle \\
&\times \left[\langle 0 | j_\nu^L(0)_{e_0 f_0} | A, \theta_1, b_1, a_1; \dots; \right. \\
&\quad \left. A, \theta_M, b_M a_M; P, \theta_{M+1}, a_{M+1}, b_{M+1}; \dots; P, \theta_{2M}, a_{2M}, b_{2M} \rangle \right]^* .
\end{aligned}
$$

The correlation function (6.2.6) can be found exactly at large N using the form factors from Chapter 4. For general $N < \infty$, we can only calculate a large-distance approximation, using the two-particle form factor (also found in Chapter 4). At large distances, it is sufficient to compute only the first intermediate state, with one particle and one antiparticle.

We recall, from Section 4.2, the form factor with one particle and one antiparticle is

$$\begin{aligned}
\langle 0|j^L_\mu(x)_{ac} \;&|\; A,\theta_1,b_1,a_1;P,\theta_2,a_2,b_1\rangle \\
&= (p_1-p_2)_\mu\left(\delta_{a_0a_2}\delta_{c_0a_1}-\frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_2}\delta_{b_1b_2}\right)e^{-ix\cdot(p_1+p_2)} \\
&\quad\times\frac{2\pi i}{(\theta+\pi i)}\exp\int_0^\infty\frac{d\xi}{\xi}\left[\frac{-2\sinh\left(\frac{2\xi}{N}\right)}{\sinh\xi}\right. \\
&\quad\left.+\frac{4e^{-\xi}\left(e^{2\xi/N}-1\right)}{1-e^{-2\xi}}\right]\frac{\sin^2[\xi(\pi i-\theta)/2\pi]}{\sinh\xi}.
\end{aligned}\tag{6.2.7}$$

Inserting (6.2.7) into (6.2.6) and time ordering, we find

$$\begin{aligned}
D(x,y)_{acef} &= \int\frac{d\theta_1\,d\theta_2}{(2\pi)^2}m^2(\cosh\theta_1-\cosh\theta_2)^2 \\
&\quad\times\left(\delta_{aa_2}\delta_{ca_1}-\frac{1}{N}\delta_{ac}\delta_{a_1a_2}\right)\left(\delta_{ea_2}\delta_{fa_1}-\frac{1}{N}\delta_{ef}\delta_{a_1a_2}\right) \\
&\quad\times\exp\left\{-im\,\mathrm{sgn}(x^0-y^0)\left[(x^0-y^0)(\cosh\theta_1+\cosh\theta_2)\right.\right. \\
&\qquad\left.\left.-(x^1-y^1)(\sinh\theta_1+\sinh\theta_2)\right]\right\} \\
&\quad\times\left\{\frac{2\pi i}{(\theta+\pi i)}\exp\int_0^\infty\frac{d\xi}{\xi}\left[\frac{-2\sinh\left(\frac{2\xi}{N}\right)}{\sinh\xi}\right.\right. \\
&\qquad\left.\left.+\frac{4e^{-\xi}\left(e^{2\xi/N}-1\right)}{1-e^{-2\xi}}\right]\frac{\sin^2[\xi(\pi i-\theta)/2\pi]}{\sinh\xi}\right\}^2.
\end{aligned}\tag{6.2.8}$$

The color factor in (6.2.8) is

$$\left(\delta_{aa_2}\delta_{ca_1}-\frac{1}{N}\delta_{ac}\delta_{a_1a_2}\right)\left(\delta_{ea_2}\delta_{fa_1}-\frac{1}{N}\delta_{ef}\delta_{a_1a_2}\right)=\delta_{ae}\delta_{ef}-\frac{1}{N}\delta_{ac}\delta_{ef}.\tag{6.2.9}$$

The term in the right-hand side of (6.2.9) proportional to $\frac{1}{N}$ does not contribute when we plug (6.2.8) back into (6.2.5), because the field Φ is traceless, so we will ignore this term from now on.

We evaluate $iS^{(2)}(\Phi)$ using coordinates X^μ, r^μ, defined by $x^\mu = X^\mu + \frac{1}{2}r^\mu$, and $y^\mu = X^\mu - \frac{1}{2}r^\mu$. We then use the derivative expansion for $X \gg r$:

$$\begin{aligned}\Phi(x) &= \Phi(X) + \frac{r^\mu}{2}\partial_\mu\Phi(X) + \frac{r^\mu r^\nu}{8}\partial_\mu\partial_\nu\Phi(X) + \dots,\\ \Phi(y) &= \Phi(X) - \frac{r^\mu}{2}\partial_\mu\Phi(X) + \frac{r^\mu r^\nu}{8}\partial_\mu\partial_\nu\Phi(X) \pm \dots,\end{aligned} \tag{6.2.10}$$

where ∂_μ denotes $\partial/\partial X^\mu$. This derivative expansion is valid at large distances. The quadratic contribution to the effective action is

$$\begin{aligned}iS^{(2)} &= -\frac{i}{2}\int d^2X d^2r\, D\left(X+\frac{r}{2}, X-\frac{r}{2}\right)_{acef}\\ &\quad\times\Phi\left(X+\frac{r}{2}\right)_{ac}\Phi\left(X-\frac{r}{2}\right)_{ef}.\end{aligned} \tag{6.2.11}$$

We substitute (6.2.10) into (6.2.11) and find

$$\begin{aligned} iS^{(2)} \;=\; & -\frac{i}{2}\int d^2X d^2r \int \frac{d\theta_1\, d\theta_2}{(2\pi)^2} m^2(\cosh\theta_1 - \cosh\theta_2)^2 \delta_{ae}\delta_{cf} \\ & \times \exp\left\{-im\,\mathrm{sgn}(r^0)\left[(r^0)(\cosh\theta_1 + \cosh\theta_2)\right.\right. \\ & \qquad \left.\left. -(r^1)(\sinh\theta_1 + \sinh\theta_2)\right]\right\} \\ & \times \left\{ \frac{2\pi i}{(\theta+\pi i)} \exp \int_0^\infty \frac{d\xi}{\xi}\left[\frac{-2\sinh\left(\frac{2\xi}{N}\right)}{\sinh\xi}\right.\right. \\ & \qquad \left.\left. + \frac{4e^{-\xi}\left(e^{2\xi/N}-1\right)}{1-e^{-2\xi}}\right] \frac{\sin^2[\xi(\pi i - \theta)/2\pi]}{\sinh\xi} \right\}^2 \\ & \times \left(\Phi(X)_{ac} + \frac{r^\mu}{2}\partial_\mu \Phi(X)_{ac} + \frac{r^\mu r^\nu}{8}\partial_\mu\partial_\nu\Phi(X)_{ac}\right) \\ & \times \left(\Phi(X)_{ef} - \frac{r^\mu}{2}\partial_\mu \Phi(X)_{ef} + \frac{r^\mu r^\nu}{8}\partial_\mu\partial_\nu\Phi(X)_{ef}\right). \end{aligned} \tag{6.2.12}$$

We keep only terms quadratic in r in (6.2.12) and then integrate out the r variable. Only the terms proportional to $(r^1)^2$ give a non-vanishing contribution in (6.2.12). After this integration, we find the effective action:

$$
\begin{aligned}
S_{\text{eff}}(\Phi) &= \int d^2x \frac{1}{2}\Phi\partial_1^2\Phi \\
&\times \left\{ 1 - \lambda^2 \frac{Nm}{2(2\pi)^2} \int d\theta_1 d\theta_2 \frac{\sinh^2\left(\frac{\theta_1+\theta_2}{2}\right)\sinh^2\left(\frac{\theta_1-\theta_2}{2}\right)}{\cosh\left(\frac{\theta_1+\theta_2}{2}\right)\cosh\left(\frac{\theta_1-\theta_2}{2}\right)} \right. \\
&\times \delta''\left(2m\cosh\left(\frac{\theta_1+\theta_2}{2}\right)\sinh\left(\frac{\theta_1-\theta_2}{2}\right)\right) \\
&\times \frac{4\pi^2}{(\theta_1-\theta_2)^2+\pi^2} \exp 2\int_0^\infty \frac{d\xi}{\xi}\left[\frac{-2\sinh\left(\frac{2\xi}{N}\right)}{\sinh\xi}\right. \\
&\quad \left.\left. + \frac{4e^{-\xi}\left(e^{2\xi/N}-1\right)}{1-e^{-2\xi}}\right] \frac{\sin^2[\xi(\pi i-(\theta_1-\theta_2))/2\pi]}{\sinh\xi} \right\} \\
&- \lambda^2 \sum_{x^2} \int d^2x \left[g_0^2\, q(x^0)\,\Phi(x^0,u^1,u^2) - g_0^2\, q'(x^0)\Phi(x^0,v^1,v^2)\right].
\end{aligned}
$$

We can now read off the renormalized string tension σ^H, by integrating out the auxiliary field Φ:

$$
\begin{aligned}
\sigma^H &= \lambda^2 \frac{g_0^2}{a^2} C_N \left\{ 1 - \left[\lambda^2 \frac{Nm}{2(2\pi)^2} \int d\theta_1 d\theta_2 \frac{\sinh^2\left(\frac{\theta_1+\theta_2}{2}\right)\sinh^2\left(\frac{\theta_1-\theta_2}{2}\right)}{\cosh\left(\frac{\theta_1+\theta_2}{2}\right)\cosh\left(\frac{\theta_1-\theta_2}{2}\right)} \right.\right. \\
&\quad \times \delta''\left(2m\cosh\left(\frac{\theta_1+\theta_2}{2}\right)\sinh\left(\frac{\theta_1-\theta_2}{2}\right)\right) \\
&\quad \times \frac{4\pi^2}{(\theta_1-\theta_2)^2+\pi^2} \exp 2\int_0^\infty \frac{d\xi}{\xi}\left[\frac{-2\sinh\left(\frac{2\xi}{N}\right)}{\sinh\xi}\right. \\
&\quad \left.\left.\left. + \frac{4e^{-\xi}\left(e^{2\xi/N}-1\right)}{1-e^{-2\xi}}\right] \frac{\sin^2[\xi(\pi i-(\theta_1-\theta_2))/2\pi]}{\sinh\xi} \right] \right\}^{-1}.
\end{aligned}
$$

After the integration over θ_1 and θ_2, the string tension is

$$\sigma^H = \lambda^2 \frac{g_0^2}{a^2} C_N \left\{ 1 - \lambda^2 \frac{N}{3m^3(2\pi)^2} \exp 2 \int_0^\infty \frac{d\xi}{\xi} \left[\frac{-2\sinh\left(\frac{2\xi}{N}\right)}{\sinh\xi} + \frac{4e^{-\xi}\left(e^{2\xi/N} - 1\right)}{1 - e^{-2\xi}} \right] \frac{\sin^2\left(\frac{i\xi}{2}\right)}{\sinh\xi} \right\}^{-1} . \tag{6.2.13}$$

The string tension (6.2.13) generalizes the result from Refence [2] from $N = 2$, to general $N > 2$.

In the next section we compute the string tension for a quark-antiquark pair separated in the x^2 direction, rather than x^1. We call this the vertical string tension σ^V.

6.3 The vertical string tension

In this section we calculate the string tension, σ^V, for a quark-antiquark pair separated only in the x^2 direction. This calculation has been done before in Reference [3] for the $SU(2)$ gauge group. We show here how to generalize this result for $N > 2$ using the form factors from Chapter 4.

If we place a static quark at the space point u^1, u^2, and an antiquark at u^1, v^2, with $u^2 > v^2$, The residual Gauss's Law (6.2.3) requires that there be at least one sigma model particle in each x^2 layer, for $u^2 > x^2 > v^2$. The left-handed color index of a particle at x^2 is contracted with the right-handed color of the particle at $x^2 + a$. The left-handed color index of the particle at $u^2 - a$ and the right-handed color of the particle at $v^2 + a$ are contracted with the color indices of the quark at u^2, and the antiquark at v^2, respectively. The physical state then resembles a color-singlet string of sigma model particles, whose endpoints are the quarks. The vertical string tension is obtain by calculating the energy of this string,

$$\sigma^V = \lim_{|u^2-v^2|\to\infty} \frac{E_{\text{string}}}{|u^2-v^2|}.$$

The first approximation is obtained by assuming the energy of the string is only given by the mass of the sigma model particles, such that $E_{\text{string}} = \frac{m}{a}|u^2-v^2|$, so $\sigma^V = m/a$.

Corrections to the vertical string tension are found by calculating the contributions to the energy of the string from the Hamiltonian $\lambda^2 H_1$. As in Reference [3], we will use a non relativistic approximation, where the sigma model particles have momentum much smaller than their mass, so we will ignore any creation or annihilation of particles.

The projection of the Hamiltonian onto the non relativistic string state is

$$H = \sum_{x^2=v^2}^{u^2} \left\{ m + \int \frac{dp}{2\pi}\frac{p^2}{2m}\mathfrak{A}_P^\dagger(p)_{ab}\mathfrak{A}_P(p)_{ab} \right\} + \lambda^2 H_1,$$

where $\mathfrak{A}_P^\dagger(p)_{ab}$, and $\mathfrak{A}_P(p)_{ab}$ are the sigma model particle creation and annihilation operators, respectively, and

$$\begin{aligned} H_1 = {} & \sum_{x^2}\int dx^1 \int dy^1 \frac{1}{4g_0^2 a}|x^1-y^1| \\ & \times \left[j_0^L(x^1,x^2) - j_0^R(x^1,x^2-a) + q_b\delta(x^1-u^1)\delta_{x^2u^2} - q_b'\delta(x^1-u^1)\delta_{x^2v^2} \right] \\ & \times \left[j_0^L(y^1,x^2) - j_0^R(y^1,x^2-a) + q_b\delta(y^1-u^1)\delta_{x^2u^2} - q_b'\delta(y^1-u^1)\delta_{x^2v^2} \right], \end{aligned} \tag{6.3.1}$$

where we have again eliminated the auxiliary field, Φ.

