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QR-LIKE ALGORITHMS FOR GENERALIZED SEMISEPARABLE MATRICES

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Abstract. A novel class $\mathcal{C}_n$ of $n \times n$ structured matrices is introduced which includes diagonal and tridiagonal plus rank-one matrices, arrowhead matrices and diagonal plus semiseparable matrices. Algorithms of QR type are devised for the computation of the eigenvalues and eigenvectors of the matrices in this class. The QR iteration applied to an initial matrix $A \in \mathcal{C}_n$ can be implemented using only $O(n)$ arithmetic operations and $O(n)$ storage per step. Numerical experiments confirm the effectiveness and the robustness of our approach.

Key words. Eigenvalues, eigenvectors, QR iteration, weakly semiseparable matrices, companion matrices, polynomial roots

AMS subject classifications. 65F15, 65H17

1. Introduction. Our subject is the efficient computation of the eigendecomposition for a large class of $n \times n$ structured matrices by means of a novel adaptation of the classical QR algorithm. Due to the structure of these matrices, their n^2 entries can be completely described by $O(n)$ parameters.

Because of its robustness, the QR iteration is usually the method of choice for finding the eigendecomposition of a general matrix $A \in \mathbb{C}^{n\times n}$ numerically (see [21, 16, 2] for a general treatment of eigenproblems). Each QR step amounts to computing a QR factorization followed by a matrix multiplication; for a general matrix $A = A_0$, the cost is $O(n^3)$ arithmetic operations (ops for short) per step. Generally the algorithm does not promise to perform much faster for structured matrices A, because the structure is rapidly lost in QR iteration, and, therefore, the cost per step rapidly reaches the worst case estimate. Some remarkable exceptions: A is a Hermitian tridiagonal matrix ($O(n)$ ops per step) or a Hessenberg matrix ($O(n^2)$ ops per step).

Our interest in structured eigenproblems originates from our search for numerically reliable methods for approximating all roots of a high degree polynomial $p(z)$ (see [5]). One may reduce this problem to the computation of the eigenvalues of a matrix whose characteristic polynomial has the same zeros as $p(z)$. We refer to such a matrix as a generalized companion matrix for $p(z)$.

In theory it is possible to obtain a generalized companion matrix A in tridiagonal form based on a variant of the Euclidean algorithm [6, 24] but this reduction suffers from severe numerical instability. Alternatively, an upper Hessenberg matrix having $p(z)$ as its characteristic polynomial can be generated in a very simple and stable way as the associated Frobenius (companion) matrix. In particular, MATLAB[1] employs the QR algorithm for the computation of the eigenvalues of a Frobenius matrix to approximate the zeros of a polynomial given by its coefficients. However, as Cleve

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[1] MATLAB is a registered trademark of The Mathworks, Inc.

Moler has pointed out [19], this method may not be the best possible because

> "it uses order n^2 storage and order n^3 time. An algorithm designed specifically for polynomial roots might use order n storage and n^2 time."

Other generalized companion matrices have been also proposed for devising new polynomial root-finders or for rephrasing the known functional iterations into a matrix setting. In [3] generalized companion matrices of the form $A = T + \boldsymbol{u}\boldsymbol{v}^H$, where $T \in \mathbb{C}^{n\times n}$ is a Hermitian tridiagonal matrix, were considered for dealing with polynomials expressed with respect to an orthogonal polynomial basis satisfying a three-terms recursion. Matrices of the form $A = D + \boldsymbol{u}\boldsymbol{v}^H$, where D is a diagonal matrix, were introduced in [11, 7] and used in [5] to provide a matrix formulation of the Weierstrass (Durand-Kerner) method and in [15] for the design of an $O(n^3)$ eigenvalue algorithm for approximating polynomial roots. Furthermore, a root-finding method was devised in [18] which is based on matrix computations applied to a generalized companion matrix of arrowhead form first studied in [14]. Eigenvalue problems for arrowhead matrices were also studied in [1, 20, 22].

In this paper we present fast QR-based algorithms for the computation of the eigenvalues and eigenvectors of some classes of generalized companion matrices and, hence, *ad fortiori* for finding the roots of a polynomial. QR algorithm does not preserve the properties of these matrix classes, but we have found an appropriate generalization. That is, we determined a nontrivial novel class of structured matrices which we denote $\mathcal{C}_n$ and which has the following desired properties:

a) it includes these classes of generalized companion matrices as its subclasses;
b) QR steps are closed in this new class, that is, they preserve its structure;
c) due to this structure, each QR iteration on a matrix in this class can be performed in linear arithmetic time using linear memory space.

This means that we developed a QR-like numerically robust algorithm which solves the eigenproblem for a generalized companion matrix (and actually for some other interesting and well studied classes of structured matrices) by using linear memory space and quadratic time (if we assume a constant number of QR steps per eigenvalue). This is the main result of our paper.

Let us specify our novel class of $n \times n$ structured matrices, denoted by $\mathcal{C}_n$. A matrix $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ belongs to $\mathcal{C}_n$ if there exist real numbers $d_1, \ldots, d_n$, complex numbers $t_2, \ldots, t_{n-1}$, and four vectors $\boldsymbol{u} = [u_1, \ldots, u_n]^T \in \mathbb{C}^n$, $\boldsymbol{v} = [v_1, \ldots, v_n]^T \in \mathbb{C}^n$, $\boldsymbol{z} = [z_1, \ldots, z_n]^T \in \mathbb{C}^n$ and $\boldsymbol{w} = [w_1, \ldots, w_n]^T \in \mathbb{C}^n$ such that

$$
\left\{
\begin{array}{ll}
a_{i,i} = d_i + z_i\overline{w_i}, & 1 \le i \le n; \\
a_{i,j} = u_i \underline{t^{\times}_{i,j}} \overline{v_j}, & 1 \le j < i,\ 2 \le i \le n; \\
a_{i,j} = \overline{u_j} t^{\times}_{j,i} v_i + z_i\overline{w_j} - \overline{z_j} w_i, & 1 \le i < j,\ 2 \le j \le n,
\end{array}
\right.
\tag{1.1}
$$

where $t^{\times}_{i,j} = t_{i-1} \ldots t_{j+1}$ for $i-1 \ge j+1$ and, otherwise, $t^{\times}_{i,i-1} = 1$. Different choices for the elements defining the structured representation of $A \in \mathcal{C}_n$ lead to distinct classes of generalized companion matrices.

For $\boldsymbol{z} = \boldsymbol{u}$, $\boldsymbol{w} = \boldsymbol{v}$ and $t_i = 1$, $i = 2, \ldots, n-1$, the class $\mathcal{C}_n$ contains the diagonal plus rank-one matrices of the form $A = D + \boldsymbol{u}\boldsymbol{v}^H$ with $D = \mathrm{diag}[d_1, \ldots, d_n] \in \mathbb{R}^{n\times n}$. Under the assumptions $\boldsymbol{w} = w\boldsymbol{e}_n$ and $t_i = 0$, $i = 2, \ldots, n-1$, A reduces to a Hermitian tridiagonal matrix plus the rank-one correction $w\boldsymbol{z}\boldsymbol{e}_n^T$. If $\boldsymbol{w} = \boldsymbol{z}$ and $t_i \ne 0$ for $i = 2, \ldots, n-1$, then A is a Hermitian diagonal-plus-semiseparable (dpss) matrix [17]. Finally, for $v_2 = \ldots = v_n = 0$ and $z_2 = \ldots = z_n = 0$, A turns into an arrowhead matrix. Since for every matrix in $\mathcal{C}_n$ its strictly lower triangular part is a submatrix of a rank-one matrix and its strictly upper triangular part is a submatrix of a rank-three matrix, we find that $\mathcal{C}_n$ is a subset of weakly semiseparable matrices [23], which includes all Hermitian semiseparable ones. Roughly speaking, a matrix A is said to be weakly semiseparable if its strictly upper and strictly lower triangular parts are the submatrices of low rank matrices. Motivated by their cited properties, we refer to the matrices satisfying (1.1) as to generalized semiseparable matrices. The study of computational properties of semiseparable-like matrices has recently received a great interest, especially because they share many features of banded matrices (see [12, 13, 8, 9, 10, 25] and the references therein).

Our main results are the following. First, we show that the QR iteration applied to a generalized semiseparable matrix $A = A_0$ generates a sequence $\{A_k\}$ of matrices such that each A_k is also a generalized semiseparable matrix. Then we prove that for a generalized semiseparable matrix $A \in \mathbb{C}^{n\times n}$ one step of QR iteration can be carried out at the cost of $O(n)$ ops. Hence, the QR algorithm allows one to compute the eigenvalues of a generalized companion matrix of the form (1.1) at a linear cost per step which yields the overall computational cost of $O(n^2)$ ops (assuming a constant number of iterations per eigenvalue).

To our knowledge, the only known result of this kind is the proof by D. Fasino that, if the input is a positive definite dpss matrix, then its diagonal-plus-semiseparable structure is maintained under the QR iteration. Fasino's proof, as communicated to us, relies upon the equivalence between LR iteration and QR iteration for positive definite matrices. Since such an equivalence does not hold in general, our proofs are basically different from Fasino's.

We organize the paper as follows. In Sect. 2 we set up notations. In Sect. 3 we prove the invariance of the generalized semiseparable structure (1.1) under the QR iteration. In Sect. 4 we develop fast algorithms of linear cost per QR step applied to a generalized semiseparable matrix. By combining these results, in Sect. 5 we summarize our QR-based algorithm and then show and discuss the results of extensive numerical experiments. Finally, conclusion and discussion are the subjects in Sect. 6.

2. Notation. We use capital letters for matrices and lower boldface letters for vectors.

For any $n \times n$ matrix B we adopt the MATLAB notation $B[i:j, k:l]$ for the principal submatrix of B with entries having row and column indices in the ranges i through j and k through l, respectively. We also denote by $A = (a_{i,j}) = \mathrm{triu}(B, p)$ the upper triangular portion of B such that $a_{i,j} = b_{i,j}$ for $j - i \geq p$, and $a_{i,j} = 0$ elsewhere. Analogously, the $n \times n$ matrix $A = \mathrm{tril}(B, p)$ is formed by the lower triangular portion of B such that $a_{i,j} = b_{i,j}$ for $j - i \leq p$, and $a_{i,j} = 0$ elsewhere.

Given two sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^m$, then $\{a_i\}_{i=1}^n \odot \{b_i\}_{i=1}^m$ is the sequence obtained by concatenating them, i.e.,

$$\{a_i\}_{i=1}^n \odot \{b_i\}_{i=1}^m = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}.$$

Let $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ be a generalized semiseparable matrix of the form given in (1.1). One has

$$\operatorname{tril}(A,-1)=\begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_2\overline{v_1} & 0 & \ddots & \ddots & \vdots \\ u_3t_2\overline{v_1} & u_3\overline{v_2} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ u_nt_{n-1}\dots t_2\overline{v_1} & u_nt_{n-1}\dots t_3\overline{v_2} & \dots & u_n\overline{v_{n-1}} & 0 \end{bmatrix}, \tag{2.1}$$

where $\overline{v_j}$ denotes the complex conjugate of v_j. Given the elements $u_2,\dots,u_n$, $t_2,\dots,t_{n-1}$ and $\overline{v_1},\dots,\overline{v_{n-1}}$, we denote by $L(\{u_i\}_{i=2}^n, \{\overline{v_i}\}_{i=1}^{n-1}, \{t_i\}_{i=2}^{n-1})$ the lower triangular matrix on the right hand side of (2.1). Moreover, the matrix

$$R(\{\overline{u_i}\}_{i=2}^n, \{v_i\}_{i=1}^{n-1}, \{\overline{t_i}\}_{i=2}^{n-1}) = (L(\{u_i\}_{i=2}^n, \{\overline{v_i}\}_{i=1}^{n-1}, \{t_i\}_{i=2}^{n-1}))^H,$$

is the upper triangular matrix of parameters $\overline{u_2},\dots,\overline{u_n}$, $v_1,\dots,v_{n-1}$ and $\overline{t_2},\dots,\overline{t_{n-1}}$. We also denote by $R'(\cdot)$ the submatrix of $R(\cdot)$ obtained after deletion of its first column and last row. Analogously, $L'(\cdot)$ is the submatrix of $L(\cdot)$ obtained after deletion of its first row and last column. Observe that $L(\{u_i\}_{i=2}^n, \{\overline{v_i}\}_{i=1}^{n-1}, 0)$ is a lower bidiagonal matrix with zero diagonal entries and subdiagonal entries equal to $\eta_i = u_i\overline{v_{i-1}}$, $2\le i\le n$. Such a matrix is denoted by $\mathrm{Bidiag}[\eta_i]$.

Let $\boldsymbol{x}_i=[z_i,w_i]$ and $\boldsymbol{y}_i=[w_i,-z_i]$ for $i=1,\dots,n$. It is easily verified that

$$\operatorname{triu}(A,1)-R(\{\overline{u_i}\}_{i=2}^n, \{v_i\}_{i=1}^{n-1}, \{\overline{t_i}\}_{i=2}^{n-1}) = \begin{bmatrix} 0 & \boldsymbol{x}_1\boldsymbol{y}_2^H & \dots & \boldsymbol{x}_1\boldsymbol{y}_n^H \\ & \ddots & \ddots & \vdots \\ & & \ddots & \boldsymbol{x}_{n-1}\boldsymbol{y}_n^H \\ & & & 0 \end{bmatrix}. \tag{2.2}$$

For given row vectors $\boldsymbol{x}_1,\dots,\boldsymbol{x}_{n-1}$ and $\boldsymbol{y}_2,\dots,\boldsymbol{y}_{n-1}$, $U(\{\boldsymbol{x}_i\}_{i=1}^{n-1}, \{\boldsymbol{y}_i\}_{i=2}^n)$ is the matrix on the right hand side of (2.2). The matrix $U'(\cdot)$ is the submatrix of $U(\cdot)$ obtained after deletion of its first column and last row.

The QR factorization of a matrix can be computed by using Givens rotations. Denote by $\mathcal{G}(\gamma)$ the 2×2 complex Givens rotation of parameter $\gamma\in\mathbb{C}\cup\{\infty\}$ given by

$$\mathcal{G}(\gamma)=\begin{bmatrix} 1 & \gamma \\ -\bar\gamma & 1\end{bmatrix}/\sqrt{1+|\gamma|^2} = \begin{bmatrix} \phi & \psi \\ -\bar\psi & \phi\end{bmatrix} \quad \gamma,\psi\in\mathbb{C},\ \phi\in\mathbb{R},\ |\psi|^2+|\phi|^2=1,$$

and

$$\mathcal{G}(\infty)=\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}.$$

Observe that it is always possible to fix the value of γ in such a way that $\mathcal{G}(\gamma)$ transforms a vector $[a,b]^T\in\mathbb{C}^2$ into a vector of the form $[\rho,0]^T$ with $|\rho|=\|\,[a,b]^T\,\|_2$. If $a\neq 0$, we set $\bar\gamma=b/a$ and otherwise choose $\gamma=\infty$. Then, define the $n\times n$ Givens rotation $\mathcal{G}_{k,k+1}(\gamma)$ of parameter γ in coordinates k and $k+1$ by means of

$$\mathcal{G}_{k,k+1}(\gamma)=I_{k-1}\oplus\mathcal{G}(\gamma)\oplus I_{n-k-1}=\begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & \mathcal{G}(\gamma) & 0 \\ 0 & 0 & I_{n-k-1}\end{bmatrix}.$$

3. Invariance of the generalized semiseparable structure under the QR iteration. In this section we show that the generalized semiseparable structure (1.1) of a matrix $A \in \mathbb{C}^{n\times n}$ is maintained under the QR iteration for the computation of its eigenvalues.