We now now need to find the expectation value

$$\langle \text{string}|H_1|\text{string}\rangle, \tag{6.3.2}$$

where the state $|\text{string}\rangle$ has a sigma-model particle for every x^2, whose center of mass is located at $x^1 = z(x^2)$. To evaluate (6.3.2), we need matrix elements of the form

$$\begin{aligned}\langle P, z_1, a_1, b_1|j_0^C(x)_{ac}|P, z_2, a_2, b_2\rangle &= \int \frac{dp_1}{2\pi}\frac{1}{\sqrt{2E_1}}\int\frac{dp_2}{2\pi}\frac{1}{\sqrt{2E_2}} \\ &\times e^{-ip_1\cdot(z_1-x)+ip_2\cdot(z_2-x)}\langle P,\theta_1,a_1,b_1|j_0^C(x)_{ac}|P,\theta_2,a_2,b_2\rangle,\end{aligned} \tag{6.3.3}$$

where the matrix element on the right hand side of (6.3.3) is the two particle form factor from Section 4.2 (with the incoming antiparticle crossed to an outgoing particle), and $C = L, R$. By applying crossing symmetry on the form factor (4.2.19), we find

$$\begin{aligned}&\langle P,\theta_1,a_1,b_1|j_0^C(x)_{ac}|P,\theta_2,a_2,b_2\rangle \\ &= (p_1+p_2)_0\mathfrak{D}^C_{a\,c\,a_1a_2b_1b_2} \\ &\times\frac{2\pi i}{\theta+2\pi i}\exp\int_0^\infty\frac{d\xi}{\xi}\left[\frac{-2\sinh\left(\frac{2\xi}{N}\right)}{\sinh\xi}+\frac{4e^{-\xi}\left(e^{\frac{2\xi}{N}}-1\right)}{1-e^{-2\xi}}\right]\frac{\sin^2[\xi\theta/2\pi]}{\sinh\xi},\end{aligned}$$

where

$$\begin{aligned}\mathfrak{D}^L_{a\,c\,a_1a_2b_1b_2} &= \delta_{a\,a_2}\delta_{c\,a_1}\delta_{b_1b_2}-\frac{1}{N}\delta_{a\,c}\delta_{a_1a2}\delta_{b_1b_2}, \\ \mathfrak{D}^R_{a\,c\,a_1a_2b_1b_2} &= \delta_{a\,b_2}\delta_{c\,b_1}\delta_{a_1a_2}-\frac{1}{N}\delta_{a\,c}\delta_{a_1a_2}\delta_{b_1b_2}.\end{aligned}$$

Taking the non-relativistic limit, we find

$$\begin{aligned}&\frac{1}{\sqrt{2E_1}}\frac{1}{\sqrt{2E_2}}\langle P,\theta_1,a_1,b_1|j_0^C(x)_{a\,c}|P,\theta_2,a_2,b_2\rangle\\&\approx \mathfrak{D}^C_{a\,c\,a_1a_2b_1b_2}\exp -\frac{A_N}{m^2}(p_1-p_2)^2.\end{aligned}$$

where

$$\begin{aligned}A_N &= \int_0^\infty \frac{d\xi}{4\pi^2}\frac{\xi}{\sinh\xi}\left[\sinh\left(\frac{2\xi}{N}\right)-2\left(e^{2\xi/N}-1\right)\right]\\&= \frac{1}{16}\pi^2\left[2\pi^2-3\psi^{(1)}\left(\frac{1}{2}-\frac{1}{N}\right)-\psi^{(1)}\left(\frac{1}{2}+\frac{1}{N}\right)\right],\end{aligned}$$

for $N>2$, where $\psi^{(n)}(x)=d^{n+1}\ln\Gamma(x)/dx^{n+1}$ is the n-th polygamma function.

The matrix element (6.3.3) is then

$$\begin{aligned}&\langle P,z_1,a_1,b_1|j_0^C(x)_{ac}|P,z_2,a_2,b_2\rangle\\&=\sqrt{\frac{m^2}{2\pi A_N}}\mathfrak{D}^C_{a\,c\,a_1a_2b_1b_2}\exp\left[-\frac{m^2}{4A_N}\left(\frac{z_1+z_2}{2}-y\right)^2\right]\delta(z_1-z_2).\end{aligned} \tag{6.3.4}$$

This means that the color of a particle is a Gaussian distribution in the non relativistic limit. In this sense, they are not point-like particles, but the color is smeared over space.

We now use (6.3.4) to write the effective Hamiltonian of the non-relativistic string. This is given by the projection of the Hamiltonian (6.3.1) onto the state $|\text{string}\rangle$, which has a sigma-model particle at each x^2 layer, located at the point $z^1(x^2)$, for $u^2>x^2>v^2$, a static quark at u^1, u^2, and an antiquark at u^1, v^2. The string Hamiltonian is

$$H_{\rm string} = \frac{m}{a}(v^2 - u^2) - \frac{1}{2m}\sum_{x^2=v^2}^{u^2-a} \frac{\partial^2}{\partial z^1(x^2)^2} + \lambda^2 V_{\rm bulk} + \lambda^2 V_{\rm ends},$$

where

$$\begin{aligned} V_{\rm bulk} \;\; = \;\; & -\frac{m^2}{8\pi A_N}\frac{1}{g_0^2 a^2}\sum_{x^2=v^2+a}^{u^2-a}\int dx^1\, dy^1 |x^1 - y^1| \\ & \times \left\{ e^{-\frac{m^2}{4A_N}\left[z^1(x^2)-x^1\right]^2} \mathfrak{D}^L(x^2)_{a\,c\,a_1a_2b_1b_2} \right. \\ & \left. - e^{-\frac{m^2}{4A_N}\left[z^1(x^2-a)-x^1\right]^2} \mathfrak{D}^R(x^2-a)_{a\,c\,a_1a_2b_1b_2} \right\} \\ & \times \left\{ e^{-\frac{m^2}{4A_N}\left[z^1(x^2)-y^1\right]^2} \mathfrak{D}^L(x^2)_{a\,c\,a_2a_1b_2b_1} \right. \\ & \left. - e^{-\frac{m^2}{4A_N}\left[z^1(x^2-a)-y^1\right]^2} \mathfrak{D}^R(x^2-a)_{a\,c\,a_2a_1b_2b_1} \right\}, \end{aligned} \tag{6.3.5}$$

and

$$\begin{aligned}
V_{\rm ends} \;=\; & -\frac{1}{4g_0^2a^2}\int dx^1dy^1|x^1-y^1|\left\{\sqrt{\frac{m^2}{2\pi A_N}}e^{-\frac{m^2}{4A_N}\left[z^1(v^2)-x^1\right]^2}\mathfrak{D}^R(v^2)_{a\,c\,a_1a_2b_1b_2}\right.\\
&\left.+\delta(x^2-v^1)q'_{a\,c}4\pi\delta_{a_1a_2}\delta_{b_1b_2}\right\}\\
&\times\left\{\sqrt{\frac{m^2}{2\pi A_N}}e^{-\frac{m^2}{4A_N}\left[z^1(v^2)-y^1\right]^2}\mathfrak{D}^R(v^2)_{a\,c\,a_1a_2b_1b_2}\right.\\
&\left.+\delta(y^1-u^1)q'_{a\,c}4\pi\delta_{a_1a_2}\delta_{b_1b_2}\right\}\\
&-\frac{1}{4g_0^2a^2}\int dx^1dy^1|x^1-y^1|\\
&\times\left\{\sqrt{\frac{m^2}{2\pi A_N}}e^{-\frac{m^2}{4A_N}\left[z^1(u^2-a)-x^1\right]^2}\mathfrak{D}^L(u^2-a)_{a\,c\,a_1a_2b_1b_2}\right.\\
&\left.+\delta(x^2-u^1)q'_{a\,c}4\pi\delta_{a_1a_2}\delta_{b_1b_2}\right\}\\
&\times\left\{\sqrt{\frac{m^2}{2\pi A_N}}e^{-\frac{m^2}{4A_N}\left[z^1(u^2-a)-y^1\right]^2}\mathfrak{D}^L(u^2-a)_{a\,c\,a_1a_2b_1b_2}\right.\\
&\left.+\delta(y^2-u^1)q'_{a\,c}4\pi\delta_{a_1a_2}\delta_{b_1b_2}\right\}. \qquad (6.3.6)
\end{aligned}$$

Enforcing the residual Gauss's law (6.2.3) on (6.3.5) and (6.3.6), implies

$$\begin{aligned}
&\int dx^1\left\{-\sqrt{\frac{m^2}{2\pi A_N}}e^{-\frac{m^2}{4A_N}\left[z^1(x^2)-x^1\right]^2}\mathfrak{D}^L(x^2)_{a\,c\,a_1a_2b_1b_2}\right.\\
&\left.+\sqrt{\frac{m^2}{2\pi A_N}}e^{-\frac{m^2}{4A_N}\left[z^1(x^2-a)-x^1\right]^2}\mathfrak{D}^R(x^2-a)_{a\,c\,a_1a_2b_1b_2}\right\}\Psi=0, \qquad (6.3.7)
\end{aligned}$$

for $u^2 > x^2 > v^2$, and

$$\begin{aligned}
&\int dx^1 \sqrt{\frac{m^2}{2\pi A_N}} \left\{ e^{-\frac{m^2}{4A_N}\left[z^1(v^2)-x^1\right]^2} \mathfrak{D}^R(v^2)_{a\,c\,a_1 a_2 b_1 b_2} \right. \\
&\qquad \left. -q'_{a\,c}\delta(x^1-u^1)4\pi\delta_{a_1a_2}\delta_{b_1b_2} \right\} \Psi = 0, \\
&\int dx^1 \sqrt{\frac{m^2}{2\pi A_N}} \left\{ e^{-\frac{m^2}{4A_N}\left[z^1(u^2-a)-x^1\right]^2} \mathfrak{D}^L(u^2-a)_{a\,c\,a_1 a_2 b_1 b_2} \right. \\
&\qquad \left. -q_{a\,c}\delta(x^1-u^1)4\pi\delta_{a_1a_2}\delta_{b_1b_2} \right\} \Psi = 0, \qquad (6.3.8)
\end{aligned}$$

respectively. The constraint (6.3.7) is satisfied by identifying $\mathfrak{D}^L(x^2)_{a\,c\,a_1a_2b_1b_2} = \mathfrak{D}^R(x^2 - a)_{a\,c\,a_1a_2b_1b_2}$. The constraint (6.3.8) is satisfied by identifying $\mathfrak{D}^R(v^2)_{a\,c\,a_1a_2b_1b_2} = q'_{a\,c}4\pi\delta_{a_1a_2}\delta_{b_1b_2}$, and $\mathfrak{D}^L(u^2-a)_{a\,c\,a_1a_2b_1b_2} = q_{a\,c}4\pi\delta_{a_1a_2}\delta_{b_1b_2}$. Using this, we can eliminate the color degrees of freedom from (6.3.5) and (6.3.6).

Now we want to integrate out the variables x^1 and y^1 from equations (6.3.5) and (6.3.6). The integrals involved are:

$$\begin{aligned}
\int dx^1 dy^1 |x^1 - y^1| e^{-\frac{m^2}{4A_N}\left[(x^1)^2+(y^1)^2\right]} &= \frac{4\sqrt{2\pi}A_N^{3/2}}{m^3}, \\
\int dx^1 dy^1 |x^1 - y^1| e^{-\frac{m^2}{4A_N}\left[(x^1+r)^2+(y^1)^2\right]} &= \frac{4\sqrt{2\pi}A_N^{3/2}}{m^3} P(r), \\
\int dx^1 |x^1 - u^1| e^{-\frac{m^2}{4A_N}\left[x^1-z^1(U^2)\right]^2} &= \frac{2A_N}{m^2} P\left[\sqrt{2}z^1(u^2) - \sqrt{2}u^1\right],
\end{aligned}$$

Where $P(r)$ is a function for which we do not have an exact analytic expression, but its behavior for small and large r is

$$P(r) = \begin{cases} 1 + \frac{m^2 r^2}{4A_N}, & r << \frac{1}{m}, \\ \sqrt{\frac{\pi}{2A_N}} m|r|, & r >> \frac{1}{m}. \end{cases} \qquad (6.3.9)$$

After integrating out x^1, and y^1, the string Hamiltonian is

$$\begin{aligned} H_{\text{string}} &= \frac{m}{a}(u^2 - v^2) - \frac{1}{2m} \sum_{x^2=v^2}^{u^2-a} \frac{\partial^2}{\partial z^1(x^2)^2} \\ &\quad - \frac{\lambda^2 N(N^2-1)}{m\, g_0^2 a^2} \sqrt{\frac{A_N}{2\pi}} \sum_{x^2=v^2+a}^{u^2} \left\{1 - P\left[z^1(x^2) - z^1(x^2-a)\right]\right\} \\ &\quad - \frac{\lambda^2 N(N^2-1)}{m\, g_0^2 a^2} \sqrt{\frac{A_N}{2\pi}} \Big(1 + P\left\{\sqrt{2}\left[z^1(v^2) - u^1\right]\right\} \\ &\qquad + P\left\{\sqrt{2}\left[z^1(u^2-a) - u^1\right]\right\}\Big), \end{aligned} \tag{6.3.10}$$

where we have used

$$\left(\mathfrak{D}^C\right)^2 = N\left(N^2 - 1\right).$$

The potential energy between a static quark-antiquark pair is then determined by finding the ground state of the Hamiltonian (6.3.10).

We further simplify the Hamiltonian (6.3.10) using the small-gradient approximation. That is, in the non-relativistic limit (when the sigma model mass gap is taken to be very large), we expect that the sigma-model particles in two adjacent x^2 layers are close to each other in the x^1 direction. Specifically, we assume $|z^1(x^2) - z^1(x^2-a)| << m^{-1}$. At the endpoints of the string, we also assume $|z^1(v^2) - u^1| << m^{-1}$, and $|z^1(u^2-a) - u^1| << m^{-1}$.

Using Eq. (6.3.9), the small-gradient approximation gives the Hamiltonian

$$\begin{aligned} H_{\text{string}} &= \frac{\lambda^2 N(N^2-1)}{m\,g_0^2 a^2}\sqrt{\frac{A_N}{2\pi}} + \frac{m}{a}(u^2-v^2) - \frac{1}{2m}\sum_{x^2=v^2}^{v^2-a}\frac{\partial^2}{\partial z^1(x^2)^2} \\ &+ \frac{\lambda^2 N(N^2-1)}{4m g_0^2 a^2}\sqrt{\frac{1}{2\pi A_N}}\sum_{x^2=v^2+a}^{u^2-a}\left[z^1(x^2)-z^1(x^2-a)\right]^2 \\ &+ \frac{\lambda^2 N(N^2-1)}{2m g_0^2 a^2}\sqrt{\frac{1}{2\pi A_N}}\left\{\left[z^1(v^2)-u^1\right]^2+\left[z^1(u^2-a)-u^1\right]^2\right\}. \end{aligned} \tag{6.3.11}$$

The first term in the Hamiltonian (6.3.11) is just a constant with no physical significance, so we will ignore it from now on. The Hamiltonian (6.3.11) is equivalent $Q=(u^2-v^2)/a$ coupled harmonic oscillators. The ground-state energy is then given by

$$E_0 = mQ - \frac{\lambda\sqrt{N(N^2-1)}}{g_0 a}\left(\frac{1}{2\pi A_N}\right)^{\frac{1}{4}}\sum_{q=0}^{Q}\sin\frac{\pi q}{2Q}. \tag{6.3.12}$$

Using the Euler summation formula, for large Q:

$$\begin{aligned} \sum_{q=0}^{Q}F\left(\frac{q}{Q}\right) &= Q\int_0^1 dx F(x) - \frac{1}{2}\left[F(1)-F(0)\right] \\ &+\frac{1}{12Q}\left[F'(1)-F'(0)\right]+\mathcal{O}\left(\frac{1}{Q^2}\right), \end{aligned}$$

the ground-state energy (6.3.12) becomes (dropping any constants that do not depend on

Q)

$$\begin{aligned} E_0 &= \left[\frac{m}{a} - \frac{2\lambda\sqrt{N(N^2-1)}}{\pi g_0 a^2}\left(\frac{1}{2\pi A_N}\right)^{\frac{1}{4}}\right] L \\ &\quad + \frac{\pi}{24}\frac{\lambda\sqrt{N(N^2-1)}}{g_0}\left(\frac{1}{2\pi A_N}\right)^{\frac{1}{4}}\frac{1}{L} + \mathcal{O}\left(\frac{1}{L^2}\right). \end{aligned} \tag{6.3.13}$$

where the distance between the quark and antiquark is $L = Qa$.

We can easily read the vertical string tension off (6.3.13):

$$\sigma^V = \frac{m}{a} - \frac{2\lambda\sqrt{N(N^2-1)}}{\pi g_0 a^2}\left(\frac{1}{2\pi A_N}\right)^{\frac{1}{4}}. \tag{6.3.14}$$

There is also a Coulomb-like term in the quark-antiquark potential, which is proportional to $1/L$.

Chapter 7

Anisotropic renormalization group for gauge theories

7.1 Classical longitudinal rescaling

This chapter contains material previously published in [54] and [55].

The longitudinal rescaling of coordinates described in Chapter 2 is completely classical. In a quantum field theory, a rescaling of coordinates changes the energy scales of the theory, and a renormalization group procedure is needed. We consider the effect of longitudinal rescaling of (3+1)-dimensional quantum electrodynamics (QED) in Sections (7.1-7.5). We study the longitudinal rescaling of (3+1)-dimensional QCD in Sections (7.6-7.9).

In QED, the Abelian gauge field with Lorentz components A_μ, $\mu = 0,1,2,3$, transforms as $A^{0,3} \to \lambda^{-1}A^{0,3}$ and $A^{1,2} \to A^{1,2}$. The action of the Maxwell gauge field is $S_G = -\frac{1}{4g^2}\int d^4x F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, indices are raised with the usual Minkowski metric and g is the bare electric charge. Under longitudinal rescaling, the gauge action becomes

$$S_G \longrightarrow \frac{1}{4g^2}\int d^4x \left(F_{01}^2 + F_{02}^2 - F_{13}^2 - F_{23}^2 + \lambda^{-2}F_{03}^2 - \lambda^2 F_{12}^2\right). \tag{7.1.1}$$

The massless Dirac action transforms as

$$S_{\text{Dirac}} \longrightarrow i\int d^4x\, \bar{\psi}\left[\lambda^{-1}\gamma^0 D_0 + \lambda^{-1}\gamma^3 D_3 + \gamma^1 D_1 + \gamma^2 D_2\right]\psi,$$

where $D_\mu\psi = \partial_\mu\psi + iA_\mu\psi$ is the covariant derivative of ψ. If we make an additional rescaling of the spinor field $\psi \to \lambda^{\frac{1}{2}}\psi$ and $\bar{\psi} \to \lambda^{\frac{1}{2}}\bar{\psi}$, we obtain

$$S_{\text{Dirac}} \longrightarrow i\int d^4x\, \bar{\psi}\left[\gamma^0 D_0 + \gamma^3 D_3 + \lambda\gamma^1 D_1 + \lambda\gamma^2 D_2\right]\psi. \tag{7.1.2}$$

In the quantum theory, anomalous powers of λ will appear in the rescaled action (7.1.1) (7.1.2).

In the quantum theory, classical rescalings are no longer possible. The best-known is example is the effect of a dilatation on a classically conformal invariant field theory. Quantum corrections violate this classical symmetry.

One can imagine cutting off the quantum field theory in the ultraviolet by a cubic lattice, with lattice spacing a. Rescaling changes the lattice spacing of longitudinal coordinates to λa, but does not change the lattice spacing of transverse coordinates, making the cutoff anisotropic. We therefore use a two-step process, where we first integrate out high-longitudinal momentum degrees of freedom, then restore isotropy with longitudinal rescaling. Instead of a lattice, we use a sharp-momentum cutoff and Wilson's renormalization procedure, to integrate out high-momentum modes [56]. This was done in reference [57] for pure

Yang-Mills theory.

In the next section we review basic Wilsonian renormalization. Then we examine the renormalization of QED first with a spherical momentum cutoff and then with aspherical cutoffs, which treat longitudinal momenta and transverse momenta differently. We then find the quantum corrections to the QED action.

7.2 Wilsonian renormalization

We Wick rotate to obtain the standard Euclidean metric, so that the action is

$$S = \int d^4x (\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \not{D} \psi).$$

where raising and lowering of indices is done with the Euclidean metric and where the slash on a vector quantity J_μ is $\not{J} = \gamma^\mu J_\mu$, where γ^μ are the Euclidean Dirac matrices.

We choose cutoffs Λ and $\tilde{\Lambda}$ to be real positive numbers with units of cm^{-1} and b and $\tilde{b}$ to be two dimensionless real numbers, such that $b \geq 1$ and $\tilde{b} \geq 1$. These quantities satisfy $\Lambda > \tilde{\Lambda}$ and that $\Lambda^2/b \geq \tilde{\Lambda}^2/\tilde{b}$. We introduce the ellipsoid in momentum space $\mathbb{P}$, which is the set of points p, such that $bp_L^2 + p_\perp^2 < \Lambda^2$. We define the smaller ellipsoid $\tilde{\mathbb{P}}$ to be the set of points p, such that $\tilde{b}p_L^2 + p_\perp^2 < \tilde{\Lambda}^2$. Finally, we define $\mathbb{S}$ to be the shell between the two ellipsoidal surfaces $\mathbb{S} = \mathbb{P} - \tilde{\mathbb{P}}$.

We split our fields into "slow" and "fast" pieces:

$$\psi(x) = \tilde{\psi}(x) + \varphi(x),\ \bar{\psi}(x) = \tilde{\bar{\psi}}(x) + \bar{\varphi}(x),\ A_\mu(x) = \tilde{A}_\mu(x) + a_\mu(x), \tag{7.2.1}$$

where the Fourier components of $\psi(x)$, $\bar{\psi}(x)$ and $A_\mu(x)$ vanish outside the ellipsoid $\mathbb{P}$, the

Fourier components of the slow fields $\tilde{\psi}(x)$, $\tilde{\bar{\psi}}(x)$ and $\tilde{A}_\mu(x)$ vanish outside the inner ellipsoid $\tilde{\mathbb{P}}$, and the Fourier components of the fast fields $\varphi(x)$, $\bar{\varphi}(x)$ and $a_\mu(x)$ vanish outside of the shell $\mathbb{S}$. Explicitly

$$
\begin{aligned}
\tilde{\psi}(x) &= \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\psi(p)e^{-ip\cdot x},\ \varphi(x) = \int_{\mathbb{S}} \frac{d^4p}{(2\pi)^4}\psi(p)e^{-ip\cdot x},\\
\tilde{\bar{\psi}}(x) &= \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\bar{\psi}(p)e^{ip\cdot x},\ \bar{\varphi}(x) = \int_{\mathbb{S}} \frac{d^4p}{(2\pi)^4}\bar{\psi}(p)e^{ip\cdot x},\\
\tilde{A}_\mu(x) &= \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}A_\mu(p)e^{-ip\cdot x},\ \ a_\mu(x) = \int_{\mathbb{S}} \frac{d^4p}{2(\pi)^4}A_\mu(p)e^{-ip\cdot x}.
\end{aligned}
$$

We denote the field strength of the slow fields by $\tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu$. The functional integral with the ultraviolet cutoff Λ and anisotropy parameter b is

$$
Z = \int_{\mathbb{P}} \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}Ae^{-S}. \tag{7.2.2}
$$

There is no renormalization of a gauge-fixing parameter, because we do not impose a gauge condition on the slow gauge field. We do impose a Feynman gauge condition on the fast gauge field. As we show below, counterterms must be included in the action to maintain gauge invariance. We expect that renormalizability of the the field theory implies that these have the same form at each loop order; we have not proved this, however.

Before integrating over the fast fields, we must expand the action to second order in these fields. This expansion is

$$
S = \tilde{S} + S_0 + S_1 + S_2 + S_3,
$$

where

$$\begin{aligned}
\tilde{S} &= \int d^4x(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + i\tilde{\bar{\psi}}\not{D}\,\tilde{\psi}), \\
S_0 &= \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{1}{2}q^2 a_\mu(-q)a^\mu(q) + i\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\bar{\varphi}(-q)\not{q}\,\varphi(q), \\
S_1 &= -\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\left[\bar{\varphi}(q)\not{A}(p)\varphi(-q-p)\right], \\
S_2 &= -\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\left[\bar{\psi}(p)\not{a}(q)\varphi(-q-p)\right], \\
S_3 &= -\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\left[\bar{\varphi}(-q-p)\not{a}(q)\psi(p)\right].
\end{aligned} \tag{7.2.3}$$

Notice that $S_2^* = S_3$. Henceforth, we drop the tildes on the slow fields, denoting these by ψ, $\bar{\psi}$ and A_μ, but we keep the tilde on the slow action $\tilde{S}$.

The functional integral (7.2.2) may be written as

$$Z = \int_{\tilde{\mathbb{P}}} \mathcal{D}\,\psi\mathcal{D}\bar{\psi}\mathcal{D}A\; e^{-\tilde{S}} \int_{\mathbb{S}} \mathcal{D}\varphi\mathcal{D}\bar{\varphi}\mathcal{D}a\; e^{-S_0}e^{-S_I} \tag{7.2.4}$$

where the integral of the interaction Lagranigian is $S_I = S_1 + S_2 + S_3$. To evaluate 7.6.1, we use the fast-field propagators

$$\begin{aligned}
\langle a_\mu(p)a_\nu(q)\rangle &= \frac{g^2}{q^2}g_{\mu\nu}\delta^{(4)}(p+q)(2\pi)^4, \\
\langle \varphi(p)\bar{\varphi}(q)\rangle &= \frac{-i\not{q}}{q^2}\delta^{(4)}(p+q)(2\pi)^4,
\end{aligned} \tag{7.2.5}$$

where the brackets $\langle\,\rangle$ mean

$$\langle Q\rangle = \left(\int_{\mathbb{S}} \mathcal{D}\varphi\mathcal{D}\bar{\varphi}\mathcal{D}a\; e^{-S_0}\right)^{-1}\int_{\mathbb{S}} \mathcal{D}\varphi\mathcal{D}\bar{\varphi}\mathcal{D}a\; Q\; e^{-S_0}. \tag{7.2.6}$$

for any quantity Q. We will ignore an overall free-energy renormalization from the first factor in (7.2.6). We use the connected-graph expansion

$$\begin{aligned} \langle e^{-S_I} \rangle &= \exp[-\langle S_I \rangle + \frac{1}{2}(\langle S_I^2 \rangle \\ &- \langle S_I \rangle^2) - \frac{1}{3!}(\langle S_I^3 \rangle - 3\langle S_I^2 \rangle \langle S_I \rangle + \langle S_I \rangle^3) + \cdots], \end{aligned} \tag{7.2.7}$$

for the interaction S_I. The terms in the exponent of (7.2.7) are straightforward to evaluate using (7.2.5). We find

$$\langle S_1 \rangle = 0$$

and

$$\frac{1}{2}\langle S_1^2 \rangle = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \, \Pi^{\mu\nu}(p) A_\mu(p) A_\nu(-p),$$

where the polarization tensor $\Pi^{\mu\nu}(p)$ is defined as

$$\Pi^{\mu\nu}(p) = \frac{1}{2} \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \mathrm{Tr} \left[\frac{\not{q}}{q^2} \gamma^\mu \frac{\not{q} + \not{p}}{(q+p)^2} \gamma^\nu \right]. \tag{7.2.8}$$

Similarly

$$\langle S_2 \rangle = \langle S_3 \rangle = 0$$

and

$$\langle S_2 S_3 \rangle = \langle S_3 S_2 \rangle = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \left[\frac{-(\not{p} + \not{q})}{(p+q)^2} \gamma^\mu \frac{g^2}{q^2} \psi(p) \bar{\psi}(-p) \gamma_\mu \right].$$

Thus

$$\frac{1}{2}(\langle S_2 S_3\rangle + \langle S_3 S_2\rangle) = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\Sigma(p)\bar{\psi}(p)\psi(-p),$$

where the self-energy correction $\Sigma(p)$ is

$$\begin{aligned}\Sigma(p) &= g^2\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\left[\gamma^\mu\frac{i(\not{p}+\not{q})}{(p+q)^2}\gamma_\mu\frac{1}{q^2}\right] = \\ &- 2g^2\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\left[\frac{i(\not{p}+\not{q})}{q^2(q+p)^2}\right].\end{aligned} \tag{7.2.9}$$

From the cubic term in (7.2.7), we find

$$\begin{aligned}-\frac{1}{3!}(\langle S_I^3\rangle &- 3\langle S_I^2\rangle\langle S_I\rangle + \langle S_I\rangle^3) = -\frac{1}{3!}\langle S_I^3\rangle = -\langle S_1 S_2 S_3\rangle \\ &= \int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\int_{\tilde{\mathbb{P}}}\frac{d^4q}{(2\pi)^4}\bar{\psi}(p)\Gamma^\mu(p,q)A_\mu(q)\psi(-p-q),\end{aligned}$$

where the vertex correction $\Gamma^\mu(p,q)$ is

$$\Gamma^\mu(p,q) = 2g^2\int_{\mathbb{S}}\frac{d^4k}{(2\pi)^4}\frac{\not{k}\gamma^\mu(\not{k}+\not{q})}{(k-p)^2(k+q)^2k^2}. \tag{7.2.10}$$

7.3 Spherical momentum cutoffs

The cutoffs of our theory become isotropic if $b, \tilde{b} = 1$. Then the region $\mathbb{P}$ is a sphere in momentum space, whose elements p_μ satisfy $p^2 < \Lambda^2$. The region $\tilde{\mathbb{P}}$ is also a sphere, whose elements q_μ satisfy $q^2 < \tilde{\Lambda}^2$. The region $\mathbb{S}$ is a spherical shell, $\mathbb{S} = \mathbb{P} - \tilde{\mathbb{P}}$.