The QR algorithm applied to the matrix $A = A_0$ defines a sequence of similar matrices according to the following rule:

$$\begin{cases} A_s - \sigma_s I_n = Q_s R_s \\ A_{s+1} - \sigma_s I_n = R_s Q_s, \quad s \geq 0, \end{cases} \tag{3.1}$$

where Q_s is unitary, R_s is upper triangular, I_n denotes the identity matrix of order n and σ_s is a parameter called the *shift parameter*. The first equation of (3.1) yields a QR factorization of the matrix $A_s - \sigma_s I_n$. Under quite mild assumptions the matrix A_s tends to an upper triangular or, at least, a block upper triangular form thus yielding information about the eigenvalues of A.

Whenever the matrix R_s in (3.1) is nonsingular, it is easily found that $A_{s+1} = R_s A_s R_s^{-1}$, and this relation allows one to prove that the structure of the lower triangular portion of $A_0 \in \mathcal{C}_n$ is maintained at any step of QR iteration (3.1).

THEOREM 3.1. *Let A_s be the matrix obtained in s steps of QR iteration (3.1) applied to $A = A_0 \in \mathcal{C}_n$. For a given integer $\tilde{s} \geq 0$, assume that $R_0, \ldots, R_{\tilde{s}}$ are nonsingular so that, for $0 \leq s \leq \tilde{s}$, we can write $A_{s+1} = R_s A_s R_s^{-1}$. Then each matrix A_s, $0 \leq s \leq \tilde{s}+1$, satisfies*

$$\mathrm{tril}(A_s, -1) = L(\{u_i^{(s)}\}_{i=2}^{n}, \{\overline{v_i^{(s)}}\}_{i=1}^{n-1}, \{t_i^{(s)}\}_{i=2}^{n-1}), \tag{3.2}$$

for suitable numbers $u_2^{(s)}, \ldots, u_n^{(s)}$, $v_1^{(s)}, \ldots, v_{n-1}^{(s)}$ and $t_2^{(s)}, \ldots, t_{n-1}^{(s)}$.

Proof. The proof is by induction on $s \leq \tilde{s}+1$. The case $s = 0$ follows from $A_0 \in \mathcal{C}_n$. Assume that the strictly lower triangular part of A_s, $s \leq \tilde{s}$, is such that (3.2) holds for suitable $u_2^{(s)}, \ldots, u_n^{(s)}$, $v_1^{(s)}, \ldots, v_{n-1}^{(s)}$ and $t_2^{(s)}, \ldots, t_{n-1}^{(s)}$, and then prove the theorem for $s+1$. Write $R_s = (r_{i,j}^{(s)})$, $R_s^{-1} = W_s = (w_{i,j}^{(s)})$ and $A_{s+1/2} = A_s W_s$. $A_{s+1/2}$ is obtained by linearly combining the columns of A_s. Hence, $\mathrm{tril}(A_{s+1/2}, -1)$ admits the following representation:

$$\mathrm{tril}(A_{s+1/2}, -1) = L(\{u_i^{(s+1/2)}\}_{i=2}^{n}, \{\overline{v_i^{(s+1/2)}}\}_{i=1}^{n-1}, \{t_i^{(s+1/2)}\}_{i=2}^{n-1}),$$

where $u_j^{(s+1/2)} = u_j^{(s)}$, $t_j^{(s+1/2)} = t_j^{(s)}$ and, moreover,

$$\begin{aligned} \overline{v_1^{(s+1/2)}} &= w_{1,1}^{(s)} \overline{v_1^{(s)}}, \\ \overline{v_j^{(s+1/2)}} &= \textstyle\sum_{k=2}^{j} w_{k-1,j}^{(s)} t_j^{(s)} \ldots t_k^{(s)} \overline{v_{k-1}^{(s)}} + w_{j,j}^{(s)} \overline{v_j^{(s)}}, \quad j = 2, \ldots, n-1. \end{aligned} \tag{3.3}$$

Analogously, the rows of $A_{s+1} = R_s A_{s+1/2}$ are linear combinations of the rows of $A_{s+1/2}$. In this way, one deduces that $\mathrm{tril}(A_{s+1}, -1)$ can also be represented in a similar form given by

$$\mathrm{tril}(A_{s+1}, -1) = L(\{u_i^{(s+1)}\}_{i=2}^{n}, \{\overline{v_i^{(s+1)}}\}_{i=1}^{n-1}, \{t_i^{(s+1)}\}_{i=2}^{n-1}),$$

where $v_j^{(s+1)} = v_j^{(s+1/2)}$, $t_j^{(s+1)} = t_j^{(s+1/2)} = t_j^{(s)}$ and

$$\begin{aligned} u_n^{(s+1)} &= r_{n,n}^{(s)} u_n^{(s)}, \\ u_{n-j}^{(s+1)} &= \textstyle\sum_{k=0}^{j-1} r_{n-j,n-k}^{(s)} t_{n-j}^{(s)} \ldots t_{n-k-1}^{(s)} u_{n-k}^{(s)} + r_{n-j,n-j}^{(s)} u_{n-j}^{(s)}, \quad j = 1, \ldots, n-2. \end{aligned} \tag{3.4}$$

□

REMARK 3.2. *By the latter theorem, we can represent the strictly lower triangular part of* A_{s+1} *by replacing* $\boldsymbol{u}^{(s)} = [u_1^{(s)}, \ldots, u_n^{(s)}]^T$ *and* $\boldsymbol{v}^{(s)} = [v_1^{(s)}, \ldots, v_n^{(s)}]$ *with* $\boldsymbol{u}^{(s+1)} = [u_1^{(s+1)}, \ldots, u_n^{(s+1)}]^T$ *and* $\boldsymbol{v}^{(s+1)} = [v_1^{(s+1)}, \ldots, v_n^{(s+1)}]^T$ *given by* (3.4) *and* (3.3), *for* $v_j^{(s+1)} = v_j^{(s+1/2)}$, *respectively, and having unchanged the scalars* $t_2^{(s+1)} = t_2^{(s)}, \ldots, t_{n-1}^{(s+1)} = t_{n-1}^{(s)}$. *Recursively, this defines a representation of* $\mathrm{tril}(A_{s+1}, -1)$ *of the form*

$$\mathrm{tril}(A_{s+1}, -1) = L(\{u_i^{(s+1)}\}_{i=2}^n, \{\overline{v_i^{(s+1)}}\}_{i=1}^{n-1}, \{t_i^{(0)}\}_{i=2}^{n-1}),$$

for suitable $u_2^{(s+1)}, \ldots, u_n^{(s+1)}$ *and* $v_1^{(s+1)}, \ldots, v_{n-1}^{(s+1)}$. *If* $t_i = t_i^{(0)} \neq 0$ *for* $i = 2, \ldots, n-1$, *then* $\mathrm{tril}(A_0, -1)$ *is the strictly lower triangular part of a rank-one matrix and, therefore, the same holds for each matrix* A_s, *i.e.,*

$$\mathrm{tril}(A_s, -1) = L(\{u_i^{(s)}\}_{i=2}^n, \{\overline{v_i^{(s)}}\}_{i=1}^{n-1}, \{t_i^{(0)}\}_{i=2}^{n-1}) = L(\{\hat{u}_i^{(s)}\}_{i=2}^n, \{\overline{\hat{v}_i^{(s)}}\}_{i=1}^{n-1}, \{1\}_{i=2}^{n-1}).$$

One may try to obtain a more robust representation of $\mathrm{tril}(A_s, -1)$ *by varying the parameters* $u_i^{(s)}$, $v_i^{(s)}$ *and* $t_i^{(s)}$.

To complete our description of the structure of the matrices A_s generated by (3.1) for $A_0 \in \mathcal{C}_n$, it remains to describe $\mathrm{triu}(A_s, 0)$. To do this, we first point out the remarkable fact that $B = A - \boldsymbol{z}\boldsymbol{w}^H$ is a Hermitian matrix, for any matrix $A \in \mathcal{C}_n$ given by (1.1). Thus, $A = A_0$ is a Hermitian matrix plus a rank-one correction. Moreover, since for any $s \geq 0$ $A_{s+1} = P_s^H A_0 P_s$, where $P_s = Q_0 \ldots Q_s$ is a unitary matrix, the same property also holds for all the matrices A_s produced by the QR iteration. By combining this observation with Theorem 3.1, we arrive at the following result.

THEOREM 3.3. *Let* A_s, $s = 1, \ldots, \tilde{s}+1$, *be the matrices generated at the first* $\tilde{s}+1$ *iterations by the* QR *scheme (3.1) starting with* $A = A_0 \in \mathcal{C}_n$ *of the form (1.1), where* $R_0, \ldots, R_{\tilde{s}}$ *are assumed to be nonsingular. Then, each* A_s, *with* $0 \leq s \leq \tilde{s}+1$, *belongs to* $\mathcal{C}_n$. *That is, for* $0 \leq s \leq \tilde{s}+1$, *there exist real numbers* $d_1^{(s)}, \ldots, d_n^{(s)}$, *complex numbers* $t_2^{(s)}, \ldots, t_{n-1}^{(s)}$, *and four* n*-vectors* $\boldsymbol{u}^{(s)} = [u_1^{(s)}, \ldots, u_n^{(s)}]^T \in \mathbb{C}^n$, $\boldsymbol{v}^{(s)} = [v_1^{(s)}, \ldots, v_n^{(s)}]^T \in \mathbb{C}^n$, $\boldsymbol{z}^{(s)} = [z_1^{(s)}, \ldots, z_n^{(s)}]^T \in \mathbb{C}^n$ *and* $\boldsymbol{w}^{(s)} = [w_1^{(s)}, \ldots, w_n^{(s)}]^T \in \mathbb{C}^n$ *such that* $A_s = (a_{i,j}^{(s)})$ *admits the following representation:*

$$\begin{cases} a_{i,i}^{(s)} = d_i^{(s)} + z_i^{(s)}\overline{w_i^{(s)}}, & 1 \leq i \leq n; \\ a_{i,j}^{(s)} = u_i^{(s)} t_{i,j}^{(s)\times}\overline{v_j^{(s)}}, & 1 \leq j < i, \ 2 \leq i \leq n; \\ a_{i,j}^{(s)} = \overline{u_j^{(s)} t_{j,i}^{(s)\times}}\, v_i^{(s)} + z_i^{(s)}\overline{w_j^{(s)}} - \overline{z_j^{(s)}} w_i^{(s)}, & 1 \leq i < j, \ 2 \leq j \leq n, \end{cases} \tag{3.5}$$

where $t_{i,j}^{(s)\times} = t_{i-1}^{(s)} \ldots t_{j+1}^{(s)}$ *for* $i-1 \geq j+1$ *and, otherwise,* $t_{i,i-1}^{(s)\times} = 1$.

Proof. For $s = 0$ the claim follows from $A \in \mathcal{C}_n$ by setting $\boldsymbol{u}^{(0)} = \boldsymbol{u}$, $\boldsymbol{v}^{(0)} = \boldsymbol{v}$, $\boldsymbol{z}^{(0)} = \boldsymbol{z}$, $\boldsymbol{w}^{(0)} = \boldsymbol{w}$, $d_i^{(0)} = d_1$, $1 \leq i \leq n$, and $t_i^{(0)} = t_i$, $2 \leq i \leq n-1$. Recall that $A_0 - \boldsymbol{z}^{(0)}\boldsymbol{w}^{(0)H} = B_0$ is a Hermitian matrix. For $s > 0$ the second equality in (3.5) is established in Theorem 3.1. Moreover, since

$$A_{s+1} = P_s^H A_0 P_s = P_s^H (B_0 + \boldsymbol{z}^{(0)}\boldsymbol{w}^{(0)H}) P_s = P_s^H B_0 P_s + \boldsymbol{z}^{(s+1)}\boldsymbol{w}^{(s+1)H}, \quad s \geq 0,$$

we find that, for $s = 0, \ldots, \tilde{s}$, A_{s+1} is a rank-one correction of the Hermitian matrix $B_{s+1} = P_s^H B_0 P_s$. From this, we characterize the diagonal and superdiagonal entries of A_s. We first deduce that

$$a_{i,j}^{(s)} - z_i^{(s)}\overline{w_j^{(s)}} = \overline{a_{j,i}^{(s)}} - \overline{z_j^{(s)}}w_i^{(s)}, \quad 1 \le i,j \le n. \tag{3.6}$$

For $i < j$, we have $a_{j,i}^{(s)} = u_j^{(s)} t_{j,i}^{(s)\times} \overline{v_i^{(s)}}$, $\overline{a_{j,i}^{(s)}} = \overline{u_j^{(s)} t_{j,i}^{(s)\times}} v_i^{(s)}$. Substitute the latter expression into (3.6) and obtain

$$a_{i,j}^{(s)} = \overline{u_j^{(s)} t_{j,i}^{(s)}}^{\times} v_i^{(s)} + z_i^{(s)}\overline{w_j^{(s)}} - \overline{z_j^{(s)}}w_i^{(s)}, \quad i \le j.$$

Otherwise, if $i = j$, then from (3.6) one deduces that the imaginary part of $a_{i,i}^{(s)}$ coincides with that of $z_i^{(s)}\overline{w_i^{(s)}}$, and so we can write

$$a_{i,i}^{(s)} = d_i^{(s)} + z_i^{(s)}\overline{w_i^{(s)}}, \quad 1 \le i \le n,$$

for suitable real numbers $d_1^{(s)}, \dots, d_n^{(s)}$. $\square$

REMARK 3.4. *From the latter theorem we find that there exists a representation of A_{s+1} of the form (3.5) such that*

$$\boldsymbol{z}^{(s+1)} = P_s^H \boldsymbol{z}^{(0)} = Q_s^H \boldsymbol{z}^{(s)}$$

and

$$\boldsymbol{w}^{(s+1)^H} = \boldsymbol{w}^{(0)^H} P_s = \boldsymbol{w}^{(s)^H} Q_s.$$

These equations provide simple rules for updating the vectors $\boldsymbol{z}^{(s)}$ and $\boldsymbol{w}^{(s)}$ at each step of the QR iteration.

Theorem 3.3 means that the matrices A_s generated at the first iterations in the QR scheme (3.1) applied to $A = A_0 \in \mathcal{C}_n$ inherit the generalized semiseparable structure (3.5) of their ancestor A_0. If, for a certain index $\hat{s}$, $R_{\hat{s}}$ and $A_{\hat{s}} - \sigma_{\hat{s}} I_n$ are singular, then $\sigma_{\hat{s}}$ is an eigenvalue of A_0, and a deflation technique should be employed. When working in finite precision arithmetic, deflation is also used if the entries of the matrix $A_{\hat{s}}$ satisfy a suitable stopping criterion. Let $A_{\hat{s}}[1:n-k, 1:n-k] \in \mathbb{C}^{(n-k)\times(n-k)}$ be the leading principal submatrix of $A_{\hat{s}}$ obtained from $A_{\hat{s}}$ by deleting its last k rows and columns. It is easily seen that $A_{\hat{s}}[1:n-k, 1:n-k]$ admits a representation similar to the one provided by Theorem 3.3. Such a representation is found simply by truncating the corresponding representation of the matrix $A_{\hat{s}}$ of larger size. Hence, $A_{\hat{s}}[1:n-k, 1:n-k] \in \mathcal{C}_{n-k}$ and, therefore, all the matrices generated by means of the QR scheme (3.1) applied to $A_0 \in \mathcal{C}_n$ for the computation of its eigenvalues still satisfy (3.5).