The polarization tensor (7.2.8) may be written

$$\Pi^{\mu\nu}(p) = tr\left[\frac{1}{2}\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{q_\alpha(q_\beta+p_\beta)}{q^2(q+p)^2}\right].$$

We expand the integrand in powers of p to second order, to find

$$\begin{gathered}\Pi^{\mu\nu}(p) = tr\left\{\tfrac{1}{2}\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta \tfrac{2\pi^2}{(2\pi)^4}\left[\int dq \tfrac{q^3\delta_{\alpha\beta}}{4q^2} - \int dq \tfrac{q^3p^2\delta_{\alpha\beta}}{4q^4}\right.\right.\\ \left.\left. -\int dq \tfrac{q^3 p_\beta p^\gamma \delta_{\alpha\gamma}}{2q^4} + \int dq \tfrac{q^3}{6q^4}p^\gamma p^\delta(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\delta}\delta_{\gamma\beta}+\delta_{\alpha\gamma}\delta_{\beta\delta})\right]\right\}. \qquad (7.3.1)\end{gathered}$$

To obtain (7.3.1), we have used

$$\int_{\mathbb{S}} d^4q\, q_\alpha q_\beta = 2\pi^2\int_{\tilde{\Lambda}}^{\Lambda} dq\, q^3\frac{\delta_{\alpha\beta}q^2}{4}$$

and

$$\int_{\mathbb{S}} d^4q\, q_\alpha q_\beta q_\gamma q_\delta = \frac{1}{24}\int_{\mathbb{S}} d^4q\, q^4(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\delta}\delta_{\gamma\beta}+\delta_{\alpha\gamma}\delta_{\beta\delta}),$$

which follow from $\mathcal{O}(4)$ symmetry. Thus the polarization tensor is

$$\begin{aligned}\Pi^{\mu\nu}(p) &= tr\left\{\frac{1}{2}\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\frac{1}{8\pi^2}\left[\frac{\delta_{\alpha\beta}}{8}(\Lambda^2-\tilde{\Lambda}^2)\right.\right.\\ &- \left.\left.\frac{1}{12}(\delta_{\alpha\beta}p^2+2p_\alpha p_\beta)\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right)\right]\right\}. \qquad (7.3.2)\end{aligned}$$

We must remove non-gauge-invariant terms, namely those quadratic in the cutoffs, with counterterms. The remaining logarithmically-divergent part of (7.3.2) is

$$\begin{aligned}\hat{\Pi}^{\mu\nu}(p) &= \Pi_{\mu\nu}(p) - \Pi_{\mu\nu}(0)\\ &= tr\left[\frac{1}{2}\gamma^{\mu}\gamma^{\alpha}\gamma^{\nu}\gamma^{\beta}\frac{1}{8\pi^2}\left(\frac{-1}{12}\right)(\delta_{\alpha\beta}p^2 + 2p_\alpha p_\beta)\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right)\right]\\ &= \frac{e^2}{12\pi^2}(g^{\mu\nu}p^2 - p^\mu p^\nu)\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right).\end{aligned}$$

This gauge-invariant contribution satisfies $p^\mu\hat{\Pi}_{\mu\nu}(p) = 0$. The contribution to the action associated with this term is

$$\frac{1}{2}\langle S_1^2\rangle = \int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\frac{1}{12\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right)(g^{\mu\nu}p^2 - p^\mu p^\nu)A_\mu(-p)A_\nu(p). \tag{7.3.3}$$

Equation (7.3.3) gives the effective coupling $\tilde{g}$ for the theory with cutoff $\tilde{\Lambda}$:

$$\frac{1}{4\tilde{g}^2} = \frac{1}{4g^2} + \frac{1}{12\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right).$$

The self-energy correction (7.6.7) is

$$\Sigma(p) = -2g^2\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{(\not{q} + \not{p})}{q^2(q+p)^2}.$$

We expand the integrand of $\Sigma(p)$ in powers of p, which gives

$$\Sigma(p) = -2g^2\frac{\gamma^\alpha}{8\pi^2}\int dq\left[\frac{-p_\alpha}{2q} + \frac{p_\alpha}{q}\right] = -\frac{g^2\not{p}}{8\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right). \tag{7.3.4}$$

The vertex correction (7.6.8) is

$$\Gamma^{\mu}(p,q) = 2g^2\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}\int_{\mathbb{S}}\frac{d^4k}{(2\pi)^4}\frac{k_{\alpha}(k_{\beta}+q_{\beta})}{(k-p)^2(k+q)^2k^2}.$$

Expanding the integrand in powers of p,

$$\Gamma^{\mu}(p,q) = 2g^2\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}\int_{\mathbb{S}}\frac{d^4k}{(2\pi)^4}\left[\frac{\delta_{\alpha\beta}}{4k^4} - \frac{\delta_{\alpha\beta}p^2}{4k^6} - \frac{\delta_{\alpha\beta}q^2}{4k^6} + \frac{\delta_{\alpha\gamma}}{2k^6}(q_{\beta}p^{\gamma} - q_{\beta}q^{\gamma}) \right.$$
$$\left. - \frac{\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}}{6k^6}(q^{\gamma}p^{\delta} - q^{\gamma}q^{\delta} - p^{\gamma}p^{\delta})\right].$$

We retain only the divergent part of $\Gamma^{\mu}(p,q)$, namely the first term:

$$\Gamma^{\mu}(p,q) = 2g^2\frac{\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}}{8\pi^2}\frac{\delta_{\alpha\beta}}{4}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) = \frac{-g^2\gamma^{\mu}}{8\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right). \tag{7.3.5}$$

7.4 Ellipsoidal momentum cutoffs

Next we consider the more general ellipsoidal case. The integration over $\mathbb{S}$ is done by changing variables from q_{μ} to two variables u and w, with units of momentum squared, and two angles θ and ϕ. These variables are defined by

$$q_1 = \sqrt{u}\cos\theta,\; q_2 = \sqrt{u}\sin\theta,\; q_3 = \sqrt{w-u}\cos\phi,\; q_0 = \sqrt{w-u}\sin\phi.$$

The integration over these variables is

$$\int_{\mathbb{S}} d^4q \;=\; \frac{1}{4}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \left[\int_0^{\tilde{\Lambda}^2} du \int_{\tilde{b}^{-1}\tilde{\Lambda}^2+(1-\tilde{b}^{-1})u}^{b^{-1}\Lambda^2+(1-b^{-1})u} dw \right.$$
$$\left. + \int_{\tilde{\Lambda}^2}^{\Lambda^2} du \int_u^{b^{-1}\Lambda^2+(1-b^{-1})u} dw \right].$$

We have a $\mathcal{O}(2)\times\mathcal{O}(2)$ symmetry, generated by the translations $\theta \to \theta + d\theta$ and $\phi \to \phi + d\phi$, rather than $\mathcal{O}(4)$ symmetry.

Our three corrections are expressed in terms of the integrals

$$A_{\alpha\beta} = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{q_\alpha q_\beta}{q^4},$$
$$B_{\alpha\beta} = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{q_\alpha q_\beta}{q^6},$$
$$C_{\alpha\beta\gamma\delta} = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{q_\alpha q_\beta q_\gamma q_\delta}{q^8}$$

and

$$D = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{1}{q^4}. \tag{7.4.1}$$

By inspection we write

$$\Pi^{\mu\nu}(p) = tr\left[\tfrac{1}{2}\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\left[A_{\alpha\beta} + 4C_{\alpha\beta\gamma\delta}p^\gamma p^\delta - p^2 B_{\alpha\beta} - 2B_{\alpha\gamma}p_\beta p^\gamma\right]\right],$$
$$\Sigma(p) = -2g^2\gamma^\alpha\left[-2B_{\alpha\gamma}p^\gamma + p_\alpha D\right]$$

and

$$\Gamma^\mu = 2g^2\gamma^\alpha\gamma^\mu\gamma^\beta B_{\alpha\beta}.$$

We use C and D to denote Lorentz indices taking only the values 1 and 2. We use Ω and Ξ to denote Lorentz indices taking only the values 3 and 0. The integration is straightforward, though tedious. We present only the results:

$$
\begin{aligned}
A_{CD} &= \frac{\delta_{CD}}{32\pi^2}\Lambda^2\left[1+\frac{b}{(b-1)^2}(1-b+\ln b)\right] \\
&\quad -\frac{\delta_{CD}}{32\pi^2}\tilde{\Lambda}^2\left[1+\frac{\tilde{b}}{(\tilde{b}-1)^2}(1-\tilde{b}+\ln\tilde{b})\right], \\
A_{\Omega\Xi} &= \frac{\delta_{\Omega\Xi}}{32\pi^2}\left[\Lambda^2\left(\frac{1}{b-1}-\frac{\ln b}{(b-1)^2}\right)-\tilde{\Lambda}^2\left(\frac{1}{b-1}-\frac{\ln\tilde{b}}{(\tilde{b}-1)^2}\right)\right], \\
A_{C\Omega} &= 0, \\
B_{CD} &= \frac{\delta_{CD}}{32\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right)-\frac{\delta_{CD}}{64\pi^2}\left[\frac{b^2\ln b}{(b-1)^2}-\frac{b}{b-1}\right] \\
&\quad +\frac{\delta_{CD}}{64\pi^2}\left[\frac{\tilde{b}^2\ln\tilde{b}}{(\tilde{b}-1)^2}-\frac{\tilde{b}}{\tilde{b}-1}\right], \\
B_{\Omega\Xi} &= \frac{\delta_{\Omega\Xi}}{32\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right)-\frac{\delta_{\Omega\Xi}}{64\pi^2}\left[\frac{b(b-2)\ln b}{(b-1)^2}+\frac{b}{b-1}\right] \\
&\quad +\frac{\delta_{\Omega\Xi}}{64\pi^2}\left[\frac{\tilde{b}(\tilde{b}-2)\ln\tilde{b}}{(\tilde{b}-1)^2}+\frac{\tilde{b}}{\tilde{b}-1}\right], \\
B_{C\Omega} &= 0, \\
C_{CCCC} &= \frac{1}{64\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right)-\frac{1}{128\pi^2}\frac{b^3}{(b-1)^3}\left(\ln b-\frac{2(b-1)}{b}+\frac{(b-1)(b+1)}{2b^2}\right) \\
&\quad +\frac{1}{128\pi^2}\frac{\tilde{b}^3}{(\tilde{b}-1)^3}\left(\ln\tilde{b}-\frac{2(\tilde{b}-1)}{\tilde{b}}+\frac{(\tilde{b}-1)(\tilde{b}+1)}{2\tilde{b}}\right), \\
C_{1122} &= \frac{C_{CCCC}}{3}, \\
C_{\Omega\Omega\Omega\Omega} &= \frac{1}{64\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right)-\frac{1}{128\pi^2}\left[\frac{b^3}{(b-1)^3}\left(\ln b-\frac{2(b-1)}{b}+\frac{(b-1)(b+1)}{2b^2}\right)\right. \\
&\quad \left.+\frac{3b\ln b}{b-1}-\frac{3b^2\ln b}{(b-1)^2}+\frac{3b}{b-1}\right] \\
&\quad +\frac{1}{128\pi^2}\left[\frac{\tilde{b}^3}{(\tilde{b}-1)^3}\left(\ln\tilde{b}-\frac{2(\tilde{b}-1)}{\tilde{b}}+\frac{(\tilde{b}-1)(\tilde{b}+1)}{2\tilde{b}^2}\right)\right. \\
&\quad \left.+\frac{3\tilde{b}\ln\tilde{b}}{\tilde{b}-1}-\frac{3\tilde{b}^2\ln\tilde{b}}{(\tilde{b}-1)^2}+\frac{3\tilde{b}}{\tilde{b}-1}\right],
\end{aligned}
$$

$$\begin{aligned}
C_{0033} &= \frac{C_{\Omega\Omega\Omega\Omega}}{3}, \\
C_{CC\Omega\Omega} &= \frac{1}{192\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) - \frac{1}{384\pi^2}\left[\frac{-2b^2\ln b}{(b-1)^3} + \frac{b^2\ln b + 2b}{(b-1)^2}\right] \\
&\quad + \frac{1}{384\pi^2}\left[\frac{-2\tilde{b}^2\ln\tilde{b}}{(\tilde{b}-1)^3} + \frac{\tilde{b}^2\ln\tilde{b} + 2\tilde{b}}{(\tilde{b}-1)^2}\right], \\
D &= \frac{1}{8}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) - \frac{1}{16\pi^2}\left[\frac{b\ln b}{b-1} - \frac{\tilde{b}\ln\tilde{b}}{\tilde{b}-1}\right].
\end{aligned} \tag{7.4.2}$$

Setting $b = \tilde{b}$ in (7.4.2), we recover the results from the spherical integration done in Section III. We simplify by setting $b = 1$ and $\tilde{b} \approx 1$, using the expansion $\tilde{b} = 1 + \ln\tilde{b} + \frac{\ln^2\tilde{b}}{2} + \frac{\ln^3\tilde{b}}{3!} + \cdots$ and $\ln\tilde{b} = \ln\tilde{b} - \frac{\ln^2\tilde{b}}{2} + \frac{\ln^3\tilde{b}}{3} - \frac{\ln^4\tilde{b}}{4} + \cdots$, dropping terms of second order in $\ln\tilde{b}$.