4. Efficient implementation of the QR iteration for generalized semiseparable matrices. We have already shown that if $A_0 = (a_{i,j}^{(0)}) \in \mathbb{C}^{n\times n}$ is of the form (3.5) then the matrix A_1 generated by the first step of (3.1) admits a similar representation. In this way, the first step of the QR iterative process (3.1) reduces to the computation of real numbers $d_1^{(1)}, \dots, d_n^{(1)}$, complex numbers $t_2^{(1)}, \dots, t_{n-1}^{(1)}$, and the entries of the vectors $\boldsymbol{u}^{(1)} \in \mathbb{C}^n$, $\boldsymbol{v}^{(1)} \in \mathbb{C}^n$, $\boldsymbol{z}^{(1)} \in \mathbb{C}^n$ and $\boldsymbol{w}^{(1)} \in \mathbb{C}^n$ which define $A_1 = (a_{i,j}^{(1)})$ according to (3.5). To perform this task efficiently, in this section we investigate the structural properties of the QR factorization of $A_0 - \sigma_0 I_n$ with $A_0 \in \mathcal{C}_n$.

For the sake of notational simplicity we assume that $\sigma_0 = 0$. As usual, the unitary matrix Q_0 can be constructed as a product of Givens rotations suitably chosen to annihilate specific entries of A_0. In particular, by exploiting the structure of $\mathrm{tril}(A_0, -1)$, we express Q_0 as the product of $2n-3$ Givens rotations. The following two-step procedure is used to compute a QR factorization $A_0 = Q_0 R_0$ of the matrix $A_0 \in \mathcal{C}_n$:

1) A_0 is reduced to an upper Hessenberg matrix H_0 by:

$$H_0 = \mathcal{G}_{2,3}(\gamma_{n-2}) \dots \mathcal{G}_{n-1,n}(\gamma_1) A_0; \tag{4.1}$$

2) H_0 is transformed into an upper triangular matrix R_0 according to:

$$R_0 = \mathcal{G}_{n-1,n}(\gamma_{2n-3}) \dots \mathcal{G}_{1,2}(\gamma_{n-1}) H_0. \tag{4.2}$$

From (4.1) and (4.2) we find that

$$R_0 = \mathcal{G}_{n-1,n}(\gamma_{2n-3}) \dots \mathcal{G}_{1,2}(\gamma_{n-1}) \mathcal{G}_{2,3}(\gamma_{n-2}) \dots \mathcal{G}_{n-1,n}(\gamma_1) A_0.$$

It follows that

$$Q_0^H = \mathcal{G}_{n-1,n}(\gamma_{2n-3}) \dots \mathcal{G}_{1,2}(\gamma_{n-1}) \mathcal{G}_{2,3}(\gamma_{n-2}) \dots \mathcal{G}_{n-1,n}(\gamma_1) \tag{4.3}$$

is the desired unitary matrix such that $Q_0^H A_0 = R_0$.

Once the upper triangular factor R_0 and the Givens rotations $\mathcal{G}_{j,j+1}(\gamma_k)$ are known, the matrix A_1 can be determined by

$$A_1 = R_0 (\mathcal{G}_{n-1,n}(\gamma_1))^H \dots (\mathcal{G}_{2,3}(\gamma_{n-2}))^H (\mathcal{G}_{1,2}(\gamma_{n-1}))^H \dots (\mathcal{G}_{n-1,n}(\gamma_{2n-3}))^H.$$

The composite scheme for the transition from A_0 to A_1 can be summarized as follows:

$$A_0 \xrightarrow{1} H_0 \xrightarrow{2} R_0 \xrightarrow{3} A_1. \tag{4.4}$$

We prove that both H_0 and R_0 can be described in terms of linear structures, i.e., structures requiring $O(n)$ parameters, and, moreover, these structures can be managed efficiently and robustly. In addition, Q_0 does not need to be formed explicitly but it can be implicitly defined as the product of $2n-3$ Givens rotation matrices $\mathcal{G}_{j,j+1}(\gamma_k)$. Combining these facts enables us to devise a robust algorithm having linear complexity for the scheme (4.4).

The remaining part of this section is divided into three subsections, each of them is devoted to a single step in (4.4).

4.1. Reduction to the Hessenberg form. The upper Hessenberg matrix H_0 is defined by (4.1), where the Givens rotation matrices $\mathcal{G}_{2,3}(\gamma_{n-2}), \dots, \mathcal{G}_{n-1,n}(\gamma_1)$ are chosen to zero the respective entries of $\mathrm{tril}(A_0, -2)$. To do this, the parameters γ_j can be determined as follows. Choose the first element γ_1 to yield

$$\mathcal{G}(\gamma_1) \begin{bmatrix} u_{n-1}^{(0)} \\ u_n^{(0)} t_{n-1}^{(0)} \end{bmatrix} = \begin{bmatrix} \widehat{u}_{n-1}^{(0)} \\ 0 \end{bmatrix}. \tag{4.5}$$

Similarly, choose the successive entries $\gamma_2, \dots, \gamma_{n-2}$ to yield

$$\mathcal{G}(\gamma_j) \begin{bmatrix} u_{n-j}^{(0)} \\ \widehat{u}_{n-j+1}^{(0)} t_{n-j}^{(0)} \end{bmatrix} = \begin{bmatrix} \widehat{u}_{n-j}^{(0)} \\ 0 \end{bmatrix}, \quad 2 \le j \le n-2. \tag{4.6}$$

To describe the effects of pre-multiplying A_0 by $\mathcal{G}_{n-1,n}(\gamma_1), \dots, \mathcal{G}_{2,3}(\gamma_{n-2})$ we first investigate the structure of the unitary matrix

$$\widehat{Q}_0 = \mathcal{G}_{2,3}(\gamma_{n-2}) \dots \mathcal{G}_{n-1,n}(\gamma_1) = \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \widehat{Q} \end{array}\right]. \tag{4.7}$$

It is easy to verify that $\widehat{Q}_0$ is an upper Hessenberg matrix. A more careful examination of the matrix $\mathrm{triu}(\widehat{Q}_0, 0)$ also reveals its semiseparable structure.

THEOREM 4.1. *The matrix $\widehat{Q} \in \mathbb{C}^{(n-1)\times(n-1)}$ in (4.7) admits the following representation:*

$$\widehat{Q} = \begin{bmatrix} \tilde{\phi}\phi_{n-2} & \tilde{\phi}\psi_{n-2}\phi_{n-3} & \dots & \dots & \tilde{\phi}\psi_{n-2}\dots\psi_1\hat{\phi} \\ -\overline{\psi}_{n-2} & \phi_{n-2}\phi_{n-3} & \dots & \dots & \phi_{n-2}\psi_{n-3}\dots\psi_1\hat{\phi} \\ 0 & -\overline{\psi}_{n-3} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\overline{\psi}_1 & \phi_1\hat{\phi} \end{bmatrix}, \tag{4.8}$$

where for the sake of notational simplicity we may write $\tilde{\phi} = \hat{\phi} = 1$.

Proof. The proof is by induction. The matrix

$$\mathcal{G}(\gamma_1) = \mathcal{G}_{n-1,n}(\gamma_1)[n-1:n, n-1:n] \in \mathbb{C}^{2\times 2}$$

satisfies (4.8). Assume that the matrix

$$\mathcal{G} = (\mathcal{G}_{n-k,n-k+1}(\gamma_k)\dots\mathcal{G}_{n-1,n}(\gamma_1))[n-k:n, n-k:n] \in \mathbb{C}^{(k+1)\times(k+1)},$$

is of the form (4.8), namely,

$$\mathcal{G} = \begin{bmatrix} \tilde{\phi}\phi_k & \tilde{\phi}\psi_k\phi_{k-1} & \dots & \dots & \tilde{\phi}\psi_k\dots\psi_1\hat{\phi} \\ -\overline{\psi}_k & \phi_k\phi_{k-1} & \dots & \dots & \phi_k\psi_{k-1}\dots\psi_1\hat{\phi} \\ 0 & -\overline{\psi}_{k-1} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\overline{\psi}_1 & \phi_1\hat{\phi} \end{bmatrix},$$

and prove the theorem for

$$\widehat{\mathcal{G}} = (\mathcal{G}_{n-k-1,n-k}(\gamma_{k+1})\dots\mathcal{G}_{n-1,n}(\gamma_1))[n-k-1:n, n-k-1:n] \in \mathbb{C}^{(k+2)\times(k+2)}.$$

The claim immediately follows from the matrix equation

$$\widehat{\mathcal{G}} = (\mathcal{G}(\gamma_{k+1}) \oplus I_k)\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathcal{G} \end{bmatrix}.$$

∎

Write $\boldsymbol{x}_i^{(0)} = [z_i^{(0)}, w_i^{(0)}]$ and $\boldsymbol{y}_i^{(0)} = [w_i^{(0)}, -z_i^{(0)}]$ for $i = 1, \dots, n$, and

$$A_0 = B_0 + C_0,$$

where $B_0 = (b_{i,j}^{(0)}) \in \mathbb{C}^{n\times n}$ and $C_0 = (c_{i,j}^{(0)}) \in \mathbb{C}^{n\times n}$ are defined by

(4.9)

$$B_0 = L(\{u_i^{(0)}\}_{i=2}^n, \{\overline{v_i^{(0)}}\}_{i=1}^{n-1}, \{t_i^{(0)}\}_{i=2}^{n-1}) + \mathrm{diag}[a_{i,i}^{(0)}] + U(\{\boldsymbol{x}_i^{(0)}\}_{i=1}^{n-1}, \{\boldsymbol{y}_i^{(0)}\}_{i=2}^n),$$

and

$$C_0 = R(\{\overline{u_i^{(0)}}\}_{i=2}^{n}, \{v_i^{(0)}\}_{i=1}^{n-1}, \{\overline{t_i^{(0)}}\}_{i=2}^{n-1}), \tag{4.10}$$

respectively. Now observe that

$$\widehat{Q}_0 A_0 = \widehat{Q}_0 B_0 + \widehat{Q}_0 C_0,$$

The following results specify the structure of $\widehat{Q}_0 B_0$. The proofs are similar to that one of Theorem 4.3 and are omitted here since they involve some tedious calculations.

The process of forming $\widehat{Q}_0 L(\{u_i^{(0)}\}_{i=2}^{n}, \{\overline{v_i^{(0)}}\}_{i=1}^{n-1}, \{t_i^{(0)}\}_{i=2}^{n-1})$ is illustrated for a generic 5×5 matrix as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \times & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 \end{bmatrix} \overset{\mathcal{G}(\gamma_1)}{\rightarrow} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \times & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 \\ 0 & 0 & 0 & \times & 0 \end{bmatrix} \overset{\mathcal{G}(\gamma_2)}{\rightarrow} \tag{4.11}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & 0 \\ 0 & 0 & 0 & \times & 0 \end{bmatrix} \overset{\mathcal{G}(\gamma_3)}{\rightarrow} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & 0 \\ 0 & 0 & 0 & \times & 0 \end{bmatrix}.$$

THEOREM 4.2. *Let* $\zeta_j = \phi_{n-j} \widehat{u}_{j+1}^{(0)} \overline{v_j^{(0)}}$, $1 \le j \le n-1$, $\phi_{n-1} = 1$, *and* $\eta_j = \psi_{n-j} \widehat{u}_{j+1}^{(0)} \overline{v_j^{(0)}}$, $1 \le j \le n-1$, $\psi_{n-1} = 1$, *where the elements* $\widehat{u}_{j+1}^{(0)}$ *are defined by* (4.5) *and* (4.6) *with* $\widehat{u}_n^{(0)} = u_n^{(0)}$. *The matrix* $\widehat{Q}_0 L(\{u_i^{(0)}\}_{i=2}^{n}, \{\overline{v_i^{(0)}}\}_{i=1}^{n-1}, \{t_i^{(0)}\}_{i=2}^{n-1})$ *has the following representation:*

$$\left[\begin{array}{c|c} \mathbf{0}^T & 0 \\ \hline \mathrm{diag}[\zeta_i] + R(\{\eta_i\}_{i=2}^{n-1}, \{\phi_{n-i}\}_{i=1}^{n-2}, \{\psi_{n-i}\}_{i=2}^{n-2}) & \mathbf{0} \end{array}\right].$$

For demonstration, consider the first two steps of the process (4.11). We have

$$\mathcal{G}(\gamma_1) \begin{bmatrix} u_4^{(0)} \overline{v_3^{(0)}} & 0 \\ u_5^{(0)} t_4^{(0)} \overline{v_4^{(0)}} & u_5^{(0)} \overline{v_4^{(0)}} \end{bmatrix} = \begin{bmatrix} \widehat{u}_4^{(0)} \overline{v_3^{(0)}} & \eta_4 \\ 0 & \zeta_4 \end{bmatrix}.$$

At the second step the matrix $\mathcal{G}(\gamma_2)$ acts on the third and the fourth row of the matrix obtained after the pre-multiplication by $I_3 \oplus \mathcal{G}(\gamma_1)$. It follows that

$$\left[\begin{array}{c|c} \mathcal{G}(\gamma_2) & \mathbf{0} \\ \hline \mathbf{0}^T & 1 \end{array}\right] \left[\begin{array}{c|cc} u_3^{(0)} \overline{v_2^{(0)}} & \multicolumn{2}{c}{\mathbf{0}^T} \\ \hline \widehat{u}_4^{(0)} t_3^{(0)} \overline{v_3^{(0)}} & \widehat{u}_4^{(0)} \overline{v_3^{(0)}} & \eta_4 \\ \mathbf{0} & 0 & \zeta_4 \end{array}\right] =$$
$$\begin{bmatrix} \widehat{u}_3^{(0)} \overline{v_2^{(0)}} & \eta_3 & \eta_4 \psi_2 \\ 0 & \zeta_3 & \eta_4 \phi_2 \\ 0 & 0 & \zeta_4 \end{bmatrix}.$$

Let us demonstrate the construction of the matrix $\widehat{Q}_0 \operatorname{diag}[a_{1,1}^{(0)}, \dots, a_{n.n}^{(0)}]$ for a generic 5×5 matrix:

$$
\begin{array}{l}
\begin{bmatrix} \times & 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 \\ 0 & 0 & \times & 0 & 0 \\ 0 & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & 0 & \times \end{bmatrix} \overset{\mathcal{G}(\gamma_1)}{\rightarrow} \begin{bmatrix} \times & 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 \\ 0 & 0 & \times & 0 & 0 \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \overset{\mathcal{G}(\gamma_2)}{\rightarrow} \\
\begin{bmatrix} \times & 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \overset{\mathcal{G}(\gamma_3)}{\rightarrow} \begin{bmatrix} \times & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}.
\end{array} \tag{4.12}
$$

THEOREM 4.3. *The matrix* $\widehat{Q}_0 \operatorname{diag}[a_{1,1}^{(0)}, \dots, a_{n.n}^{(0)}]$ *has the following structure:*

$$
\left[\begin{array}{c|c} a_{1,1}^{(0)} & \mathbf{0}^T \\ \hline \mathbf{0} & \operatorname{Bidiag}[\eta_i] + R'(\{\zeta_i\}_{i=2}^{n}, \{\phi_{n-i}\}_{i=1}^{n-1}, \{\psi_{n-i}\}_{i=2}^{n-1}) \end{array}\right]
$$

where $\eta_i = -\overline{\psi_{n-i-1}} a_{i+1,i+1}^{(0)}$, $1 \le i \le n-2$, $\zeta_i = \phi_{n-i} a_{i,i}^{(0)}$, $1 \le i \le n$, $\phi_0 = 1$, $\phi_{n-1} = 1$.

Let us describe the first two steps of (4.12). We have

$$
\mathcal{G}(\gamma_1) \begin{bmatrix} a_{4,4}^{(0)} & 0 \\ 0 & a_{5,5}^{(0)} \end{bmatrix} = \begin{bmatrix} \zeta_4 & \psi_1 \zeta_5 \\ \eta_3 & \phi_1 \zeta_5 \end{bmatrix},
$$

and

$$
\left[\begin{array}{c|c} \mathcal{G}(\gamma_2) & \mathbf{0} \\ \hline \mathbf{0}^T & 1 \end{array}\right] \left[\begin{array}{c|cc} a_{3,3}^{(0)} & \multicolumn{2}{c}{\mathbf{0}^T} \\ \hline \mathbf{0} & \zeta_4 & \psi_1\zeta_5 \\ & \eta_3 & \phi_1 \zeta_5 \end{array}\right] =
\begin{bmatrix} \zeta_3 & \psi_2\zeta_4 & \psi_2\psi_1\zeta_5 \\ \eta_2 & \phi_2\zeta_4 & \phi_2\psi_1\zeta_5 \\ 0 & \eta_3 & \phi_1\zeta_5 \end{bmatrix}.
$$

The matrix $\widehat{Q}_0 U(\{\boldsymbol{x}_i^{(0)}\}_{i=1}^{n-1}, \{\boldsymbol{y}_i^{(0)}\}_{i=2}^{n})$ is an upper triangular matrix. Its structure is revealed by the following observation demonstrated for a generic 5×5 matrix. Consider the first step, where the matrix $U(\{\boldsymbol{x}_i^{(0)}\}_{i=1}^{n-1}, \{\boldsymbol{y}_i^{(0)}\}_{i=2}^{n})$ is pre-multiplied by $\mathcal{G}_{4,5}(\gamma_1) = I_3 \oplus \mathcal{G}(\gamma_1)$,

$$
\begin{bmatrix} 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{\mathcal{G}(\gamma_1)}{\rightarrow} \begin{bmatrix} 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \\ 0 & 0 & 0 & 0 & \times \end{bmatrix}. \tag{4.13}
$$

Then it is easily observed that the matrix on the right-hand side of (4.13) is equal to $U'(\{\widehat{\boldsymbol{x}}_i^{(0)}\}_{i=1}^{5}, \{\widehat{\boldsymbol{y}}_i^{(0)}\}_{i=1}^{5})$, for suitable vectors $\widehat{\boldsymbol{x}}_i^{(0)}$ and $\widehat{\boldsymbol{y}}_i^{(0)}$, plus a matrix having only one nonzero entry in position $(4, 4)$.

The same strategy applies to the matrices generated at the intermediate steps, so that we arrive at the next result.

THEOREM 4.4. *The matrix* $\widehat{Q}_0 U(\{\boldsymbol{x}_i^{(0)}\}_{i=1}^{n-1}, \{\boldsymbol{y}_i^{(0)}\}_{i=2}^{n})$ *admits a representation as*

$$\left[\begin{array}{c|ccccc} 0 & \boldsymbol{x}_1^{(0)}\boldsymbol{y}_2^{(0)H} & \dots & \dots & \dots & \boldsymbol{x}_1^{(0)}\boldsymbol{y}_n^{(0)H} \\ \hline \mathbf{0} & \multicolumn{5}{c}{U'(\{\widehat{\boldsymbol{x}}_i^{(0)}\}_{i=2}^{n}, \{\boldsymbol{y}_i^{(0)}\}_{i=2}^{n}) + R'(\{\zeta_i\}_{i=2}^{n}, \{\phi_{n-i}\}_{i=1}^{n-1}, \{\psi_{n-i}\}_{i=2}^{n-1})} \end{array}\right].$$

The vectors $\widehat{\boldsymbol{x}}_i^{(0)}$ *are generated by the following two-step procedure subjected to the initialization* $\tilde{\boldsymbol{x}}_n^{(0)} = \mathbf{0}$*:*

for $j = 1 \ : \ n-2$
 1. $\widehat{\boldsymbol{x}}_{n-j+1}^{(0)} = -\overline{\psi_j}\boldsymbol{x}_{n-j}^{(0)} + \phi_j \tilde{\boldsymbol{x}}_{n-j+1}^{(0)}$;
 2. $\tilde{\boldsymbol{x}}_{n-j}^{(0)} = \phi_j \boldsymbol{x}_{n-j}^{(0)} + \psi_j \tilde{\boldsymbol{x}}_{n-j+1}^{(0)}$;

end
$\widehat{\boldsymbol{x}}_2^{(0)} = \tilde{\boldsymbol{x}}_2^{(0)}$.

The elements ζ_i *are defined by* $\zeta_n = 0$ *and* $\zeta_i = -\tilde{\boldsymbol{x}}_i^{(0)}\boldsymbol{y}_i^{(0)H}$, $2 \le i \le n-1$.

To illustrate the theorem, consider the first two steps of the construction of $\widehat{Q}_0 U(\{\boldsymbol{x}_i^{(0)}\}_{i=1}^{n-1}, \{\boldsymbol{y}_i^{(0)}\}_{i=2}^{n})$ for $n = 5$. We find that

$$\mathcal{G}(\gamma_1)\begin{bmatrix} 0 & \boldsymbol{x}_4^{(0)}\boldsymbol{y}_5^{(0)H} \\ 0 & \zeta_5 \end{bmatrix} = \begin{bmatrix} \zeta_4 + \tilde{\boldsymbol{x}}_4^{(0)}\boldsymbol{y}_5^{(0)H} & \zeta_5\psi_1 + \tilde{\boldsymbol{x}}_4^{(0)}\boldsymbol{y}_5^{(0)H} \\ 0 & \zeta_5\phi_1 + \widehat{\boldsymbol{x}}_5^{(0)}\boldsymbol{y}_5^{(0)H} \end{bmatrix},$$

and

$$\left[\begin{array}{c|c} \mathcal{G}(\gamma_2) & \mathbf{0} \\ \hline \mathbf{0}^T & 1 \end{array}\right]\left[\begin{array}{c|cc} 0 & \boldsymbol{x}_3^{(0)}\boldsymbol{y}_4^{(0)H} & \boldsymbol{x}_3^{(0)}\boldsymbol{y}_5^{(0)H} \\ \hline \mathbf{0} & \zeta_4 + \tilde{\boldsymbol{x}}_4^{(0)}\boldsymbol{y}_4^{(0)H} & \zeta_5\psi_1 + \tilde{\boldsymbol{x}}_4^{(0)}\boldsymbol{y}_5^{(0)H} \\ & 0 & \zeta_5\phi_1 + \widehat{\boldsymbol{x}}_5^{(0)}\boldsymbol{y}_5^{(0)H} \end{array}\right] =$$
$$\begin{bmatrix} \zeta_3 + \tilde{\boldsymbol{x}}_3^{(0)}\boldsymbol{y}_4^{(0)H} & \zeta_4\psi_2 + \tilde{\boldsymbol{x}}_3^{(0)}\boldsymbol{y}_4^{(0)H} & \zeta_5\psi_1\psi_2 + \tilde{\boldsymbol{x}}_3^{(0)}\boldsymbol{y}_5^{(0)H} \\ 0 & \zeta_4\phi_2 + \widehat{\boldsymbol{x}}_4^{(0)}\boldsymbol{y}_4^{(0)H} & \zeta_5\psi_1\phi_2 + \widehat{\boldsymbol{x}}_4^{(0)}\boldsymbol{y}_5^{(0)H} \\ 0 & 0 & \zeta_5\phi_1 + \widehat{\boldsymbol{x}}_5^{(0)}\boldsymbol{y}_5^{(0)H} \end{bmatrix}.$$

By combining these results together, we finally arrive at a structured representation of the matrix $\widehat{Q}_0 B_0$.

THEOREM 4.5. *The matrix* $\widehat{Q}_0 B_0$ *is defined as follows:*

$$\widehat{Q}_0 B_0 = \left[\begin{array}{c|ccc} a_{1,1}^{(0)} & \boldsymbol{x}_1^{(0)}\boldsymbol{y}_2^{(0)H} & \dots & \boldsymbol{x}_1^{(0)}\boldsymbol{y}_n^{(0)H} \\ \hline \widehat{u}_2^{(0)}\overline{v_1^{(0)}} & & & \\ 0 & & \widehat{B}_0 & \\ \vdots & & & \end{array}\right],$$

where

$$\widehat{B}_0 = \mathrm{Bidiag}[\eta_2, \dots, \eta_{n-1}] + U'(\{\widehat{\boldsymbol{x}}_i^{(0)}\}_{i=2}^{n}, \{\boldsymbol{y}_i^{(0)}\}_{i=2}^{n}) + R'(\{\zeta_i\}_{i=2}^{n}, \{\phi_{n-i}\}_{i=1}^{n-1}, \{\psi_{n-i}\}_{i=2}^{n-1}).$$

That is, we have

$$
\widehat{B}_0 = \begin{bmatrix} \widehat{\boldsymbol{x}}_2^{(0)} \boldsymbol{y}_2^{(0)H} & \dots & \dots & \widehat{\boldsymbol{x}}_2^{(0)} \boldsymbol{y}_n^{(0)H} \\ \eta_2 & \widehat{\boldsymbol{x}}_3^{(0)} \boldsymbol{y}_3^{(0)H} & \dots & \widehat{\boldsymbol{x}}_3^{(0)} \boldsymbol{y}_n^{(0)H} \\ & \ddots & \ddots & \vdots \\ 0 & \dots & \eta_{n-1} & \widehat{\boldsymbol{x}}_n^{(0)} \boldsymbol{y}_n^{(0)H} \end{bmatrix} + \tag{4.14}
$$
$$
\begin{bmatrix} \phi_{n-1}\zeta_2 & \phi_{n-1}\psi_{n-2}\zeta_3 & \dots & \phi_{n-1}\psi_{n-2}\dots\psi_1\zeta_n \\ 0 & \zeta_3\phi_{n-2} & \dots & \phi_{n-2}\psi_{n-3}\dots\psi_1\zeta_n \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \phi_1\zeta_n \end{bmatrix}, \quad (\phi_{n-1} = 1).
$$

The subdiagonal entries η_j are defined by

$$
\eta_j = -\overline{\psi_{n-j}} a_{j,j}^{(0)} + \phi_{n-j} \widehat{u}_{j+1}^{(0)} \overline{v_j^{(0)}}, \quad j = 2, \dots, n-1. \tag{4.15}
$$

The remaining elements ζ_j are determined according to the following equation:

$$
\begin{aligned} \zeta_j &= \phi_{n-j} a_{j,j}^{(0)} + \psi_{n-j} \widehat{u}_{j+1}^{(0)} \overline{v_j^{(0)}} - \tilde{\boldsymbol{x}}_j^{(0)} \boldsymbol{y}_j^{(0)H}, \quad j = 2, \dots, n-1, \\ \zeta_n &= a_{n,n}^{(0)}, \end{aligned} \tag{4.16}
$$

where $\tilde{\boldsymbol{x}}_j^{(0)}$ and $\widehat{\boldsymbol{x}}_j^{(0)}$ are given as in Theorem 4.4.

Summing up, we represent the matrix $\widehat{Q}_0 B_0$ by means of a data structure of linear size whose elements are computed at a linear cost. The same clearly holds for the upper Hessenberg matrix H_0 conveniently described as

$$
H_0 = \left[\begin{array}{c|ccc} a_{1,1}^{(0)} & \boldsymbol{x}_1^{(0)} \boldsymbol{y}_2^{(0)H} & \dots & \boldsymbol{x}_1^{(0)} \boldsymbol{y}_n^{(0)H} \\ \hline \widehat{u}_2^{(0)} \overline{v_1^{(0)}} & & & \\ 0 & & \widehat{B}_0 & \\ \vdots & & & \end{array} \right] + \left[\begin{array}{c|c} 1 & \boldsymbol{0} \\ \hline \boldsymbol{0} & \widehat{Q} \end{array} \right] C_0, \tag{4.17}
$$

where $\widehat{B}_0$, $\widehat{Q}$ and C_0 are given by (4.14), (4.8) and (4.10), respectively.

4.2. Reduction to the upper triangular form. We reduce the matrix $H_0 = (h_{i,j}^{(0)})$ of (4.17) to the upper triangular form $R_0 = (r_{i,j}^{(0)})$ by applying Givens rotations $\mathcal{G}_{n-1,n}(\gamma_{2n-3}), \dots, \mathcal{G}_{1,2}(\gamma_{n-1})$ to annihilate the entries located on the first subdiagonal. The first Givens rotation $\mathcal{G}_{1,2}(\gamma_{n-1})$ is determined by the vector equation

$$
\mathcal{G}(\gamma_{n-1}) \begin{bmatrix} a_{1,1}^{(0)} \\ \widehat{u}_2^{(0)} \overline{v_1^{(0)}} \end{bmatrix} = \begin{bmatrix} r_{1,1}^{(0)} \\ 0 \end{bmatrix}.
$$

If we pre-multiply H_0 by $\mathcal{G}_{1,2}(\gamma_1)$, we obtain that

$$
\mathcal{G}_{1,2}(\gamma_1) H_0 = \left[\begin{array}{c|ccc} r_{1,1}^{(0)} & \widetilde{\boldsymbol{x}}_1^{(0)} \boldsymbol{y}_2^{(0)} + \psi_{n-1}\zeta_2 & \dots & \widetilde{\boldsymbol{x}}_1^{(0)} \boldsymbol{y}_n^{(0)H} + \psi_{n-1}..\psi_1\zeta_n \\ \hline \boldsymbol{0} & & \widetilde{B}_0 & \end{array} \right] + \widetilde{Q}_0 C_0,
$$

where $\widetilde{\boldsymbol{x}}_1^{(0)} = \phi_{n-1}\boldsymbol{x}_1^{(0)} + \psi_{n-1}\widehat{\boldsymbol{x}}_2^{(0)}$,

$$\widetilde{B}_0 = \begin{bmatrix} \widehat{\boldsymbol{x}}_2^{(0)}\boldsymbol{y}_2^{(0)H} & \dots & \dots & \widehat{\boldsymbol{x}}_2^{(0)}\boldsymbol{y}_n^{(0)H} \\ \eta_2 & \widehat{\boldsymbol{x}}_3^{(0)}\boldsymbol{y}_3^{(0)H} & \dots & \widehat{\boldsymbol{x}}_3^{(0)}\boldsymbol{y}_n^{(0)H} \\ & \ddots & \ddots & \vdots \\ 0 & \dots & \eta_{n-1} & \widehat{\boldsymbol{x}}_n^{(0)}\boldsymbol{y}_n^{(0)H} \end{bmatrix} + \begin{bmatrix} \phi_{n-1}\zeta_2 & \phi_{n-1}\psi_{n-2}\zeta_3 & \dots & \phi_{n-1}\psi_{n-2}\dots\psi_1\zeta_n \\ 0 & \zeta_3\phi_{n-2} & \dots & \phi_{n-2}\psi_{n-3}\dots\psi_1\zeta_n \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \phi_1\zeta_n \end{bmatrix},$$

with $\widehat{\boldsymbol{x}}_2^{(0)} = \phi_{n-1}\widehat{\boldsymbol{x}}_2^{(0)} - \overline{\psi_{n-1}}\boldsymbol{x}_1^{(0)}$, and, moreover,

$$\widetilde{Q}_0 = \begin{bmatrix} \tilde{\phi}\phi_{n-1} & \tilde{\phi}\psi_{n-1}\phi_{n-2} & \dots & \dots & \tilde{\phi}\psi_{n-1}\dots\psi_1\hat{\phi} \\ -\overline{\psi_{n-1}} & \phi_{n-1}\phi_{n-2} & \dots & \dots & \phi_{n-1}\psi_{n-2}\dots\psi_1\hat{\phi} \\ 0 & -\overline{\psi}_{n-2} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\overline{\psi}_1 & \phi_1\hat{\phi} \end{bmatrix},$$

with $\tilde{\phi} = \hat{\phi} = 1$. Note that $\widetilde{B}_0$ and $\widetilde{Q}_0$ have a similar structure. Indeed,

$$\widetilde{Q}_0 = \mathrm{Bidiag}[-\overline{\psi_i}] + R'(\{\phi_{n-i}\}_{i=1}^{n-1} \odot \{1\}, \{1\} \odot \{\phi_{n-i}\}_{i=1}^{n-1}, \{\psi_{n-i}\}_{i=1}^{n-1}),$$

and

$$\widetilde{B}_0 = \mathrm{Bidiag}[\eta_j] + R'(\{\zeta_i\}_{i=2}^{n}, \{\phi_{n-i}\}_{i=1}^{n-1}, \{\psi_{n-i}\}_{i=1}^{n-1}) + U'(\{\widehat{\boldsymbol{x}}_i^{(0)}\}_{i=2}^{n}, \{\boldsymbol{y}_i^{(0)}\}_{i=2}^{n}).$$

To see how pre-multiplication by $\mathcal{G}_{2,3}(\gamma_n), \dots \mathcal{G}_{n-1,n}(\gamma_{2n-3})$ modifies these matrices, it suffices to consider the process for $\widetilde{Q}_0$. First suppose that the parameters $\gamma_n, \dots, \gamma_{2n-3}$ are already known. Later on, we present a method for computing them during the triangulation process.