The vertex correction is

$$\begin{aligned}
&\Gamma^\mu(p,q) = 2g^2\gamma^\alpha\gamma^\mu\gamma^\beta B_{\alpha\beta} \\
&= 2g^2\left(\frac{-\gamma^\mu}{16\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) - \frac{\gamma^\mu}{16\pi^2}\ln\tilde{b} + \frac{g^{C\mu}\gamma_C}{32\pi^2}\frac{5}{6}\ln\tilde{b} + \frac{g^{\Omega\mu}\gamma_\Omega}{32\pi^2}\frac{7}{6}\ln\tilde{b}\right).
\end{aligned} \tag{7.4.3}$$

The self-energy correction is

$$\begin{aligned}
\Sigma(p) &= -2g^2\gamma^\alpha[-2B_{\alpha\beta}p^\beta + p_\alpha D] \\
&= -2g^2\left[\frac{\gamma^\mu p_\mu}{16\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) + \frac{1}{32\pi^2}\ln\tilde{b}\frac{\gamma^C p_C}{6} - \frac{1}{32\pi^2}\ln\tilde{b}\frac{\gamma^\Omega p_\Omega}{6}\right].
\end{aligned} \tag{7.4.4}$$

In the next section, we show how these affect the effective action.

The most general gauge-field action which is quadratic in A_μ, is $\mathcal{O}(2)\times\mathcal{O}(2)$ invariant and gauge invariant, to leading order is

$$S_{\text{quadratic}} = \int_{\mathbb{P}} \frac{d^4p}{(2\pi)^4} A(-p)^T\left[a_1M_1(p) + a_2M_2(p) + a_3M_3(p)\right]A(p),$$

where

$$M_1(p) = \begin{pmatrix} p_2^2 & -p_1p_2 & 0 & 0 \\ -p_1p_2 & p_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_2(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_0^2 & -p_3p_0 \\ 0 & 0 & -p_3p_0 & p_3^2 \end{pmatrix},$$

$$M_3(p) = \begin{pmatrix} p_L^2 & 0 & -p_1p_3 & -p_1p_0 \\ 0 & p_L^2 & -p_2p_3 & -p_2p_0 \\ -p_1p_3 & -p_2p_3 & p_\perp^2 & 0 \\ -p_1p_0 & -p_2p_0 & 0 & p_\perp^2 \end{pmatrix}, \tag{7.4.5}$$

and a_1, a_2 and a_3 are real numbers. Any part of the polarization tensor that cannot be expressed in terms of these matrices (i.e. $\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} A_\mu(-p)\Pi_{\mu\nu}(p)A_\nu(p) - S_{\text{quadratic}}$) must be removed with counterterms. After some work we find

$$\begin{aligned} \Pi^{\mu\nu}(p) &= \operatorname{tr}\left[\frac{1}{2}\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\left[A_{\alpha\beta} + 4C_{\alpha\beta\gamma\delta}p^\gamma p^\delta - p^2B_{\alpha\beta} - 2B_{\alpha\gamma}p_\beta p^\gamma\right]\right] \\ &= \frac{1}{12\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right)(p^2\mathbf{1} - pp^T)^{\mu\nu} + \frac{5\ln\tilde{b}}{48\pi^2}(p^2\mathbf{1} - pp^T)^{\mu\nu} \\ &\quad + \frac{\ln\tilde{b}}{128\pi^2}\left[\frac{8}{9}M_3 + \frac{40}{9}M_2 - \frac{104}{9}M_1\right. \\ &\quad \left. + \frac{8}{3}\begin{pmatrix} \left(\frac{17}{6}p_\perp^2 + \frac{4}{3}p_L^2\right)\mathbf{1}_{2\times 2} & 0 \\ 0 & -\left(\frac{7}{6}p_L^2 + \frac{14}{3}p_\perp^2\right)\mathbf{1}_{2\times 2} \end{pmatrix}\right]^{\mu\nu}. \end{aligned} \tag{7.4.6}$$

This determines a_1, a_2 and a_3, so that

$$\begin{aligned} S_{\text{diff}} &= \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi^2)} A(-p)^T M_{\text{diff}} A(p) \\ &= \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^2} A(-p)^T \Pi A(p) - S_{\text{quadratic}} \end{aligned} \tag{7.4.7}$$

is maximally non-gauge invariant. The matrix M_{diff} is the last diagonal matrix in (7.6.6). The quantity S_{diff} is proportional to the local counterterms to include in the action. We find

$$\begin{aligned} a_1 &= \frac{1}{12\pi^2} \ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) + \left(\frac{5}{48\pi^2} - \frac{1}{128\pi^2}\frac{104}{9}\right) \ln \tilde{b}, \\ a_2 &= \frac{1}{12\pi^2} \ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) + \left(\frac{5}{48\pi^2} + \frac{1}{128\pi^2}\frac{40}{9}\right) \ln \tilde{b}, \end{aligned}$$

and

$$a_3 = \frac{1}{12\pi^2} \ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) + \left(\frac{5}{48\pi^2} + \frac{1}{128\pi^2}\frac{8}{9}\right) \ln \tilde{b}$$

In the next section, we show how the action changes under renormalization. We then rescale to restore the isotropy.

7.5 The rescaled effective action

We define the effective action S', which contains the effects of integrating out the fast fields, by

$$Z = \int_{\tilde{\mathbb{P}}} \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \, e^{-S'} = \int_{\tilde{\mathbb{P}}} \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \, e^{-\tilde{S}} \int_{\mathbb{S}} \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \mathcal{D}a \, e^{-S_0} e^{-R},$$

where $S' = \int d^4x\,[\mathcal{L}_{\text{Fermion}} + \mathcal{L}_{\text{vertex}} + \mathcal{L}_{\text{gauge}}] = \int d^4x\,[\mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{gauge}}]$. To one loop

$$\mathcal{L}_{\text{Fermion}} = \bar{\psi} i(\partial\!\!\!/ - \Sigma(\partial))\psi,$$
$$\mathcal{L}_{\text{vertex}} = \bar{\psi}(\gamma^\mu - \Gamma^\mu)A_\mu\psi$$

and

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + A_\mu(\sum_{i=1}^{3} a_i M_i^{\mu\nu}(\partial))A_\nu.$$

Explicitly, $\mathcal{L}_{\text{vertex}}$ is

$$\begin{aligned}\mathcal{L}_{\text{vertex}} = \bar{\psi}\Big[\gamma^C\Big(1 + \tfrac{g^2}{8\pi^2}\ln\Big(\tfrac{\Lambda}{\tilde{\Lambda}}\Big) + \tfrac{g^2}{8\pi^2}\ln\tilde{b} - \tfrac{5g^2}{96\pi^2}\ln\tilde{b}\Big)A_C \\ +\gamma^\Omega\Big(1 + \tfrac{g^2}{8\pi^2}\ln\Big(\tfrac{\Lambda}{\tilde{\Lambda}}\Big) + \tfrac{g^2}{8\pi^2}\ln\tilde{b} - \tfrac{7g^2}{96\pi^2}\ln\tilde{b}\Big)A_\Omega\Big]\psi \\ = R\bar{\psi}\Big[\gamma^C A_C + \lambda^{\frac{g^2}{24\pi^2\tilde{R}}}\gamma^\Omega A_\Omega\Big]\psi,\end{aligned}$$

where

$$R = \tilde{R} + \left(\frac{g^2}{8\pi^2} - \frac{5g^2}{96\pi^2}\right)\ln\tilde{b} \approx \tilde{R}\tilde{b}^{\frac{7g^2}{96\pi^2\tilde{R}}} = \tilde{R}\lambda^{-\frac{7g^2}{48\pi^2\tilde{R}}},$$

and

$$\tilde{R} = 1 + \frac{g^2}{8\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right),$$

for small $\ln\tilde{b}$, where we have identified $\tilde{b} = \lambda^{-2}$.

The term $\mathcal{L}_{\text{Fermion}}$, which contains the self-energy correction:

$$\begin{aligned}\mathcal{L}_{\text{Fermion}} = \bar{\psi} i \Big[& \gamma^C \partial_C \left(1 + \tfrac{g^2}{8\pi^2} \ln\left(\tfrac{\Lambda}{\tilde{\Lambda}}\right) + \tfrac{g^2}{8\pi^2} \ln \tilde{b} - \tfrac{5g^2}{96\pi^2} \ln \tilde{b} - \tfrac{g^2}{16\pi^2} \ln \tilde{b}\right) \\ & + \gamma^\Omega \partial_\Omega \left(1 + \tfrac{g^2}{8\pi^2} \ln\left(\tfrac{\Lambda}{\tilde{\Lambda}}\right) + \tfrac{g^2}{8\pi^2} \ln \tilde{b} - \tfrac{5g^2}{96\pi^2} \ln \tilde{b} - \tfrac{g^2}{12\pi^2} \ln \tilde{b}\right)\Big] \psi \\ & = R' \bar{\psi} i \left[\gamma^C \partial_C + \lambda^{\frac{g^2}{24\pi^2 \bar{R}}} \gamma^\Omega \partial_\Omega\right] \psi,\end{aligned}$$

where

$$R' = R \tilde{b}^{-\frac{g^2}{16\pi^2 \bar{R}}} = R \lambda^{\frac{g^2}{8\pi^2 \bar{R}}}.$$

For consistency, we write $\mathcal{L}_{\text{vertex}}$ in terms of R',

$$\mathcal{L}_{\text{vertex}} = R' \lambda^{-\frac{g^2}{8\pi^2 \bar{R}}} \bar{\psi} \left[\gamma^C A_C + \lambda^{\frac{g^2}{24\pi^2 \bar{R}}} \gamma^\Omega A_\Omega\right] \psi.$$

We must rescale the gauge field by

$$\lambda^{\frac{-g^2}{8\pi^2 \bar{R}}} A_\mu \to A_\mu, \tag{7.5.1}$$

to express $\mathcal{L}_{\text{Dirac}} = \mathcal{L}_{\text{Fermion}} + \mathcal{L}_{\text{vertex}}$ in terms of a covariant derivative. This rescaling also affects $\mathcal{L}_{\text{gauge}}$. We now have

$$\mathcal{L}_{\text{Dirac}} = R' \bar{\psi} i \left[\gamma^C D_C + \lambda^{\frac{g^2}{24\pi^2 \bar{R}}} \gamma^\Omega D_\Omega\right] \psi.$$

Rescaling the spinor field by

$$R' \lambda^{-1+\frac{g^2}{24\pi^2 \bar{R}}} \bar{\psi} \psi \to \bar{\psi} \psi,$$

gives us the form

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi} i \left[\lambda^{1-\frac{g^2}{24\pi^2 \tilde{R}}} \gamma^C D_C + \gamma^\Omega D_\Omega \right] \psi. \tag{7.5.2}$$

Including vacuum-polarization corrections, $\mathcal{L}_{\text{gauge}}$ becomes

$$\begin{aligned}\mathcal{L}_{\text{gauge}} = &\left(\tfrac{1}{4g^2} + \tfrac{1}{12\pi^2} \ln\left(\tfrac{\Lambda}{\tilde{\Lambda}}\right) + \tfrac{1}{9\pi^2} \ln \tilde{b} \right) (F_{01}^2 + F_{02}^2 + F_{13}^2 + F_{23}^2) \\ &+ \left(\tfrac{1}{4g^2} + \tfrac{1}{12\pi^2} \ln\left(\tfrac{\Lambda}{\tilde{\Lambda}}\right) + \tfrac{1}{9\pi^2} \ln \tilde{b} + \tfrac{1}{36\pi^2} \ln \tilde{b} \right) F_{03}^2 \\ &+ \left(\tfrac{1}{4g^2} + \tfrac{1}{12\pi^2} \ln\left(\tfrac{\Lambda}{\tilde{\Lambda}}\right) + \tfrac{1}{9\pi^2} \ln \tilde{b} - \tfrac{7}{72\pi^2} \ln \tilde{b} \right) F_{12}^2.\end{aligned}$$

We introduce the effective coupling g_{eff},

$$\frac{1}{g_{\text{eff}}^2} = \frac{1}{\tilde{g}^2} + \frac{4}{9\pi^2} \ln \tilde{b} \approx \frac{1}{\tilde{g}^2} \tilde{b}^{\frac{4\tilde{g}^2}{9\pi^2}} = \frac{1}{\tilde{g}^2} \lambda^{\frac{-8}{9\pi^2}\tilde{g}^2},$$

where

$$\frac{1}{\tilde{g}^2} = \frac{1}{g^2} + \frac{1}{3\pi^2} \ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right). \tag{7.5.3}$$

Then

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4g_{\text{eff}}^2} \left(F_{01}^2 + F_{02}^2 + F_{13}^2 + F_{23}^2 + \lambda^{-\frac{2}{9\pi^2}\tilde{g}^2} F_{03}^2 + \lambda^{\frac{7}{9\pi^2}\tilde{g}^2} F_{12}^2 \right).$$