Let us begin with the generation of the matrix

$$\mathcal{G}_{n-1,n}(\gamma_{2n-3})\dots\mathcal{G}_{2,3}(\gamma_n)R'(\{\phi_{n-i}\}_{i=1}^{n-1} \odot \{1\}, \{1\} \odot \{\phi_{n-i}\}_{i=1}^{n-1}, \{\psi_{n-i}\}_{i=1}^{n-1}),$$

which we demonstrate for a generic 5×5 matrix:

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \end{bmatrix} \xrightarrow{\mathcal{G}(\gamma_5)} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \end{bmatrix} \xrightarrow{\mathcal{G}(\gamma_6)} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times \end{bmatrix} \xrightarrow{\mathcal{G}(\gamma_7)} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix}. \tag{4.18}$$

THEOREM 4.6. *The matrix* $G_{n-1,n}(\gamma_{2n-3})\dots\mathcal{G}_{2,3}(\gamma_n)R'(\{\phi_{n-i}\}_{i=1}^{n-1}\odot\{1\},\{1\}\odot\{\phi_{n-i}\}_{i=1}^{n-1},\{\psi_{n-i}\}_{i=1}^{n-1})$ *admits the following representation:*

$$\left[\begin{array}{c|c} \phi_{n-1} & \psi_{n-1}\phi_{n-2} \quad \dots \quad \psi_{n-1}\dots\psi_1\hat{\phi} \\ \hline \mathbf{0} & \mathrm{diag}[\rho_2,\dots,\rho_n]+F \end{array}\right],$$

where F *is given by*

$$R(\{\phi_{n-i}\}_{i=3}^{n-1}\odot\{1\},\{\delta_i\}_{i=2}^{n-1},\{\psi_{n-i}\}_{i=3}^{n-1})+L(\{\phi_{n+i}\}_{i=1}^{n-3}\odot\{1\},\{\mu_i\}_{i=2}^{n-1},\{-\overline{\psi_{n+i}}\}_{i=1}^{n-3}).$$

The entries δ_j *are generated by the following two-step procedure subjected to the initialization* $\delta_1=\psi_{n-1}$, $\widehat{\delta}_2=\phi_{n-1}$*:*

for $j=2\ :\ n-1$
1. $\delta_j=\phi_{n-2+j}\widehat{\delta}_j\psi_{n-j}+\psi_{n-2+j}\phi_{n-j}$;
2. $\widehat{\delta}_{j+1}=-\overline{\psi_{n-2+j}}\widehat{\delta}_j\psi_{n-j}+\phi_{n-2+j}\phi_{n-j}$;

end

$\delta_n=\widehat{\delta}_n$.

The elements μ_j *are given by* $\mu_j=-\overline{\psi_{n-2+j}}\widehat{\delta}_j\phi_{n-j}$, $2\le j\le n-1$, *whereas the remaining diagonal entries* ρ_j *are defined by* $\rho_j=\phi_{n-2+j}\widehat{\delta}_j\phi_{n-j}$, $2\le j\le n-1$, *and* $\rho_n=\widehat{\delta}_n$.

Let us demonstrate the first two steps of the process (4.18):

$$\mathcal{G}(\gamma_5)\begin{bmatrix} \phi_4\phi_3 & \phi_4\psi_3\phi_2 \\ 0 & \phi_3\phi_2 \end{bmatrix}=\begin{bmatrix} \rho_2 & \delta_2\phi_2 \\ \mu_2 & \widehat{\delta}_3\phi_2 \end{bmatrix}$$

and

$$\left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \mathcal{G}(\gamma_6) \end{array}\right]\left[\begin{array}{cc|c} \rho_2 & \delta_2\phi_2 & \delta_2\psi_2\phi_1 \\ \mu_2 & \widehat{\delta}_3\phi_2 & \widehat{\delta}_3\psi_2\phi_1 \\ \hline 0 & 0 & \phi_2\phi_1 \end{array}\right]= \begin{bmatrix} \rho_2 & \delta_2\phi_2 & \delta_2\psi_2\phi_1 \\ \mu_2\phi_6 & \rho_3 & \delta_3\phi_1 \\ -\mu_2\overline{\psi_6} & \mu_3 & \widehat{\delta}_4\phi_1 \end{bmatrix}.$$

The construction of $\mathcal{G}_{n-1,n}(\gamma_{2n-3})\dots\mathcal{G}_{2,3}(\gamma_n)\mathrm{Bidiag}[-\overline{\psi_i}]$ for $n=5$ can be described as follows:

$$(4.19)\qquad \begin{bmatrix} 0&0&0&0&0\\ \times&0&0&0&0\\ 0&\times&0&0&0\\ 0&0&\times&0&0\\ 0&0&0&\times&0 \end{bmatrix} \overset{\mathcal{G}(\gamma_5)}{\rightarrow} \begin{bmatrix} 0&0&0&0&0\\ \times&\times&0&0&0\\ \times&\times&0&0&0\\ 0&0&\times&0&0\\ 0&0&0&\times&0 \end{bmatrix} \overset{\mathcal{G}(\gamma_6)}{\rightarrow}$$
$$\begin{bmatrix} 0&0&0&0&0\\ \times&\times&0&0&0\\ \times&\times&\times&0&0\\ \times&\times&\times&0&0\\ 0&0&0&\times&0 \end{bmatrix} \overset{\mathcal{G}(\gamma_7)}{\rightarrow} \begin{bmatrix} 0&0&0&0&0\\ \times&\times&0&0&0\\ \times&\times&\times&0&0\\ \times&\times&\times&\times&0\\ \times&\times&\times&\times&0 \end{bmatrix}.$$

THEOREM 4.7. *The matrix* $G_{n-1,n}(\gamma_{2n-3})\dots\mathcal{G}_{2,3}(\gamma_n)\mathrm{Bidiag}[-\overline{\psi_i}]$ *can be represented as follows:*

$$\left[\begin{array}{c|c} \mathbf{0}^T & 0 \\ \hline L'(\{\phi_{n+i-1}\}_{i=1}^{n-2}\odot\{1\},\{\mu_i\}_{i=1}^{n-1},\{-\overline{\psi_{n+i-1}}\}_{i=1}^{n-2})+\mathrm{Bidiag}^T[\rho_2,\dots,\rho_{n-1}] & \mathbf{0} \end{array}\right],$$

where $\mu_1 = -\overline{\psi_{n-1}}$ *and* $\mu_j = -\overline{\psi_{n-j}}\phi_{n-2+j}$, $2 \le j \le n-1$, *and* $\rho_j = -\psi_{n-2+j}\overline{\psi_{n-j}}$, $2 \le j \le n-1$.

For a generic 5×5 matrix the process (4.19) yields:

$$\mathcal{G}(\gamma_5)\begin{bmatrix} -\overline{\psi_4} & 0 \\ 0 & -\overline{\psi_3} \end{bmatrix} = \begin{bmatrix} \phi_5\mu_1 & \rho_2 \\ -\overline{\psi_5}\mu_1 & \mu_2 \end{bmatrix}$$

and

$$\left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \mathcal{G}(\gamma_6) \end{array}\right]\left[\begin{array}{cc|c} \phi_5\eta_1 & \rho_2 & 0 \\ -\overline{\psi_5}\mu_1 & \mu_2 & 0 \\ \hline 0 & 0 & -\overline{\psi_2} \end{array}\right] = \begin{bmatrix} \phi_5\mu_1 & \rho_2 & 0 \\ -\phi_6\overline{\psi_5}\mu_1 & \phi_6\mu_2 & \rho_3 \\ \overline{\psi_6\psi_5}\mu_1 & -\overline{\psi_6}\mu_2 & \mu_3 \end{bmatrix}$$

From Theorems 4.6 and 4.7 one obtains the following result, which enables us to compute the quantities γ_j, $n \le j \le 2n-3$.

THEOREM 4.8. *The matrix* $\mathrm{tril}(\mathcal{G}_{n-1,n}(\gamma_{2n-3})\dots\mathcal{G}_{2,3}(\gamma_n)\widetilde{Q}_0, 0)$ *is equal to*

$$L(\{\phi_{n+i-1}\}_{i=1}^{n-2} \odot \{1\}, \{\mu_i\}_{i=1}^{n-1}, \{-\overline{\psi_{n+i-1}}\}_{i=1}^{n-2}) + \mathrm{diag}[\rho_1, \dots, \rho_n],$$

with $\rho_1 = \phi_{n-1}$, $\rho_j = \phi_{n-2+j}\widehat{\delta}_j\phi_{n-j} - \psi_{n-2+j}\overline{\psi_{n-j}}$, $2 \le j \le n-1$, $\rho_n = \widehat{\delta}_n$, $\mu_1 = -\overline{\psi_{n-1}}$ *and* $\mu_j = -\overline{\psi_{n-j}}\phi_{n-2+j} - \overline{\psi_{n-2+j}\widehat{\delta}_j}\phi_{n-j}$, $2 \le j \le n-1$, *where* $\widehat{\delta}_j$ *are defined in Theorem 4.6.*

COROLLARY 4.9. *Let* $\widetilde{s}_2 = (\widetilde{Q}_0C_0)_{2,2}$ *and* $\widetilde{s}_j = (\mathcal{G}_{j-1,j}(\gamma_{n+j-3})\dots\mathcal{G}_{2,3}(\gamma_n)\widetilde{Q}_0C_0)_{j,j}$, $3 \le j \le n-1$. *Then* $\widetilde{s}_j = \overline{u_j^{(0)}}s_j$, $2 \le j \le n-1$, *where*

$$\begin{cases} s_2 = \mu_1 v_1^{(0)}, \\ s_j = (-\overline{\psi_{n-3+j}})\overline{t_{j-1}^{(0)}}s_{j-1} + \mu_{j-1}v_{j-1}^{(0)}, \quad j = 3, \dots, n-1, \end{cases}$$

where μ_j *are the same as in Theorem 4.8.*

This result provides a simple recursion for the computation of the elements $\widetilde{s}_j$ at a linear cost. Similarly to our study of the matrix $\mathcal{G}_{n-1,n}(\gamma_{2n-3})\dots\mathcal{G}_{2,3}(\gamma_n)\widetilde{Q}_0$, one easily deduces the following results for the matrices

$$\mathcal{G}_{n-1,n}(\gamma_{2n-3})\dots\mathcal{G}_{2,3}(\gamma_n)U'(\{\widetilde{\boldsymbol{x}}_1^{(0)}\} \odot \{\widehat{\boldsymbol{x}}_i^{(0)}\}_{i=2}^n, \{\mathbf{0}^T\} \odot \{\boldsymbol{y}_i^{(0)}\}_{i=2}^n)$$

and

$$\mathcal{G}_{n-1,n}(\gamma_{2n-3})\dots\mathcal{G}_{2,3}(\gamma_n)R'(\{0\} \odot \{\zeta_i\}_{i=2}^n, \{1\} \odot \{\phi_{n-i}\}_{i=1}^n, \{\psi_{n-i}\}_{i=1}^{n-1}),$$

which are obtained by means of the triangulation process applied to $\widetilde{B}_0$.

THEOREM 4.10. *The matrix*

$$\mathrm{triu}(\mathcal{G}_{n-1,n}(\gamma_{n-3})\dots\mathcal{G}_{2,3}(\gamma_n)U'(\{\widetilde{\boldsymbol{x}}_1^{(0)}\} \odot \{\widehat{\boldsymbol{x}}_i^{(0)}\}_{i=2}^n, \{\mathbf{0}^T\} \odot \{\boldsymbol{y}_i^{(0)}\}_{i=2}^n), 1)$$

is equal to $U(\{\widetilde{\boldsymbol{x}}_i^{(0)}\}_{i=1}^{n-1}, \{\boldsymbol{y}_i^{(0)}\}_{i=2}^n)$, *where the vectors* $\{\widetilde{x}_i^{(0)}\}_{i=2}^{n-1}$ *are generated by the following two-step procedure subjected to the initialization* $\breve{\boldsymbol{x}}_2^{(0)} = \widehat{\boldsymbol{x}}_2^{(0)}$ *:*

for $j = 2 \ : \ n-1$

1. $\widetilde{\boldsymbol{x}}_j^{(0)} = \phi_{n-2+j}\check{\boldsymbol{x}}_j^{(0)} + \psi_{n-2+j}\widehat{\boldsymbol{x}}_{j+1}^{(0)}$;
2. $\check{\boldsymbol{x}}_{j+1}^{(0)} = \phi_{n-2+j}\widehat{\boldsymbol{x}}_{j+1}^{(0)} - \overline{\psi_{n-2+j}}\check{\boldsymbol{x}}_j^{(0)}$;
end
$\widetilde{\boldsymbol{x}}_n^{(0)} = \check{\boldsymbol{x}}_n^{(0)}$.