We finally rescale the gauge field with the factor from (7.5.1),

$$F_{\mu\nu}^2 \to \lambda^{\frac{g^2}{4\pi^2 \tilde{R}}} F_{\mu\nu}^2,$$

and define a new effective coupling g'_{eff} that absorbs this factor

$$\frac{1}{g'^2_{\text{eff}}} = \frac{1}{\tilde{g}^2}\lambda^{-\frac{8}{9\pi^2}\tilde{g}^2+\frac{g^2}{4\pi^2\tilde{R}}}, \tag{7.5.4}$$

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4g'^2_{\text{eff}}}\left(F_{01}^2 + F_{02}^2 + F_{13}^2 + F_{23}^2 + \lambda^{-\frac{2}{9\pi^2}\tilde{g}^2}F_{03}^2 + \lambda^{\frac{7}{9\pi^2}\tilde{g}^2}F_{12}^2\right). \tag{7.5.5}$$

Our final result, after longitudinal rescaling and Wick-rotating back to Minkowski spacetime is

$$\begin{aligned}\mathcal{L} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{gauge}} = \bar{\psi}i\left[\lambda^{1-\frac{g^2}{12\pi^2\tilde{R}}}\gamma^C D_C + \gamma^\Omega D_\Omega\right]\psi \\ +\tfrac{1}{4g'^2_{\text{eff}}}\left(F_{01}^2 + F_{02}^2 - F_{13}^2 - F_{23}^2 + \lambda^{-2-\frac{2}{9\pi^2}\tilde{g}^2}F_{03}^2 - \lambda^{2+\frac{7}{9\pi^2}\tilde{g}^2}F_{12}^2\right).\end{aligned} \tag{7.5.6}$$

7.6 Wilsonian renormalization of QCD

We now examine the quantum longitudinal rescaling of a non-Abelian gauge theory. We start with the Euclidean action

$$S = S_{\text{gauge}} + S_{\text{Dirac}},$$

where

$$S_{\text{gauge}} = \frac{1}{4g_0^2}\int d^4x F^{\mu\nu}F_{\mu\nu},$$

$$S_{\text{Dirac}} = \int d^4x\, \bar{\psi}\gamma^\mu D_\mu \psi,$$

where now $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$.

We write the gauge field as $A_\mu = A_\mu^a t^a$ where t^a are the $SU(N)$ generators, with $a = 1, \ldots, N^2 - 1$. These generators satisfy $[t^a, t^b] = if^{abc}t^c$, and are normalized by $\operatorname{Tr} t^a t^b = \delta^{ab}$. The field strength is then $F_{\mu\nu} = F^a_{\mu\nu} t^a$, where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + if^{abc} A^b_\mu A^c_\nu$.

We split the fields into "fast" pieces that lie in the region of momentum $\mathbb{S}$ and "slow" pieces with momentum in the region $\tilde{\mathbb{P}}$:

$$\begin{gathered}
\tilde{\psi}(x) = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \psi(p) e^{-ip\cdot x},\; \varphi(x) = \int_{\mathbb{S}} \frac{d^4p}{(2\pi)^4} \psi(p) e^{-ip\cdot x}, \\
\tilde{\bar{\psi}}(x) = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \bar{\psi}(p) e^{ip\cdot x},\; \bar{\varphi}(x) = \int_{\mathbb{S}} \frac{d^4p}{(2\pi)^4} \bar{\psi}(p) e^{ip\cdot x}, \\
\tilde{A}_\mu(x) = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} A_\mu(p) e^{-ip\cdot x},\;\; a_\mu(x) = \int_{\mathbb{S}} \frac{d^4p}{2(\pi)^4} A_\mu(p) e^{-ip\cdot x}.
\end{gathered}$$

The covariant derivative and the field strength become

$$D_\mu = \partial_\mu - i\tilde{A}_\mu - ia_\mu = \tilde{D}_\mu - ia_\mu,$$

and

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} + [\tilde{D}_\mu, a_\nu] - [\tilde{D}_\nu, a_\mu] - i[a_\mu, a_\nu],$$

respectively, where $\tilde{F}_{\mu\nu} = i[\tilde{D}_\mu, \tilde{D}_\nu]$. The pure gauge action is

$$\begin{aligned} S_{\text{gauge}} &= \int d^4x \frac{1}{4g_0^2} \Big(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - 4[\tilde{D}_\mu, \tilde{F}^{\mu\nu}]a_\nu \\ &\quad + \Big([\tilde{D}_\mu, a_\nu] - [\tilde{D}_\nu, a_\mu]\Big)\Big([\tilde{D}^\mu, a^\nu] - [\tilde{D}^\nu, a^\mu]\Big) - 2i\tilde{F}^{\mu\nu}[a_\mu, a_\nu]\Big), \end{aligned}$$

to quadratic order in a_μ.

To do perturbation theory, we add a gauge-fixing term $\frac{1}{2g_0^2}\int d^4x \mathrm{Tr}\,[\tilde{D}_\mu, a_\mu]^2$ to the action. This reduces the gauge symmetry of the fast fields, and means that we must also introduce Faddeev-Popov ghost fields. The Yang-Mills action becomes

$$S_{\text{gauge}} = \frac{1}{4g_0^2}\int d^4x \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2g_0^2}\int d^4x \Big([\tilde{D}_\mu, a_\nu][\tilde{D}^\mu, a^\nu] - 2i\tilde{F}^{\mu\nu}[a_\mu, a_\nu]\Big).$$

The expansion of the quark-field action into slow and fast components is

$$S_{\text{Dirac}} = \int d^4x \left(\bar{\tilde{\psi}}\not{\partial}\tilde{\psi} + \bar{\varphi}\not{\partial}\varphi + \bar{\varphi}\tilde{\not{A}}\varphi + \bar{\tilde{\psi}}\not{a}\varphi + \bar{\varphi}\not{a}\tilde{\psi}\right).$$

The action is that of free slow and fast fields plus interaction terms

$$S = \tilde{S} + S_0 + S_I + S_{II} + S_1 + S_2 + S_3 + S_{\text{ghost}},$$

where

$$
\begin{aligned}
\tilde{S} &= \frac{1}{4g_0^4}\int d^4x \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \int d^4x \bar{\tilde{\psi}}\partial\!\!\!/\tilde{\psi}, \\
S_0 &= \frac{1}{2g_0^2}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^2} q^2 a^b_\mu(-q)a^{\mu\, b}(q) - i\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\bar{\varphi}(-q) q\!\!\!/ \varphi(q), \\
S_I &= \frac{i}{2g_0^2}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} q^\mu f_{bcd} a^b_\nu(q)\tilde{A}^c_\mu(p)a^{\nu\, d}(-q-p) \\
&+ \frac{1}{2g_0^2}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\int_{\tilde{\mathbb{P}}} \frac{d^4l}{(2\pi)^4} f_{bcd}f_{bfg} a^d_\nu(q)\tilde{A}^c_\mu(p)\tilde{A}^{\mu\, f}(l)a^{\nu\, g}(-q-p-l), \\
S_{II} &= \frac{1}{2g_0^2}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} f_{bcd} a^b_\mu(q)\tilde{F}^{\mu\nu\, c}(p)a^d_\nu(-p-q), \\
S_1 &= \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\bar{\varphi}(p) A\!\!\!/(q)\varphi(-q-p), \\
S_2 &= \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\bar{\tilde{\psi}}(p) a\!\!\!/(q)\varphi(-q-p), \\
S_3 &= S_2^* = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\bar{\varphi}(-q-p) a\!\!\!/(q)\varphi(p)
\end{aligned}
$$

and the ghost action, needed to have the correct measure on the fast field, is

$$
\begin{aligned}
S_{\text{ghost}} &= \frac{i}{g_0^2}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} q^\mu f_{bcd} G^b(q)\tilde{A}^c_\mu(p)H^d(-q-p) \\
&+ \frac{1}{2g_0^2}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\int_{\tilde{\mathbb{P}}} \frac{d^4l}{(2\pi)^4} f_{bcd}f_{bfg} G^d(q)\tilde{A}^c_\mu(p)\tilde{A}^f_\mu(l)H^g(-q-p).
\end{aligned}
$$

The interaction is therefore $S_{\text{int}} = S_I + S_{II} + S_1 + S_2 + S_3 + S_{\text{ghost}}$.

We start with the functional integral

$$
Z = \int_{\tilde{\mathbb{P}}} \mathcal{D}\tilde{\psi}\mathcal{D}\bar{\tilde{\psi}}\mathcal{D}\tilde{A}\, e^{-\tilde{S}}\int_{\mathbb{S}} \mathcal{D}\varphi\mathcal{D}\bar{\varphi}\mathcal{D}a\, e^{-S_0-S_{\text{int}}}, \tag{7.6.1}
$$

and integrate out the fast fields φ, $\bar{\varphi}$ and a, to obtain an effective action, S', defined by

$$e^{-S'} = e^{-\tilde{S}} \int_{\mathbb{S}} \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \mathcal{D}a \, e^{-S_0 - S_{\text{int}}}. \tag{7.6.2}$$

Then the functional integral in terms of slow degrees of freedom only is

$$Z = \int_{\tilde{\mathbb{P}}} \mathcal{D}\tilde{\psi} \mathcal{D}\tilde{\bar{\psi}} \mathcal{D}\tilde{A} e^{-S'}.$$

The Green's functions of the slow fields ψ, $\bar{\psi}$ and $\tilde{A}$, with action S' are unchanged from the same Green's functions in the original theory.

The effective action is given explicitly by $S' = \tilde{S} - \ln\langle e^{-S_{\text{int}}}\rangle$, which we evaluate using the connected-graph expansion, as in Section (7.2). The fast-field propagators are

$$\langle a^b_\mu(q) a^c_\nu(p)\rangle = g_0^2 \delta^{bc} \delta_{\mu\nu} \delta^4(q+p) q^{-2} (2\pi)^4, \quad \langle \varphi(p) \bar{\varphi}(q)\rangle = \frac{i\not{q}}{q^2} \delta^4(p+q)(2\pi)^4.$$

We first consider all the contributions quadratic in the slow gauge field, *i.e.* vacuum polarization. One contribution comes from the interactions S_I and S_{ghost}:

$$\begin{aligned} &\langle S_I\rangle - \tfrac{1}{2}\left(\langle S_I^2\rangle - \langle S_I\rangle^2\right) + \langle S_{\text{ghost}}\rangle - \tfrac{1}{2}\left(\langle S^2_{\text{ghost}}\rangle - \langle S_{\text{ghost}}\rangle^2\right) \\ &\quad = \tfrac{C_G}{4} \int_{\tilde{\mathbb{P}}} \tfrac{d^4p}{(2\pi)^4} \tilde{A}^b_\mu(-p) \tilde{A}^b_\nu(p) P_{\mu\nu}(p), \end{aligned}$$

where

$$P_{\mu\nu}(p) = \int_{\mathbb{S}} \frac{d^4p}{(2\pi)^4} \left[-\frac{q_\mu(p_\nu + 2q_\nu)}{4q^2(q+p)^2} + \frac{\delta_{\mu\nu}}{4q^2} \right],$$

and C_G is the quadratic Casimir operator in the adjoint representation, defined by $C_G \delta^{bh} =$

$f^{bcd}f^{hcd}$. The tensor $P_{\mu\nu}(p)$ is not symmetric under exchange of indices. We define the integral $I_\alpha(p)$ by

$$I_\alpha(p) = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \frac{p_\alpha + 2q_\alpha}{q^2(q+p)^2}$$

and notice that $I_\alpha(p)+I_\alpha(-p) = 0$ (we can see this by changing the sign of q in the integrand). We can then use this to replace the tensor $P_{\mu\nu}(p)$ by the manifestly symmetric tensor $\Pi^1_{\mu\nu}(p)$:

$$\begin{aligned} &\langle S_I\rangle - \tfrac{1}{2}\left(\langle S_I^2\rangle - \langle S_I\rangle^2\right) + \langle S_{\text{ghost}}\rangle - \tfrac{1}{2}\left(\langle S_{\text{ghost}}^2\rangle - \langle S_{\text{ghost}}\rangle^2\right) \\ &\qquad = \textstyle\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \tilde{A}^b_\mu(-p)\tilde{A}^b_\nu(p)\Pi^1_{\mu\nu}(p), \end{aligned} \tag{7.6.3}$$

where

$$\Pi^1_{\mu\nu}(p) = C_G \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\left[-\frac{(p_\mu + 2q_\nu)(p_\nu + 2q_\mu)}{8q^2(p+q)^2} + \frac{\delta_{\mu\nu}}{4q^2}\right].$$

A second contribution to vacuum polarization comes from

$$-\tfrac{1}{2}\left(\langle S_{II}^2\rangle - \langle S_{II}\rangle^2\right) = -\tfrac{C_G}{2}\textstyle\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^2} \tilde{F}^b_{\mu\nu}(-p)\tilde{F}^b_{\mu\nu}(p)\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^2}\frac{1}{q^2(p+q)^2}. \tag{7.6.4}$$

The third and final contribution to vacuum polarization comes from integration over the fast quark field:

$$-\frac{1}{2}\left(\langle S_1^2\rangle - \langle S_1^2\rangle^2\right) = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^2}\tilde{A}^b_\mu(-p)\tilde{A}^b_\nu(p)\Pi^3_{\mu\nu}(p), \tag{7.6.5}$$

where

$$\Pi^3_{\mu\nu}(p) = \frac{N_f}{2}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^2}\mathrm{Tr}\left[\frac{\not{q}}{q^2}\gamma^\mu\frac{\not{q}+\not{p}}{(q+p)^2}\gamma^\nu\right],$$

and N_f is the number of flavors. Combining (7.6.3), (7.6.4) and (7.6.5), we find the vacuum-polarizaton contribution:

$$\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^2}\tilde{A}^b_\mu(-p)\tilde{A}^b_\nu(p)\Pi_{\mu\nu}(p), \tag{7.6.6}$$

where

$$\Pi_{\mu\nu}(p) = \Pi^1_{\mu\nu}(p) + \Pi^2_{\mu\nu}(p) + \Pi^3_{\mu\nu}(p),$$

and

$$\Pi^2_{\mu\nu}(p) = (p^2\delta_{\mu\nu} - p_\mu p_\nu)C_G\int_{\mathbb{S}}\frac{-1}{2q^2(p+q)^2}.$$