Similarly, we deduce the next result:

THEOREM 4.11. *The matrix*

$$\mathrm{triu}(\mathcal{G}_{n-1,n}(\gamma_{2n-3})\dots\mathcal{G}_{2,3}(\gamma_n)R'(\{0\}\odot\{\zeta_i\}_{i=2}^n,\{1\}\odot\{\phi_{n-i}\}_{i=1}^n,\{\psi_{n-i}\}_{i=1}^{n-1}),1)$$

is equal to $R(\{\zeta_i\}_{i=2}^n,\{\delta_i\}_{i=1}^{n-1},\{\psi_{n-i}\}_{i=2}^{n-1})$, *where* $\delta_1 = \psi_{n-1}$ *and* δ_j *are determined as in Theorem 4.6.*

Let us now describe the generation of the matrix $\mathcal{G}_{n-1,n}(\gamma_{2n-3})\dots\mathcal{G}_{1,2}(\gamma_{n-1})H_0$. For demonstration, consider the second and the third step. Recall that the parameters δ_j and $\widehat{\delta}_j$ are defined in Theorem 4.6, whereas the parameters μ_j and ρ_j are given in Theorem 4.8. Observe that

$$(\mathcal{G}(\gamma_{n-1})H_0)[2:3,2:3] = \begin{bmatrix} \check{\boldsymbol{x}}_2^{(0)}\boldsymbol{y}_2^{(0)H} + \widehat{\delta}_2\zeta_2 & \check{\boldsymbol{x}}_2^{(0)}\boldsymbol{y}_3^{(0)H} + \widehat{\delta}_2\psi_{n-2}\zeta_3 \\ \eta_2 & \widehat{\boldsymbol{x}}_3^{(0)}\boldsymbol{y}_3^{(0)H} + \phi_{n-2}\zeta_3 \end{bmatrix} + \begin{bmatrix} \mu_1 & \widehat{\delta}_2\phi_{n-2} \\ 0 & -\overline{\psi_{n-2}} \end{bmatrix}\begin{bmatrix} 0 & \overline{u_2^{(0)}}v_1^{(0)} \\ 0 & 0 \end{bmatrix}.$$

Therefore, γ_n is determined by the following vector equation:

$$\mathcal{G}(\gamma_n)\begin{bmatrix} \check{\boldsymbol{x}}_2^{(0)}\boldsymbol{y}_2^{(0)H} + \widehat{\delta}_2\zeta_2 + \widetilde{s}_2 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} r_{2,2}^{(0)} \\ 0 \end{bmatrix},$$

which implies that

$$\mathcal{G}(\gamma_n)(\mathcal{G}(\gamma_{n-1})H_0)[2:3,2:3] = \begin{bmatrix} r_{2,2}^{(0)} - \phi_n\widetilde{s}_2 & \widetilde{\boldsymbol{x}}_2^{(0)}\boldsymbol{y}_3^{(0)H} + \delta_2\zeta_3 \\ \overline{\psi_n}\widetilde{s}_2 & \check{\boldsymbol{x}}_3^{(0)}\boldsymbol{y}_3^{(0)H} + \widehat{\delta}_3\zeta_3 \end{bmatrix} + \begin{bmatrix} \mu_1\phi_n & \rho_2 \\ -\overline{\psi_n}\mu_1 & \mu_2 \end{bmatrix}\begin{bmatrix} 0 & \overline{u_2^{(0)}}v_1^{(0)} \\ 0 & 0 \end{bmatrix}.$$

At the third step the value of γ_{n+1} is found from the vector equation

$$\mathcal{G}(\gamma_{n+1})\begin{bmatrix} \check{\boldsymbol{x}}_3^{(0)}\boldsymbol{y}_3^{(0)H} + \widehat{\delta}_3\zeta_3 + \widetilde{s}_3 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} r_{3,3}^{(0)} \\ 0 \end{bmatrix},$$

where

$$\widehat{s}_3 = -\overline{\psi_n}\mu_1 u_3^{(0)}\overline{t_2^{(0)}}v_1^{(0)} + \mu_2 u_3^{(0)}v_2^{(0)} = u_3^{(0)}(-\overline{\psi_n}\overline{t_2^{(0)}}s_2) + \mu_2 v_2^{(0)} = u_3^{(0)}s_2.$$

By collecting all these facts together, we finally arrive at the following theorem, which provides the desired representation of the upper triangular matrix R_0 obtained at the end of the triangulation process applied to H_0.

THEOREM 4.12. *The matrix* $R_0 = \mathcal{G}_{n-1,n}(\gamma_{2n-3})\dots\mathcal{G}_{1,2}(\gamma_{n-1})\mathcal{G}_{2,3}(\gamma_{n-2})\dots\mathcal{G}_{n-1,n}(\gamma_1)A_0 = \mathcal{G}_{n-1,n}(\gamma_{2n-3})\dots\mathcal{G}_{1,2}(\gamma_{n-1})H_0$ *admits the following representation:*

$$R_0 = R_0^{(1)} + R_0^{(2)} + Q_0^{(1)}C_0 + Q_0^{(2)}C_0,$$

where

$$R_0^{(1)} = \begin{bmatrix} r_{1,1}^{(0)} & \widetilde{\boldsymbol{x}}_1^{(0)} {\boldsymbol{y}_2^{(0)}}^H + \delta_1\zeta_2 & \dots & \widetilde{\boldsymbol{x}}_1^{(0)} {\boldsymbol{y}_n^{(0)}}^H + \delta_1\psi_{n-2}\dots\psi_1\zeta_n \\ 0 & r_{2,2}^{(0)} - \phi_n\widetilde{s}_2 & \dots & \widetilde{\boldsymbol{x}}_2^{(0)} {\boldsymbol{y}_n^{(0)}}^H + \delta_2\psi_{n-3}\dots\psi_1\zeta_n \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \widetilde{\boldsymbol{x}}_n^{(0)} {\boldsymbol{y}_n^{(0)}}^H + \delta_n\zeta_n \end{bmatrix},$$

$$R_0^{(2)} = \begin{bmatrix} 0 & & & & \\ 0 & 0 & & & \\ \vdots & (-\overline{\psi_n})\phi_{n+1}(-\widetilde{s}_2) & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & (-\overline{\psi_n})\dots(-\overline{\psi_{2n-3}})\tilde{\phi}(-\widetilde{s}_2) & \dots & (-\overline{\psi_{2n-3}})\tilde{\phi}(-\widetilde{s}_{n-1}) & 0 \end{bmatrix},$$

$$Q_0^{(1)} = \begin{bmatrix} \rho_1 & \delta_1\phi_{n-2} & \dots & \delta_1\psi_{n-2}\dots\psi_1\tilde{\phi} \\ & \ddots & \ddots & \vdots \\ & & \rho_{n-1} & \delta_{n-1}\tilde{\phi} \\ 0 & \dots & 0 & \rho_n \end{bmatrix},$$

and

$$Q_0^{(2)} = \begin{bmatrix} 0 & & & \\ \phi_n\mu_1 & 0 & & \\ \vdots & \ddots & \ddots & \\ \tilde{\phi}(-\overline{\psi_n})\dots(-\overline{\psi_{2n-3}})\mu_1 & \dots & \tilde{\phi}\mu_{n-1} & 0 \end{bmatrix}, \quad (\tilde{\phi} = 1).$$

The vectors $\widetilde{\boldsymbol{x}}_j^{(0)}$ *and* $\check{\boldsymbol{x}}_j^{(0)}$ *are given as in Theorem 4.10 according to the initial conditions*

$$\widetilde{\boldsymbol{x}}_1^{(0)} = \phi_{n-1}\boldsymbol{x}_1^{(0)} + \psi_{n-1}\widehat{\boldsymbol{x}}_2^{(0)},$$

and

$$\check{\boldsymbol{x}}_2^{(0)} = \widetilde{\boldsymbol{x}}_2^{(0)} = \phi_{n-1}\widehat{\boldsymbol{x}}_2^{(0)} - \overline{\psi_{n-1}}\boldsymbol{x}_1^{(0)}.$$

The parameters ϕ_j *and* ψ_j*,* $n-1 \le j \le 2n-3$*, defining the Givens rotations* $\mathcal{G}_{n-1,n}(\gamma_{2n-3}), \dots, \mathcal{G}_{1,2}(\gamma_{n-1})$*, are determined by the following vector equations:*

$$\mathcal{G}(\gamma_{n-1}) \begin{bmatrix} a_{1,1}^{(0)} \\ \widehat{u}_2^{(0)}\overline{v_1^{(0)}} \end{bmatrix} = \begin{bmatrix} r_{1,1}^{(0)} \\ 0 \end{bmatrix}$$

and

$$\mathcal{G}(\gamma_{n-2+j}) \begin{bmatrix} \check{\boldsymbol{x}}_j^{(0)} {\boldsymbol{y}_j^{(0)}}^H + \widehat{\delta}_j\zeta_j + \widetilde{s}_j \\ \eta_j \end{bmatrix} = \begin{bmatrix} r_{j,j}^{(0)} \\ 0 \end{bmatrix}, \quad j = 2, \dots, n-1,$$

where δ_j *are defined in Theorem 4.6, the parameters* μ_j *and* ρ_j *in Theorem 4.8, and the quantities* s_j *and* $\widetilde{s}_j$ *in Corollary 4.9.*

Theorem 4.12 provides a structural description of the matrix R_0 obtained by triangulation of the matrix $A_0 \in \mathcal{C}_n$. The next step of the QR scheme (3.1) is to compute the matrix $A_1 = R_0 Q_0$. From the results of the previous section we know that $A_1 \in \mathcal{C}_n$. Hence, our task reduces to the computation of $d_1^{(1)}, \dots, d_n^{(1)}$, $t_2^{(1)}, \dots, t_{n-1}^{(1)}$, $\boldsymbol{u}^{(1)} \in \mathbb{C}^n$, $\boldsymbol{v}^{(1)} \in \mathbb{C}^n$, $\boldsymbol{z}^{(1)} \in \mathbb{C}^n$ and $\boldsymbol{w}^{(1)} \in \mathbb{C}^n$, which define $A_1 = (a_{i,j}^{(1)})$ according to (3.5). From Remark 3.4 it follows that

$$\boldsymbol{z}^{(1)} = \mathcal{G}_{n-1,n}(\gamma_{2n-3}) \dots \mathcal{G}_{1,2}(\gamma_{n-1}) \mathcal{G}_{2,3}(\gamma_{n-2}) \dots \mathcal{G}_{n-1,n}(\gamma_1) \boldsymbol{z}^{(0)},$$

and, analogously,

$$\boldsymbol{w}^{(1)} = \mathcal{G}_{n-1,n}(\gamma_{2n-3}) \dots \mathcal{G}_{1,2}(\gamma_{n-1}) \mathcal{G}_{2,3}(\gamma_{n-2}) \dots \mathcal{G}_{n-1,n}(\gamma_1) \boldsymbol{w}^{(0)}.$$

In this way, it remains to find only the quantities $d_1^{(1)}, \dots, d_n^{(1)}$, $t_2^{(1)}, \dots, t_{n-1}^{(1)}$, $\boldsymbol{u}^{(1)} \in \mathbb{C}^n$ and $\boldsymbol{v}^{(1)} \in \mathbb{C}^n$, which specify the lower triangular part of A_1. This issue is addressed in the next subsection.

4.3. Computation of the new iterate. We have already shown how to calculate the upper triangular matrix R_0 and the unitary matrix Q_0 such that $A_0 = Q_0 R_0$. Once this factorization is known, the QR iteration (3.1) determines the new iterate A_1 as $A_1 = R_0 Q_0$. Observe that

$$\begin{aligned} A_1 &= R_0 (\mathcal{G}_{n-1,n}(\gamma_1))^H \dots (\mathcal{G}_{2,3}(\gamma_{n-2}))^H (\mathcal{G}_{1,2}(\gamma_{n-1}))^H \dots (\mathcal{G}_{n-1,n}(\gamma_{2n-3}))^H \\ &= A_{1/2} (\mathcal{G}_{1,2}(\gamma_{n-1}))^H \dots (\mathcal{G}_{n-1,n}(\gamma_{2n-3}))^H, \end{aligned}$$

with

$$A_{1/2} = (a_{i,j}^{1/2}) = R_0 (\mathcal{G}_{n-1,n}(\gamma_1))^H \dots (\mathcal{G}_{2,3}(\gamma_{n-2}))^H. \tag{4.20}$$

Let $\widehat{r}_{j,j}^{(0)}$ denote the diagonal entry of $R_0(\mathcal{G}_{n-1,n}(\gamma_1))^H \dots (\mathcal{G}_{j,j+1}(\gamma_{n-j}))^H$ in position (j,j), where $\widehat{r}_{n,n}^{(0)} = r_{n,n}^{(0)}$ and $\widehat{r}_{1,1}^{(0)} = r_{1,1}^{(0)}$. The process of forming the matrix $A_{1/2}$ is demonstrated below for a generic 5×5 matrix.

$$\begin{aligned} &\begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \widehat{r}_{5,5}^{(0)} \end{bmatrix} \overset{\mathcal{G}(\gamma_1)^H}{\rightarrow} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \widehat{r}_{4,4}^{(0)} & \times \\ 0 & 0 & 0 & \overline{\psi}_1 \widehat{r}_{5,5}^{(0)} & \phi_1 \widehat{r}_{5,5}^{(0)} \end{bmatrix} \overset{\mathcal{G}(\gamma_2)^H}{\rightarrow} \\ &\begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \widehat{r}_{3,3}^{(0)} & \times & \times \\ 0 & 0 & \overline{\psi}_2 \widehat{r}_{4,4}^{(0)} & \phi_2 \widehat{r}_{4,4}^{(0)} & \times \\ 0 & 0 & \overline{\psi_1 \psi_2} \widehat{r}_{5,5}^{(0)} & \overline{\psi}_1 \widehat{r}_{5,5}^{(0)} & \phi_1 \widehat{r}_{5,5}^{(0)} \end{bmatrix} \overset{\mathcal{G}(\gamma_3)^H}{\rightarrow} \\ &\begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \widehat{r}_{2,2}^{(0)} & \times & \times & \times \\ 0 & \overline{\psi}_3 \widehat{r}_{3,3}^{(0)} & \phi_3 \widehat{r}_{3,3}^{(0)} & \times & \times \\ 0 & \overline{\psi_2 \psi_3} \widehat{r}_{4,4}^{(0)} & \overline{\psi}_2 \phi_3 \widehat{r}_{4,4}^{(0)} & \phi_2 \widehat{r}_{4,4}^{(0)} & \times \\ 0 & \overline{\psi_1 \psi_2 \psi_3} \widehat{r}_{5,5}^{(0)} & \overline{\psi_1 \psi_2} \phi_3 \widehat{r}_{5,5}^{(0)} & \overline{\psi}_1 \phi_2 \widehat{r}_{5,5}^{(0)} & \phi_1 \widehat{r}_{5,5}^{(0)} \end{bmatrix}. \end{aligned} \tag{4.21}$$

This is extended to the next theorem.

THEOREM 4.13. *The matrix* $A_{1/2}$ *of* (4.20) *satisfies*

$$\mathrm{tril}(A_{1/2},1)=\left[\begin{array}{c|ccccc} \widehat{r}^{(0)}_{1,1} & a^{(1/2)}_{1,2} & 0 & \dots & \dots & \dots \quad 0 \\ \hline \mathbf{0} & \multicolumn{5}{c}{\mathrm{Bidiag}[a^{(0)}_{i,i+1}]^T + L'(\{\widehat{r}^{(0)}_{i,i}\}_{i=2}^n, \{1\}\odot\{\phi_{n-i}\}_{i=2}^{n-1}, \{\overline{\psi}_{n-i}\}_{i=2}^{n-1})} \end{array}\right]$$

$$=\begin{bmatrix} \widehat{r}^{(0)}_{1,1} & a^{(1/2)}_{1,2} & & & \\ 0 & \widehat{r}^{(0)}_{2,2} & a^{(1/2)}_{2,3} & & \\ 0 & \overline{\psi_{n-2}}\widehat{r}^{(0)}_{3,3} & \phi_{n-2}\widehat{r}^{(0)}_{3,3} & \ddots & \\ \vdots & \ddots & \ddots & \ddots & a^{(1/2)}_{n-1,n} \\ 0 & \overline{\psi_{n-2}\dots\psi_1}\widehat{r}^{(0)}_{n,n} & \dots & \overline{\psi_1}\phi_2\widehat{r}^{(0)}_{n,n} & \phi_1\widehat{r}^{(0)}_{n,n} \end{bmatrix}.$$

The transformation from $A_{1/2}$ to A_1 modifies the lower triangular part of $A_{1/2}$ and is next demonstrated in the case of a 5×5 matrix $A_{1/2}$ represented as at the end of the process (4.21).

$$A_{1/2} \overset{\mathcal{G}(\gamma_4)^H}{\to} \begin{bmatrix} a^{(1)}_{1,1} & \times & \times & \times & \times \\ \overline{\psi}_4\widehat{r}^{(0)}_{2,2} & \phi_4\widehat{r}^{(0)}_{2,2} & a^{(1/2)}_{2,3} & \times & \times \\ \overline{\psi_3\psi_4}\widehat{r}^{(0)}_{3,3} & \overline{\psi}_3\phi_4\widehat{r}^{(0)}_{3,3} & \times & \times & \times \\ \overline{\psi_2\psi_3\psi_4}\widehat{r}^{(0)}_{4,4} & \overline{\psi_2\psi_3}\phi_4\widehat{r}^{(0)}_{4,4} & \times & \times & \times \\ \overline{\psi_1\psi_2\psi_3\psi_4}\widehat{r}^{(0)}_{5,5} & \overline{\psi_1\psi_2\psi_3}\phi_4\widehat{r}^{(0)}_{5,5} & \times & \times & \times \end{bmatrix} \overset{\mathcal{G}(\gamma_5)^H}{\to} \tag{4.22}$$

$$\begin{bmatrix} a^{(1)}_{1,1} & \times & \times & \times & \times \\ \overline{v^{(1)}_1}\widehat{r}^{(0)}_{2,2} & a^{(1)}_{2,2} & \times & \times & \times \\ \overline{v^{(1)}_1}\overline{\psi_3}\widehat{r}^{(0)}_{3,3} & \overline{v^{(1)}_2}\widehat{r}^{(0)}_{3,3} & \eta_3\widehat{r}^{(0)}_{3,3} & a^{(1/2)}_{3,4} & \times \\ \overline{v^{(1)}_1}\overline{\psi_2\psi_3}\widehat{r}^{(0)}_{4,4} & \overline{v^{(1)}_2}\overline{\psi_2}\widehat{r}^{(0)}_{4,4} & \overline{\psi_2}\eta_3\widehat{r}^{(0)}_{4,4} & \times & \times \\ \overline{v^{(1)}_1}\overline{\psi_1\psi_2\psi_3}\widehat{r}^{(0)}_{5,5} & \overline{v^{(1)}_2}\overline{\psi_1\psi_2}\widehat{r}^{(0)}_{5,5} & \overline{\psi_1\psi_2}\eta_3\widehat{r}^{(0)}_{5,5} & \times & \times \end{bmatrix},$$

where

$$a^{(1)}_{1,1}=\phi_4\widehat{r}^{(0)}_{1,1}+\overline{\psi}_4 a^{(1/2)}_{1,2}, \quad a^{(1)}_{2,2}=\phi_5\phi_4\widehat{r}^{(0)}_{2,2}+\overline{\psi}_5 a^{(1/2)}_{2,3},$$

and

$$v^{(1)}_1=\psi_4, \quad ,v^{(1)}_2=\phi_4\phi_5\psi_3+\psi_5\phi_3, \quad \eta_3=-\psi_5\overline{\psi_3}\phi_4+\phi_5\phi_3.$$

In this way, we easily arrive at the following characterization of the lower triangular part of A_1.

THEOREM 4.14. *We have*

$$\mathrm{tril}(A_1,0)=\mathrm{diag}[a^{(1)}_{1,1},\dots,a^{(1)}_{n,n}]+L(\{u_i\}_{i=2}^n,\{\overline{v_i}\}_{i=1}^{n-1},\{t_i\}_{i=2}^{n-1})=$$

$$\begin{bmatrix} a^{(1)}_{1,1} & 0 & \dots & \dots & 0 \\ u^{(1)}_2\overline{v^{(1)}_1} & a^{(1)}_{2,2} & \ddots & \ddots & \vdots \\ u^{(1)}_3 t^{(1)}_2\overline{v^{(1)}_1} & u^{(1)}_3\overline{v^{(1)}_2} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ u^{(1)}_n t^{(1)}_{n-1}\dots t^{(1)}_2\overline{v^{(1)}_1} & u^{(1)}_n t^{(1)}_{n-1}\dots t^{(1)}_3\overline{v^{(1)}_2} & \dots & u^{(1)}_n\overline{v^{(1)}_{n-1}} & a^{(1)}_{n,n} \end{bmatrix},$$

where

$$u_j^{(1)} = \widehat{r}_{j,j}^{(0)},\ 2 \le j \le n, \quad t_j^{(1)} = \overline{\psi_{n-j}},\ 2 \le j \le n-1.$$

Moreover, the entries $a_{j,j}^{(1)}$ *and* $v_j^{(1)}$ *are defined by the following procedure:*

Set $v_1^{(1)} = \psi_{n-1}$, $a_{1,1}^{(1)} = \phi_{n-1}\widehat{r}_{1,1}^{(0)} + \overline{\psi}_{n-1}a_{1,2}^{(1/2)}$, *and* $\eta_2 = \phi_{n-1}$*;*
for $j = 2 \ : \ n-1$
1. $\overline{v_j^{(1)}} = \phi_{n-2+j}\overline{\psi_{n-j}}\eta_j + \overline{\psi_{n-2+j}}\phi_{n-j}$*;*
2. $a_{j,j}^{(1)} = \phi_{n-2+j}\eta_j\widehat{r}_{j,j}^{(0)} + \overline{\psi_{n-2+j}}a_{j,j+1}^{(1/2)}$*;*
3. $\eta_{j+1} = -\psi_{n+j-2}\overline{\psi_{n-j}}\eta_j + \phi_{n+j-2}\phi_{n-j}$.
end
Set $a_{n,n}^{(1)} = \eta_n\widehat{r}_{n,n}^{(0)}$.

This theorem says that the lower triangular part of A_1 and, therefore, the unknowns $d_1^{(1)}, \dots, d_n^{(1)}$, $t_2^{(1)}, \dots, t_{n-1}^{(1)}$, $\boldsymbol{u}^{(1)} \in \mathbb{C}^n$ and $\boldsymbol{v}^{(1)} \in \mathbb{C}^n$ can be evaluated at a linear cost whenever we know both the superdiagonal entries $a_{j,j+1}^{(1/2)}$ of $A_{1/2}$ and the elements $\widehat{r}_{j,j}^{(0)}$ emerging from the main diagonal in the construction of $A_{1/2}$. To compute these quantities efficiently, we devise suitable recurrence relations using the structural representation of R_0 provided by Theorem 4.12.

The updating of $R_0^{(1)}$, i.e., the construction of the matrix $R_0^{(1)}(\mathcal{G}_{n-1,n}(\gamma_1))^H \dots (\mathcal{G}_{2,3}(\gamma_{n-2}))^H$, can be carried out in a compact way at the cost of $O(n)$ ops by explicitly combining the columns of $R_0^{(1)}$ step by step. The updating of $R_0^{(2)}$ is also easily performed since the resulting matrix is lower triangular. Hence, we only need to specify the updating of $Q_0^{(1)}C$ and $Q_0^{(2)}C$. For the final updates of the latter matrices, we restrict ourselves to computing their diagonal and superdiagonal entries.

Recall from (4.10) that

$$C_0 = \begin{bmatrix} 0 & \overline{u_2^{(0)}}v_1^{(0)} & \overline{u_3^{(0)}t_2^{(0)}}v_1^{(0)} & \dots & \overline{u_n^{(0)}t_{n-1}^{(0)}\dots t_2^{(0)}}v_1^{(0)} \\ \vdots & \ddots & \overline{u_3^{(0)}}v_2^{(0)} & \dots & \overline{u_n^{(0)}t_{n-1}^{(0)}\dots t_3^{(0)}}v_2^{(0)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \overline{u_n^{(0)}}v_{n-1}^{(0)} \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix}$$

For demonstration consider the first two steps of the updating process applied to a 5×5 matrix C_0.

$$C_0 \overset{\mathcal{G}(\gamma_1)^H}{\to} \begin{bmatrix} 0 & \times & \times & \tilde{u}_4^{(0)}\overline{t_3^{(0)}t_2^{(0)}}v_1^{(0)} & \widehat{u}_5^{(0)}\overline{t_3^{(0)}t_2^{(0)}}v_1^{(0)} \\ 0 & 0 & \times & \tilde{u}_4^{(0)}\overline{t_3^{(0)}}v_2^{(0)} & \widehat{u}_5^{(0)}\overline{t_3^{(0)}}v_2^{(0)} \\ 0 & 0 & 0 & \tilde{u}_4^{(0)}v_3^{(0)} & \widehat{u}_5^{(0)}v_3^{(0)} \\ 0 & 0 & 0 & \eta_4 & \zeta_4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{\mathcal{G}(\gamma_2)^H}{\to}$$

$$\begin{bmatrix} 0 & \times & \tilde{u}_3^{(0)}\overline{t_2^{(0)}}v_1^{(0)} & \widehat{u}_4^{(0)}\overline{t_2^{(0)}}v_1^{(0)} & \widehat{u}_5^{(0)}\overline{t_3^{(0)}t_2^{(0)}}v_1^{(0)} \\ 0 & 0 & \tilde{u}_3^{(0)}v_2^{(0)} & \widehat{u}_4^{(0)}v_2^{(0)} & \widehat{u}_5^{(0)}\overline{t_3^{(0)}}v_2^{(0)} \\ 0 & 0 & \eta_3 & \zeta_3 & \widehat{u}_5^{(0)}v_3^{(0)} \\ 0 & 0 & \overline{\psi_2}\eta_4 & \phi_2\eta_4 & \zeta_4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.23}$$

where

$$\tilde{u}_4^{(0)} = \phi_1 \overline{u_4^{(0)}} + \overline{\psi_1 u_5^{(0)} t_4^{(0)}}, \quad \tilde{u}_3^{(0)} = \phi_2 \overline{u_3^{(0)}} + \overline{\psi_2 \tilde{u}_4^{(0)} t_3^{(0)}},$$

$$\widehat{u}_5^{(0)} = -\psi_1 \overline{u_4^{(0)}} + \phi_1 \overline{u_5^{(0)} t_4^{(0)}}, \quad \widehat{u}_4^{(0)} = -\psi_2 \overline{u_3^{(0)}} + \phi_2 \overline{\tilde{u}_4^{(0)} t_3^{(0)}},$$

and

$$\eta_4 = \overline{\psi_1 u_5^{(0)}} v_4^{(0)}, \quad \eta_3 = \overline{\psi_2 \tilde{u}_4^{(0)}} v_3^{(0)},$$

$$\zeta_4 = \phi_1 \overline{u_5^{(0)}} v_4^{(0)}, \quad \zeta_3 = \phi_2 \overline{\tilde{u}_4^{(0)}} v_3^{(0)}.$$

By proceeding in this way, we obtain the following description of $C_0(\mathcal{G}_{n-1,n}(\gamma_1))^H \ldots (\mathcal{G}_{j,j+1}(\gamma_{n-j}))^H$.

THEOREM 4.15. *We have*

$$C_0(\mathcal{G}_{n-1,n}(\gamma_1))^H \ldots (\mathcal{G}_{j,j+1}(\gamma_{n-j}))^H[1:n-1, j:n] =$$

$$\left[\begin{array}{c|cccc}
\tilde{u}_j^{(0)} \overline{t_{j-1}^{(0)} \ldots t_2^{(0)}} v_1^{(0)} & \widehat{u}_{j+1}^{(0)} \overline{t_{j-1}^{(0)} \ldots t_2^{(0)}} v_1^{(0)} & \ldots & \ldots & \widehat{u}_n^{(0)} \overline{t_{n-2}^{(0)} \ldots t_2^{(0)}} v_1^{(0)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\tilde{u}_j^{(0)} v_{j-1}^{(0)} & \widehat{u}_{j+1}^{(0)} v_{j-1}^{(0)} & \vdots & \vdots & \vdots \\
\hline
\eta_j & \zeta_j & \vdots & \vdots & \vdots \\
\overline{\psi_{n-j}} \eta_{j+1} & \phi_{n-j} \eta_{j+1} & \zeta_{j+1} & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\overline{\psi_{n-j} \ldots \psi_2} \eta_{n-1} & \overline{\psi_{n-j-1} \ldots \psi_2} \phi_{n-j} \eta_{n-1} & \ldots & \phi_2 \eta_{n-1} & \zeta_{n-1}
\end{array}\right]$$

where

$$\tilde{u}_{n-k}^{(0)} = \phi_k \overline{u_{n-k}^{(0)}} + \overline{\psi_k} \tilde{u}_{n-k+1}^{(0)} \overline{t_{n-k}^{(0)}}, \quad 1 \le k \le n-j, \ (\tilde{u}_n^{(0)} = \overline{u_n^{(0)}})),$$

$$\widehat{u}_{n-k+1}^{(0)} = -\psi_k \overline{u_{n-k}^{(0)}} + \phi_k \tilde{u}_{n-k+1}^{(0)} \overline{t_{n-k}^{(0)}}, \quad 1 \le k \le n-j,$$

and

$$\eta_{n-k} = \overline{\psi_k} \tilde{u}_{n-k+1}^{(0)} v_{n-k}^{(0)}, \quad \zeta_{n-k} = \phi_k \tilde{u}_{n-k+1}^{(0)} v_{n-k}^{(0)}, \quad 1 \le k \le n-j.$$

For $j = 2, \ldots, n-1$, let $\widehat{c}_{j,j}^{(0)}$ and $\widehat{c}_{j,j+1}^{(0)}$ denote the entries of the matrix

$$(Q_0^{(1)} + Q_0^{(2)}) C_0 (\mathcal{G}_{n-1,n}(\gamma_1))^H \ldots (\mathcal{G}_{j,j+1}(\gamma_{n-j}))^H$$

in positions (j, j) and $(j, j+1)$, respectively. The computation of these entries can be efficiently performed based on the recurrence relations which employ the generalized semiseparable structure of C_0, $Q_0^{(1)}$ and $Q_0^{(2)}$.