The quark self-energy contribution, which comes from the interactions S_2 and S_3, is

$$-\frac{1}{2}\left(\langle S_2S_3\rangle + \langle S_3S_2\rangle\right) = \int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^2}\Sigma(p)\tilde{\bar{\psi}}(p)\tilde{\psi}(p), \tag{7.6.7}$$

where

$$\Sigma(p) = 2g_0^2\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\left[\frac{i(\not{p}+\not{q})}{q^2(p+q)^2}\right].$$

The quark-gluon vertex receives a correction from

$$\begin{aligned}
&\tfrac{1}{3!}\left(\langle S_1 S_2 S_3\rangle - \langle S_2 S_3\rangle\langle S_1\rangle - \langle S_3 S_2\rangle\langle S_1\rangle\right)\\
&+\tfrac{1}{3!}\left(\langle S_I S_2 S_3\rangle - \langle S_2 S_3\rangle\langle S_I\rangle - \langle S_3 S_2\rangle\langle S_I\rangle\right)\\
&\quad = \int_{\tilde{\mathbb{P}}} \tfrac{d^4q}{(2\pi)^4} \int_{\tilde{\mathbb{P}}} \tfrac{d^4p}{(2\pi)^4} \tilde{\bar{\psi}}(p)\Gamma^{\mu\,a}(p,q)\tilde{A}^a_\mu(q)\tilde{\psi}(-q-p),
\end{aligned} \tag{7.6.8}$$

where

$$\Gamma^{\mu\,a}(p,q) = -2g_0^2 t^a \int_{\mathbb{S}} \frac{d^4k}{(2\pi)^4} \frac{\not{k}\gamma^\mu(\not{k}+\not{q})}{(k-p)^2(k+q)^2k^2}.$$

7.7 QCD with spherical cutoffs

As we did for QED, we can recover spherical cutoffs by setting $b = \tilde{b} = 1$. We expand $\Pi_{\mu\nu}(p), \Sigma(p)$ and $\Gamma^{\mu\,a}(p,q)$ in powers of the slow momenta, treating momenta in $\tilde{\mathbb{P}}$ as much smaller than momenta in $\mathbb{S}$. This gives

$$\begin{aligned}
\Pi_{\mu\nu}(p) &= \Pi^1_{\mu\nu}(p) + \Pi^2_{\mu\nu}(p) + \Pi^3_{\mu\nu}(p),\\
\Pi^1_{\mu\nu}(p) &= C_G\Big[\tfrac{\delta_{\mu\nu}}{4}E - \tfrac{1}{2}A_{\mu\nu} + \tfrac{p_\mu p_\alpha}{2}B_{\nu\alpha} + \tfrac{p_\nu p_\alpha}{2}B_{\nu\alpha} + \tfrac{p_\nu p_\alpha}{2}B_{\mu\alpha} - \tfrac{p_\mu p_\nu}{8}D\\
&\qquad + \tfrac{p^2}{2}B_{\mu\nu} - 2p_\alpha p_\beta C_{\alpha\beta\mu\nu}\Big],\\
\Pi^2_{\mu\nu}(p) &= -\tfrac{1}{2}(p^2 g_{\mu\nu} - p_\mu p_\nu)C_G D,\\
\Pi^3_{\mu\nu}(p) &= \tfrac{N_f}{2}\mathrm{Tr}\,\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\left[A_{\alpha\beta} + 4C_{\alpha\beta\gamma\delta}p^\gamma p^\delta - p^2 B_{\alpha\beta} - 2B_{\alpha\gamma}p_\beta p^\gamma\right],\\
\Sigma(p) &= 2g_0^2 N_f \gamma^\alpha\left[-2B_{\alpha\beta}p^\beta + p_\alpha D\right],\\
\Gamma^{\mu\,a} &= -2g_0^2 N_f t^a \gamma^\alpha\gamma^\mu\gamma^\beta B_{\alpha\beta},
\end{aligned} \tag{7.7.1}$$

where

$$A_{\alpha\beta} = \int_{\mathbb{S}} \tfrac{d^4q}{(2\pi)^4} \tfrac{q_\alpha q_\beta}{q^4}, \quad B_{\alpha\beta} = \int_{\mathbb{S}} \tfrac{d^4q}{(2\pi)^4} \tfrac{q_\alpha q_\beta}{q^6},$$
$$C_{\alpha\beta\gamma\delta} = \int_{\mathbb{S}} \tfrac{d^4q}{(2\pi)^4} \tfrac{q_\alpha q_\beta q_\gamma q_\delta}{q^8}, \quad D = \int_{\mathbb{S}} \tfrac{d^4q}{(2\pi)^4} \tfrac{1}{q^4}, \quad E = \int_{\mathbb{S}} \tfrac{d^4q}{(2\pi)^4} \tfrac{1}{q^2}. \tag{7.7.2}$$

The integrals (7.7.2) are invariant under $\mathcal{O}(4)$ rotation symmetry. This allows us to write

$$\int_{\mathbb{S}} d^4q\, q_\alpha q_\beta = \frac{\pi^2}{2} \int_{\tilde{\Lambda}}^{\Lambda} dq\, \delta_{\alpha\beta}\, q^2 \,, \tag{7.7.3}$$

and

$$\int_{\mathbb{S}} d^4q\, q_\alpha q_\beta q_\gamma q_\delta = \frac{1}{24} \int_{\mathbb{S}} d^4q\, q^4 (\delta_{\alpha\beta}\delta_{\gamma\delta} + \gamma_{\alpha\delta}\delta_{\gamma\beta} + \delta_{\alpha\gamma}\delta_{\beta\delta}). \tag{7.7.4}$$

Using (7.7.3) and (7.7.4) we solve (7.7.1):

$$\Pi_{\mu\nu}(p) = -\tfrac{11 C_G}{192\pi^2}(p^2\delta_{\mu\nu} - p_\mu p_\nu) \ln \tfrac{\Lambda}{\tilde{\Lambda}} + \tfrac{N_f}{12\pi^2}(p^2\delta_{\mu\nu} - p_\mu p_\nu) \ln \tfrac{\Lambda}{\tilde{\Lambda}}$$
$$+ \tfrac{C_G}{128\pi^2}(\Lambda^2 - \tilde{\Lambda}^2)\delta_{\mu\nu} - \tfrac{N_f}{16\pi^2}(\Lambda^2 - \tilde{\Lambda})\delta_{\mu\nu},$$
$$\Sigma(p) = g_0^2 N_f \tfrac{\gamma^\mu p_\mu}{8\pi^2} \ln \tfrac{\Lambda}{\tilde{\Lambda}},$$
$$\Gamma^{a\mu} = g_0^2 N_f t^a \tfrac{\gamma^\mu}{8\pi^2} \ln \tfrac{\Lambda}{\tilde{\Lambda}}. \tag{7.7.5}$$

The terms in the polarization tensor that are quadratic in the cutoffs produce corrections to the action that break gauge invariance. We can fix this problem by introducing mass counterterms in the action at each scale to cancel these. We keep the gauge invariant part of the polarization tensor, which we call $\hat{\Pi}_{\mu\nu}(p)$, and is defined by

$$\hat{\Pi}_{\mu\nu}(p) = \Pi_{\mu\nu}(p) - \Pi_{\mu\nu}(0).$$

The resulting action for the slow fields has the coupling $\tilde{g}$, given by

$$\frac{1}{4\tilde{g}^2} = \frac{1}{4g_0^2} - \frac{11C_G}{192\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{N_f}{12\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}}, \tag{7.7.6}$$

which is the standard result.

7.8 QCD with ellipsoidal cutoffs

In this section, we generalize to the case where the region $\mathbb{S}$ is an ellipsoidal shell. We have already found the results for the integrals $A_{\alpha\beta}$, $B_{\alpha\beta}$, $C_{\alpha\beta\gamma\delta}$, and D in Eq. (7.4.2). The result for the remaining integral is

$$E = \frac{1}{16\pi^2}\left(\frac{\Lambda^2\ln b}{b-1} - \frac{\tilde{\Lambda}^2\ln\tilde{b}}{\tilde{b}-1}\right).$$

We take $b = 1$ and $\tilde{b} \approx 1$. We expand $\tilde{b} = 1 + \ln\tilde{b} + \frac{\ln^2\tilde{b}}{2!} + \cdots$ and $\ln\tilde{b} = \ln\tilde{b} - \frac{\ln^2\tilde{b}}{2} + \cdots$, keeping only the first-order terms in $\ln\tilde{b}$.

The self-energy correction is

$$\Sigma(p) = 2g_0^2\left[\frac{\gamma^\mu p_\mu}{16\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{1}{32\pi^2}\ln\tilde{b}\frac{\gamma^C p_C}{6} - \frac{1}{32\pi^2}\ln\tilde{b}\frac{\gamma^\Omega p_\Omega}{6}\right]. \tag{7.8.1}$$

The vertex correction is

$$\Gamma^{\mu\,a} = -2g_0^2 t^a \left[\frac{-\gamma^\mu}{16\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{\gamma^\mu}{16\pi^2}\ln\tilde{b} + \frac{g^{C\mu}\gamma_C}{32\pi^2}\frac{5}{6}\ln\tilde{b} + \frac{g^{\Omega\mu}\gamma_\Omega}{32\pi^2}\frac{7}{6}\ln\tilde{b}\right]. \tag{7.8.2}$$

The general form of the quadratic part of the renormalized gauge field action, which is invariant under $\mathcal{O}(2)\times\mathcal{O}(2)$ and gauge symmetry is

$$S_{\text{quadratic}} = \int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}A(-p)^T[a_1M_1(p) + a_2M_2(p) + a_3M_3(p)]A(p),$$

where the matrices $M_{1,2,3}(p)$ are defined in Eq. (7.4.5), and the coefficients a_1, a_2 and a_3 are real numbers. We extract these coefficients from the polarization tensor. Any part that cannot be expressed in terms of (7.4.5) (i.e. $S_{\text{diff}} = \int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}A_\mu(-p)\Pi_{\mu\nu}(p)A_\nu(p) - S_{\text{quadratic}}$) must be removed with counterterms in the action. The coefficients a_i are selected such that S_{diff} is maximally non-gauge invariant. From the polarization tensor in Eq. (7.7.1), we find

$$\begin{aligned}
a_1 &= -\frac{11C_G}{192\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{N_f}{12\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{1}{64\pi^2}\frac{31}{9}C_G\ln\tilde{b} \\
&\quad + \left(\frac{5}{48\pi^2} - \frac{1}{128\pi^2}\frac{104}{9}\right)N_f\ln\tilde{b}, \\
a_2 &= -\frac{11C_G}{192\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{N_f}{12\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{1}{64\pi^2}\frac{67}{9}C_G\ln\tilde{b} + \\
&\quad \left(\frac{5}{48\pi^2} + \frac{1}{128\pi^2}\frac{40}{9}\right)N_f\ln\tilde{b}, \\
a_3 &= -\frac{11C_G}{192\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{N_f}{12\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{1}{64\pi^2}\frac{59}{9}C_G\ln\tilde{b} + \\
&\quad \left(\frac{5}{48\pi^2} + \frac{1}{128\pi^2}\frac{8}{9}\right)\ln\tilde{b},
\end{aligned} \tag{7.8.3}$$

and

$$
\begin{aligned}
& M_{\text{diff}} \\
= \quad & \frac{C_G \ln \tilde{b}}{64\pi^2} \begin{pmatrix} -\frac{1}{12}p_1^2 - \frac{1}{2}p_2^2 + \frac{7}{12}p + L^2 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}p_1^2 - \frac{1}{12}p_2^2 + \frac{7}{12}p_L^2 & 0 & 0 \\ 0 & 0 & \frac{7}{12}p_\perp^2 + \frac{17}{12}p_3^2 + \frac{5}{6}p_0^2 & 0 \\ 0 & 0 & 0 & \frac{7}{12}p_\perp^2 + \frac{5}{6}p_3^2 + \frac{17}{12}p_0^2 \end{pmatrix} \\
+ \quad & \frac{N_f \ln \tilde{b}}{128\pi^2}\frac{8}{3} \begin{pmatrix} \frac{17}{6}p_\perp^2 + \frac{4}{3}p_L^2 & 0 & 0 & 0 \\ 0 & \frac{17}{6}p_\perp^2 + \frac{4}{3}p_L^2 & 0 & 0 \\ 0 & 0 & -\frac{7}{6}p_L^2 - \frac{14}{3}p_\perp^2 & 0 \\ 0 & 0 & 0 & -\frac{7}{6}p_L^2 - \frac{14}{3}p_\perp^2 \end{pmatrix},
\end{aligned}
$$

where M_{diff} is defined as

$$
S_{\text{diff}} = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} A(-p)^T M_{\text{diff}} A(p).
$$

7.9 The QCD renormalized action

We next put together the results of the previous section to obtain the action S', defined in (7.6.2). This action is

$$S' = \int d^4x \mathcal{L}_{\text{quarks}} + \mathcal{L}_{\text{vertex}} + \mathcal{L}_{\text{gauge}} = \int d^4x \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{gauge}},$$

where to one loop,

$$\begin{aligned}
\mathcal{L}_{\text{quarks}} &= \bar{\tilde{\psi}} i(\partial\!\!\!/ + \Sigma(\partial))\tilde{\psi}, \\
\mathcal{L}_{\text{vertex}} &= \bar{\tilde{\psi}}(\gamma^\mu t^a + \Gamma^{\mu\, a})\tilde{A}^a_\mu \tilde{\psi},
\end{aligned}$$

and

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4g_0^2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \tilde{A}_\mu \left[\sum_{i=1}^{3} a_i M_i^{\mu\nu}(\partial)\right] A_\nu. \tag{7.9.1}$$

Substituting (7.8.2) into (7.9.1) yields

$$\begin{aligned}
\mathcal{L}_{\text{vertex}} &= R\bar{\tilde{\psi}}\left[\gamma^C\left(1 + \frac{N_f g_0^2}{8\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{N_f g_0^2}{8\pi^2}\ln\tilde{b} - \frac{N_f 5 g_0^2}{96\pi^2}\ln\tilde{b}\right)A_C \right.\\
&\quad \left. +\gamma^\Omega\left(1 + \frac{N_f g_0^2}{8\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{N_f g_0^2}{8\pi^2}\ln\tilde{b} - \frac{N_f 7 g_0^2}{96\pi^2}\ln\tilde{b}\right)A_\Omega\right]\tilde{\psi} \\
&= R\bar{\tilde{\psi}}\left(\gamma^C A_C + \lambda^{\frac{N_f g_0^2}{24\pi^2 R}}\gamma^\Omega A_\Omega\right)\tilde{\psi},
\end{aligned}$$

where

$$R = \tilde{R} + N_f\left(\tfrac{g_0^2}{8\pi^2} - \tfrac{5g_0^2}{96\pi^2}\right)\ln\tilde{b} \approx \tilde{R}\tilde{b}^{\frac{7N_f g_0^2}{96\pi^2\tilde{R}}} = \tilde{R}\lambda^{-\frac{7N_f g_0^2}{48\pi^2\tilde{R}}},$$
$$\tilde{R} = 1 + \tfrac{N_f}{8\pi^2}g_0^2\ln\tfrac{\Lambda}{\tilde{\Lambda}}, \tag{7.9.2}$$

and where we have identified $\tilde{b} = \lambda^{-2}$ and dropped terms of order $(\ln\tilde{b})^2$.