In view of Theorem 4.12, one easily finds that

$$\widehat{c}_{j,j}^{(0)} = \tilde{u}_j^{(0)}\phi_{n-2+j}[\mu_1 \textstyle\prod_{i=0}^{j-3} -\overline{\psi_{n+i}}, \mu_2 \prod_{i=1}^{j-3} -\overline{\psi_{n+i}}, \dots, \mu_{j-1}] \begin{bmatrix} \overline{t_{j-1}^{(0)} \dots t_2^{(0)}} v_1^{(0)} \\ \vdots \\ v_{j-1}^{(0)} \end{bmatrix} +$$

$$\rho_j\eta_j + \delta_j\overline{\psi_{n-j}}[\phi_{n-j-1}, \phi_{n-j-2}\psi_{n-j-1}, \dots, \tilde{\phi}\textstyle\prod_{i=1}^{n-1-j}\psi_i] \begin{bmatrix} \eta_{j+1} \\ \vdots \\ \eta_{n-1}\prod_{i=1}^{n-j-2}\overline{\psi_{n-j-i}} \\ 0 \end{bmatrix},$$

and, similarly,

$$\widehat{c}_{j,j+1}^{(0)} = \widehat{u}_{j+1}^{(0)}\phi_{n-2+j}[\mu_1 \textstyle\prod_{i=0}^{j-3} -\overline{\psi_{n+i}}, \mu_2 \prod_{i=1}^{j-3} -\overline{\psi_{n+i}}, \dots, \mu_{j-1}] \begin{bmatrix} \overline{t_{j-1}^{(0)} \dots t_2^{(0)}} v_1^{(0)} \\ \vdots \\ v_{j-1}^{(0)} \end{bmatrix} +$$

$$\rho_j\zeta_j + \delta_j\phi_{n-j}[\phi_{n-j-1}, \phi_{n-j-2}\psi_{n-j-1}, \dots, \tilde{\phi}\textstyle\prod_{i=1}^{n-1-j}\psi_i] \begin{bmatrix} \eta_{j+1} \\ \vdots \\ \eta_{n-1}\prod_{i=1}^{n-j-2}\overline{\psi_{n-j-i}} \\ 0 \end{bmatrix}.$$

Thus, the computation of $\widehat{c}_{j,j}^{(0)}$ and $\widehat{c}_{j,j+1}^{(0)}$ reduces to evaluation of two scalar products:

$$s_j^{(0)} = [\mu_1 \prod_{i=0}^{j-3} -\overline{\psi_{n+i}}, \mu_2 \prod_{i=1}^{j-3} -\overline{\psi_{n+i}}, \dots, \mu_{j-1}] \begin{bmatrix} \overline{t_{j-1}^{(0)} \dots t_2^{(0)}} v_1^{(0)} \\ \vdots \\ v_{j-1}^{(0)} \end{bmatrix},$$

and

$$q_{n-j}^{(0)} = [\phi_{n-j-1}, \phi_{n-j-2}\psi_{n-j-1}, \dots, \tilde{\phi}\prod_{i=1}^{n-1-j}\psi_i] \begin{bmatrix} \eta_{j+1} \\ \vdots \\ \eta_{n-1}\prod_{i=1}^{n-j-2}\overline{\psi_{n-j-i}} \\ 0 \end{bmatrix}.$$

Observe that

$$s_j^{(0)} = (-\overline{\psi_{n-3+j}})\overline{t_{j-1}^{(0)}} s_{j-1}^{(0)} + \mu_{j-1}v_{j-1}^{(0)}, \quad 2 \le j \le n \tag{4.24}$$

and

$$q_{n-j}^{(0)} = |\psi_{n-j-1}|^2 q_{n-j-1}^{(0)} + \phi_{n-j-1}\eta_{j+1}, \quad j = n-2, \dots, 1. \tag{4.25}$$

By using recursions (4.24) and (4.25) complemented with the initial conditions $s_1^{(0)} = 0$ and $q_1^{(0)} = 0$, we determine the entries $\widehat{c}_{j,j}^{(0)}$ and $\widehat{c}_{j,j+1}^{(0)}$ at a linear cost. Hence, the values $\widehat{r}_{j,j}^{(0)}$ as well as the superdiagonal entries of $A_{1/2}$ are also computed at a linear cost.

5. Implementation issues and numerical results. In this section we study the speed and the accuracy of the proposed algorithm. In particular, the structured QR iteration described in the previous sections has been implemented in MATLAB and then used for the computation of the eigenvalues of generalized semiseparable matrices of both small and large size. The results of extensive numerical experiments confirm the robustness and the efficiency of the proposed approach.

At each QR step (3.1) the matrix $A_s \in \mathbb{C}^{n\times n}$ is stored in a linear data structure of size $\simeq 6n$ which yields the vectors $\boldsymbol{d}^{(s)}$, $\boldsymbol{u}^{(s)}$ $\boldsymbol{v}^{(s)}$, $\boldsymbol{z}^{(s)}$, $\boldsymbol{w}^{(s)}$ and $\boldsymbol{t}^{(s)}$. The QR step (3.1) is performed as follows.

function $[\boldsymbol{d}^{(1)}, \boldsymbol{u}^{(1)}\ \boldsymbol{v}^{(1)}, \boldsymbol{z}^{(1)}, \boldsymbol{w}^{(1)}, \boldsymbol{t}^{(1)}] = \mathrm{QRSStep}(\boldsymbol{d}^{(0)}, \boldsymbol{u}^{(0)}\ \boldsymbol{v}^{(0)}, \boldsymbol{z}^{(0)}, \boldsymbol{w}^{(0)}\ \boldsymbol{t}^{(0)}, \sigma_0)$
% Compute the structured representation of A_1 generated by (3.1)
% A_1 is the matrix generated from A_0 after having performed one step
% of QR iteration with linear shift σ_0
Compute Givens rotations $\mathcal{G}_{n-1,n}(\gamma_1), \ldots, \mathcal{G}_{2,3}(\gamma_{n-2})$ by (4.5) and (4.6).
Set $a_{i,i}^{(0)} = d_i^{(0)} + z_i^{(0)}\overline{w_i^{(0)}} - \sigma_0$, $i = 1, \ldots, n$.
Find the generators of $\widehat{Q}_0 B_0$ as defined in Theorem 4.5.
Find Givens rotations $\mathcal{G}_{1,2}(\gamma_{n-1}), \ldots, \mathcal{G}_{n-1,n}(\gamma_{2n-3})$ as shown in Theorem 4.12.
Compute $\boldsymbol{z}^{(1)}$ and $\boldsymbol{w}^{(1)}$ as described in Remark 3.4.
Find the generators of $R_0^{(1)}$, $R_0^{(2)}$, $Q_0^{(1)}$ and $Q_0^{(2)}$ by means of Theorem 4.12.
Find $\boldsymbol{u}^{(1)}$ $\boldsymbol{v}^{(1)}$, $\boldsymbol{t}^{(1)}$ and $a_{i,i}^{(1)}$, $1 \le i \le n$, by using Theorem 4.14.
Evaluate $d_i^{(1)} = \mathrm{real}(a_{i,i}^{(1)} + \sigma_0 - z_i^{(1)}\overline{w_i^{(1)}})$, $i = 1, \ldots, n$.

Our implementation of `function QRSStep` requires $120\, n + O(1)$ multiplications and $28\ n + O(1)$ storage. The main program complements this routine with the following shifting strategy. At the beginning the shift parameter σ is equal to zero. If $A_s = (a_{i,j}^{(s)}) \in \mathbb{C}^{n\times n}$ satisfies

$$|a_{n,n}^{(s-1)} - a_{n,n}^{(s)}| \le 0.1|a_{n,n}^{(s-1)}|,$$

then we apply non-zero shifts by setting $\sigma_k = a_{n,n}^{(k)}$, $k = s, s+1, \ldots$. We say that $a_{n,n}^{(k)}$ provides a numerical approximation of an eigenvalue λ of A_0 whenever

$$|u_n^{(k)}| \max\{|v_{n-1}^{(k)}|, |t_{n-1}^{(k)}|\} \le eps\ |a_{n,n}^{(k)}|,$$

where eps is the machine precision, i.e., $eps \simeq 2.2 \cdot 10^{-16}$. If this condition is fulfilled, then we set $\lambda = a_{n,n}^{(k)}$ and deflate the matrix by restarting the process with the initial matrix being the leading principal submatrix of A_k of order $n-1$. After non-zero shifting has begun, we check for the convergence of the last diagonal entries of the currently computed iterate A_k. If convergence fails to occur after 15 iterations, then at the 16-th iteration we set $\sigma_k = 1.5\ (|a_{n,n}^{(k)}| + |u_n^{(k)} v_{n-1}^{(k)}|)$ and continue with non-zero shifting. If $a_{n,n}^{(k)}$ does not converge in the next 15 iterations, then the program reports failure.

Figures 5.1 and 5.2 show the results of our numerical experiments. Figure 5.1 covers our tests with unsymmetric arrowhead matrices A_0 obtained by setting $v_2^{(0)} = \ldots = v_n^{(0)} = 0$, $z_2^{(0)} = \ldots = z_n^{(0)} = 0$ and $t_2^{(0)} = \ldots = t_{n-1}^{(0)} = 1$, whereas the remaining entries $u_i^{(0)}$, $w_i^{(0)}$, $v_1^{(0)}$ and $z_1^{(0)}$ are random complex entries with real and imaginary

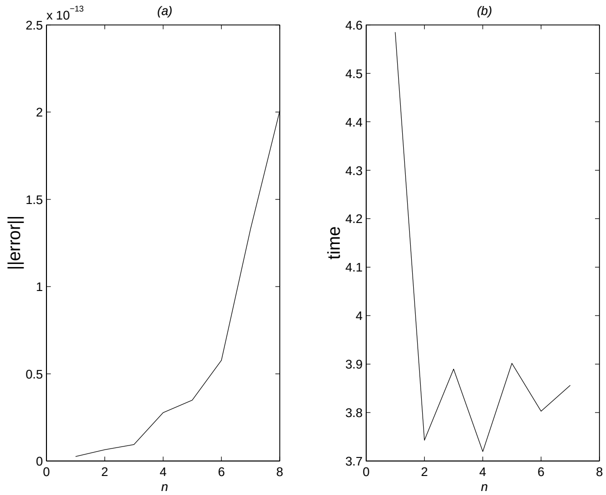

FIG. 5.1. *Eigenvalue computation for unsymmetric arrowhead matrices of size* $m(n) = 2^{2+n}$, $1 \leq n \leq 8$.

part ranging from -1 and 1 and $d_i^{(0)}$ are random real entries lying in the interval $[-1, 1]$. Figure 5.2 reports the results for Hermitian diagonal-plus-semiseparable matrices, where $\boldsymbol{w}^{(0)} = \boldsymbol{z}^{(0)}$, $t_2^{(0)} = \ldots = t_{n-1}^{(0)} = 1$, $u_i^{(0)}$ and $v_i^{(0)}$ are random complex entries with real and imaginary part ranging from -1 and 1 and $d_i^{(0)}$ are random real entries in the interval $[-1, 1]$.

Each figure contains two plots showing the values of the *errors* and running *time* for matrices A of size $m(n) = 2^{2+n}$ for $n = 1, \ldots, 8$. Our test program returns these values as the output. The error value is computed as the maximum of the minimum distance between each computed eigenvalue and the set of "true" eigenvalues computed by the function `eig` of MATLAB. For each size we carried out 100 numerical experiments. In each figure, the first plot reports the average value of the errors, and the second plot reports the ratio between the average values of running time for matrices having sizes $m(n)$ and $m(n+1)$. Since $m(n+1)/m(n) = 2$ and the proposed algorithm for computing all the eigenvalues of A is expected to have a quadratic cost, this ratio should be close to 4 for large $m(n)$.

Numerical experiments show that our algorithm for computing all eigenvalues of a generalized semiseparable matrix A is quite accurate. This algorithm does not perform any division except in the computation of the Givens rotations. In addition, since it employs unitary transformations only, coefficients growth at intermediate steps is quite moderate. Therefore, the propagation of absolute errors during the iterative process can be taken under control and this explains the robustness of the algorithm.

The pictures concerning running time confirm the effectiveness of our structured approach. According to our tests, the overall time for computing all the eigenvalues of A indeed changes roughly quadratically in n, in contrast to the classical QR method which requires $O(n^3)$ arithmetic operations and $O(n^2)$ storage.

6. Conclusion and future work. We devised a novel QR-based algorithm for computing all eigenvalues of a generalized semiseparable matrix. Structural representation of such a matrix is maintained at each step of QR iteration, which enables us to yield a linear time per step using a linear memory space. Extensive numerical

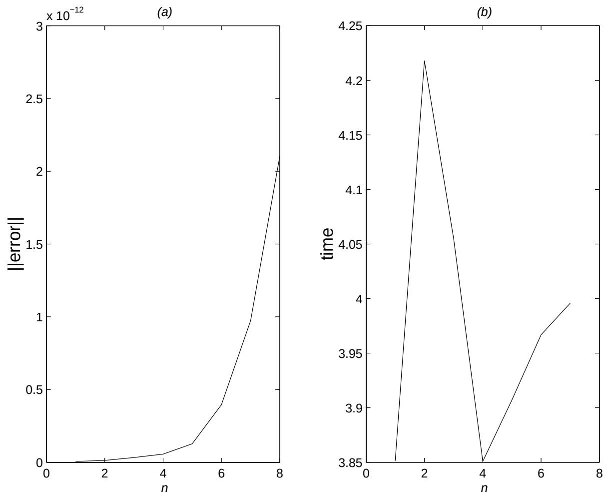

FIG. 5.2. *Eigenvalue computation for Hermitian diagonal-plus-semiseparable matrices of size* $m(n) = 2^{2+n}$, $1 \leq n \leq 8$.

experiments confirmed the effectiveness and the robustness of the proposed approach. A more refined implementation of the structured QR iteration including quadratic shifting techniques together with the optimization of the memory requirements is almost ready. This implementation should also be translated in Fortran to be compared with the LAPACK routines.

Several extensions and applications of our results are now under investigation. The application to the polynomial root-finding problem is immediate. Indeed, given a monic polynomial $p(z)$, a diagonal plus rank-one matrix A having $p(z)$ as its characteristic polynomial can be constructed at almost a linear cost. Such generalized companion matrices form a subset of the class of generalized semiseparable matrices. Our QR-like iteration applied to this matrix A provides an efficient polynomial root-finding algorithm with good convergence features. The theoretical properties as well as the numerical behavior of this QR-based polynomial root-finding algorithm will be described in a forthcoming paper.

Frobenius (companion) matrices are generally used to compute polynomial roots. Frobenius matrices do not belong to the class of generalized semiseparable matrices introduced here. Notwithstanding that, by generalizing the approach of this paper it is possible to develop an algorithm for computing the eigenvalues of Frobenius matrices which needs $O(n)$ ops per step and a linear storage [4]. However, practical experience with this algorithm shows that numerical errors can sometimes magnify so that computed results are less accurate that the ones computed by the plain QR iteration. The design of a fast and stable algorithm for Frobenius matrices is therefore an ongoing work which still deserves further investigations.

A further interesting topic is concerned with the use of our results for the computation of all the eigenvalues of a general real matrix A. It is well known that the inverse of a nonsingular upper Hessenberg matrix takes the form $R + \boldsymbol{u}\boldsymbol{v}^T$, where R is an upper triangular matrix and $\boldsymbol{u}\boldsymbol{v}^T$ is a rank-one matrix. Furthermore, some recent algorithms for the numerical treatment of symmetric diagonal plus semiseparable matrices (see [26] and the references therein) transform A into a matrix of the same form $R + \boldsymbol{u}\boldsymbol{v}^T$, without using intermediate recurrence to a Hessenberg matrix. The transformation employs only unitary matrices so it is numerically robust. Once the reduction has been carried out, the matrix $R + \boldsymbol{u}\boldsymbol{v}^T$ can be further transformed by a similarity transformation into a diagonal plus rank-one matrix B. The overall computational cost of computing B is $O(n^3)$ and the eigenvalues of B can be approximated by our algorithm at the total cost of $O(n^2)$. The second stage of the reduction from A to B, however, involves transformation matrices that generally are not unitary so that numerical difficulties could have arisen at this step. The design of a numerically stable algorithm that converts a real matrix A into a diagonal plus rank-one matrix by a similarity transformation as well as the experimental study of the numerical behavior of the latter algorithm are subjects of our work in progress.

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