We substitute (7.8.1) into (7.9.1) to find

$$\begin{aligned}\mathcal{L}_{\text{quarks}} &= \bar{\tilde{\tilde{\psi}}}\left[\gamma^C\partial_C\left(1 + \frac{N_f g_0^2}{8\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{N_f g_0^2}{8\pi^2}\ln\tilde{b} - \frac{5N_f g_0^2}{96\pi^2}\ln\tilde{b} - \frac{N_f g_0^2}{16\pi^2}\ln\tilde{b}\right)\right. \\ &\quad\left. +\gamma^\Omega\partial_\Omega\left(1 + \frac{N_f g_0^2}{8\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{N_f g_0^3}{8\pi^2}\ln\tilde{b} - \frac{5N_f g_0^2}{96\pi^2}\ln\tilde{b} - \frac{N_f g_0^2}{12\pi^2}\ln\tilde{b}\right)\right]\tilde{\psi} \\ &\approx R'\bar{\tilde{\tilde{\psi}}}\, i\left[\gamma^C\partial_C + \lambda^{\frac{N_f g_0^2}{24\pi^2\tilde{R}}}\gamma^\Omega\partial_\Omega\right]\tilde{\psi},\end{aligned}$$

where

$$R' = R\lambda^{\frac{N_f g_0^2}{8\pi^2\tilde{R}}}. \tag{7.9.3}$$

To make the effective action manifestly gauge invariant, we need to $\mathcal{L}_{\text{Dirac}} = \mathcal{L}_{\text{quarks}} + \mathcal{L}_{\text{vertex}}$ in terms of covariant derivatives. We accomplish this by redefining

$$\lambda^{-\frac{N_f g_0^2}{8\pi^2\tilde{R}}}\tilde{A}_\mu \to \tilde{A}_\mu, \quad R'\lambda^{-1+\frac{N_f g_0^2}{24\pi^2\tilde{R}}}\bar{\tilde{\tilde{\psi}}}\tilde{\psi} \to \bar{\psi}\psi, \tag{7.9.4}$$

so that

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}\, i\left(\lambda^{1-\frac{N_f g_0^2}{24\pi^2\tilde{R}}}\gamma^C D_C + \gamma^\Omega D_\Omega\right)\psi.$$

The factor absorbed by the gauge field in the rescaling (7.9.4) modifies the pure gauge action. We notice that this factor depends on the effective coupling $\frac{g_0}{\tilde{R}}$ instead of the coupling $\tilde{g}$ from (7.7.6). This is because the factor from (7.9.4) arises from the quark self energy and the vertex corrections, instead of the vacuum polariztion.

Substituting (7.8.3) into (7.9.1) gives us

$$\begin{aligned}
\mathcal{L}_{\text{gauge}} &= \frac{1}{4}\left(\frac{1}{g_0^2} - \frac{11}{48\pi^2}C_G \ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{1}{12\pi^2}N_f \ln\frac{\Lambda}{\tilde{\Lambda}}\right. \\
&\quad \left. - \frac{1}{64\pi^2}\frac{59}{9}C_G \ln\tilde{b} + \frac{1}{9\pi^2}N_f \ln\tilde{b}\right)(\tilde{F}_{01}^2 + \tilde{F}_{02}^2 + \tilde{F}_{13}^2 + \tilde{F}_{23}^2) \\
&\quad + \frac{1}{4}\left(\frac{1}{g_0^2} - \frac{11}{48\pi^2}C_G \ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{1}{12\pi^2}N_f \ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{1}{64\pi^2}\frac{31}{9}C_G \ln\tilde{b}\right. \\
&\quad \left. + \frac{1}{9\pi^2}N_f \ln\tilde{b} - \frac{7}{72\pi}N_f \ln\tilde{b}\right)\tilde{F}_{12}^2 \\
&\quad \frac{1}{4}\left(\frac{1}{g_0^2} - \frac{11}{48\pi^2}C_G \ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{1}{12\pi^2}N_f \ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{1}{64\pi^2}\frac{67}{9}C_G \ln\tilde{b}\right. \\
&\quad \left. + \frac{1}{9\pi^2}N_f \ln\tilde{b} + \frac{1}{36\pi^2}N_f \ln\tilde{b}\right)\tilde{F}_{03}^2.
\end{aligned}$$

The pure gauge Lagrangian is then

$$\begin{aligned}
\mathcal{L}_{\text{gauge}} &= \frac{1}{4g_{\text{eff}}^2}\left(\tilde{F}_{01}^2 + \tilde{F}_{02}^2 + \tilde{F}_{13}^2 + \tilde{F}_{23}^2 + \tilde{F}_{03}^2 \lambda^{\frac{C_G}{32\pi^2}\frac{8}{9}\tilde{g}^2 - \frac{2N_f}{9\pi^2}\tilde{g}}\right. \\
&\quad \left. + \tilde{F}_{12}^2 \lambda^{-\frac{C_G}{32\pi^2}\frac{28}{9}\tilde{g}^2 + \frac{7N_f}{9\pi^2}\tilde{g}^2}\right),
\end{aligned}$$

where

$$g_{\text{eff}}^2 = \tilde{g}^2 \lambda^{-\frac{C_G}{32\pi^2}\frac{59}{9}\tilde{g}^2 + \frac{8N_f}{9\pi^2}\tilde{g}^2 + \frac{N_f g_0^2}{4\pi^2 \tilde{R}}}, \tag{7.9.5}$$

and

$$\frac{1}{\tilde{g}^2} = \frac{1}{g_0^2} - \frac{11}{48\pi^2} C_G \ln \frac{\Lambda}{\tilde{\Lambda}} + \frac{N_f}{3\pi^2} \ln \frac{\Lambda}{\tilde{\Lambda}}.$$

The last term in the powers of λ in (7.9.5) comes from the redefinitions (7.9.4).

Finally, we longitudinally rescale the slow fields, after removing the tildes, and Wick rotating back to real space. The effective action is given by $S_{\text{eff}} = \int d^4x \, \mathcal{L}_{\text{eff}}$, where

$$\begin{aligned} \mathcal{L}_{\text{eff}} \;\; = \;\; & \frac{1}{4g_{\text{eff}}^2} \Bigg(F_{01}^2 + F_{02}^2 - F_{13}^2 - F_{23}^2 + \lambda^{-2+\frac{C_G}{36\pi^2}\tilde{g}^2 - \frac{2N_f}{9\pi^2}\tilde{g}^2} F_{03}^2 \\ & \qquad -\lambda^{2-\frac{7C_G}{72\pi^2}\tilde{g}^2 + \frac{7N_f}{9\pi^2}\tilde{g}^2} F_{12}^2 \Bigg) \\ & + \bar{\psi}_\alpha \, i \left(\lambda^{1-\frac{N_f g_0^2}{12\pi^2 \tilde{R}}} \gamma^C D_C + \gamma^\Omega D_\Omega \right) \psi_\alpha, \end{aligned} \tag{7.9.6}$$

where the label $\alpha = 1, 2, ..., N_f$ denotes the flavors of quarks.

Chapter 8

Discussion

In this final chapter, we present a brief discussion of the new results found in this thesis, and discuss possible future research problems.

Our first original results are the form factors and correlation functions of the principal chiral sigma model. We were able to find all the exact form factors of the Noether current and energy-momentum tensor operators in the 't Hooft large-N limit. We then calculated the exact two-point function of these operators, using the new results. This model is the only example of a field theory of propagating particles which has been completely solved (in the sense that correlation functions are known) in the planar limit. The only other theories solved in this limit are (0+1)-dimensional matrix models. There is hope that N=4 gauge theories will be completely solved in the 't Hooft limit, but this program is not yet finished [58]. The planar limit is much harder to solve than the large-N limit of iso-vector theories (such as the O(N) sigma model or the SU(N) Chiral-Gross-Neveu model).

At finite values of N, our results are much less ambitious. We are able to find only the two-excitation form factors. In the future it might be possible to calculate form factors with more particles using a more sophisticated method, like the nested off-shell Bethe Ansatz [22].

It would be interesting in the future to study PCSM in a finite volume, with periodic boundary conditions, in the 't Hooft limit. One can try to find the spectrum of energies as a function of the PCSM mass gap and the length of the x^1 direction using the Bethe ansatz, which would be greatly simplified at large N. Some work in this direction has been done for $N = 2$ and $N = 3$ by Kazakov and Leurent [59].

The simplest non-integrable model we examined was (1+1)-dimensional massive Yang-Mills theory. We saw that in the axial gauge, this model reduces to a principal chiral sigma model perturbed with a current-current interaction. The excitations are confined hadron-like states. Using the exact S-matrix of the sigma model we calculated the meson spectrum in the non-relativistic limit. As we discussed in Chapter 5, in the future we would like to calculate relativistic corrections to this spectrum. We can use the exact Noether current form factors to write a Bethe-Salpeter equation. One can compute corrections to the bound state masses in powers of $1/c$, solving the Bethe-Salpeter equation perturbatively.

In Chapter 6, we use the exact results from the PCSM to compute physical quantities in anisotropic (2+1)-dimensional Yang-Mills theory. These calculations have been done before for $N = 2$, using the S-matrix and form factors of the $O(4)$ nonlinear sigma model, in References [1], [2], [3]. In this thesis, we use our new result for the two-excitation form factor to generalize these results for $N > 2$. We computed the first corrections in powers of the rescaling parameter λ, for the string tension of a static quark-antiquark pair. The theory is not 90-degrees rotation invariant, so the string tension is different if the particles are separated in the x^1 or the x^2 direction. We also calculated the low-lying glueball spectrum. This result proved to be very similar to the meson-spectrum of (1+1)-dimensional massive Yang-Mills.

In the future we hope to examine the partition function of (2+1)-Yang-Mills Theory in powers of λ, away from the integrable limit, in the context of form factor perturbation theory [48]. This involves computing matrix elements $\langle \Psi' | H_1 | \Psi \rangle$. This is equivalent to evaluating Noether current correlation functions between the states of the principal chiral model. This is can be done with our form factors from Chapter 4.

The isotropic theory can be examined through the truncated spectrum approach. This was used by R.M. Konik and Y. Adamov to explore the 3-dimensional Ising model as an array of coupled 2-dimensional Ising chains [60]. One can discretize the spectrum of the (1+1)-dimensional models by putting them in a box of finite size. The physical states are then ordered by energy, as $|1\rangle$, $|2\rangle, \ldots, |n\rangle$, with energies $E_1 < E_2 < \cdots < E_n$, respectively, where E_n is the truncation energy.

We can define a transfer-matrix operator that describes how the system evolves in the x^2 direction, as

$$\hat{T}_{x^2-a,\, x^2} = e^{-\frac{1}{2}H_0(x^2-a)-\frac{1}{2}H_0(x^2)-\lambda^2 H_1(x^2,\, x^2-a)}.$$

In the truncated spectrum approach one can build a discrete, $n \times n$ matrix $T_{ij} = \langle i | \hat{T}_{x^2-a,\, x^2} | j \rangle$, using the set of states with energies E_i, $E_j \leq E_n$. The Yang-Mills partition function is

$$Z = \operatorname{Tr} T^{N_2},$$

where N_2 is the total number of sigma models (the size of the x^2 direction). The partition function can be computed by diagonalizing the matrix T_{ij}, which can be done numerically, or perturbatively in powers of λ. One can extract the mass spectrum this way, and examine their dependence on the truncation energy E_n.

In Chapter 7, we explored the quantum effects of longitudinal rescaling in gauge theories. By rescaling some of the longitudinal coordinates, the ultraviolet momentum cutoffs become ellipsoidal, rather than spherical. We devise an anisotropic version of Wilson's renormalization group, which illustrates how the parameters of the theory flow as the momentum cutoffs become anisotropic. We explicitly computed the quantum longitudinally rescaled actions of QED and QCD with massless fermions in 3+1 dimensions.

One obvious future project would be study the anisotropic renormalization group in 2+1 dimensions. We also hope in to future to find a gauge-invariant method of calculating the longitudinally-rescaled action. This may be possible with some version of dimensional regularization, where the number of longitudinal and transverse dimensions can vary independently. The background-field method could be used instead of Wilsonian renormalization with sharp momentum cutoffs. This would eliminate the need for counter terms to maintain gauge invariance.

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