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Abstract

In this paper, we define a Hilbert-style axiom system IPC_{CA} that conservatively extends intuitionistic propositional logic (IPC) by adding new classical atoms for which the law of excluded middle (LEM) holds. We establish completeness of IPC_{CA} with respect to an appropriate class of Kripke models. We show that IPC_{CA} is a conservative extension of both classical propositional logic (CPC) and also IPC. We further investigate the disjunction property in IPC_{CA}. In particular, we show that the disjunction property holds for every formula $A \vee B$ if either A or B does not contain classical atoms.[1]

In our previous paper, we discussed the intuitionistic version of a basic Logic of Proofs [Kurokawa, 2003]. From a semantical point of view the formulas of the form $x : F$, denoting x is a proof of F, behave as if they were classical propositions put on Kripke models of intuitionistic propositional logic. A question appeared of how we can describe a logic in which we have propositional variables of two sorts: those that behave classically and those that behave intuitionistically. What results is a logic that combines many essential features of IPC and CPC.

[1] The results of this paper were obtained in November 2003. Early in 2004, the author learned from an FOM posting about A. Sakharov's paper "Median Logic" (submitted on February 6, 2004 to the Mathematics Preprint Server), on a first-order intuitionistic logic with classical propositional atoms. In Sakharov's paper, a relevant proof system with some weak form of cut-elimination is given. Apart from a common design idea, this paper and "Median Logic" do not have a significant overlap.

The present work is not the first attempt at such a blending of the two logics. We will examine the history of the relationship between IPC and CPC.

There are numerous examples of the so-called "intermediate logics," which are obtained from IPC by adding axiom schemas that are weaker than those of classical logic. For instance, Dummett logic is obtained by adding the axiom schema $(A \to B) \vee (B \to A)$ to IPC. Theorems of this logic are valid in all linear Kripke models. For a survey of intermediate logics one could look at Chagrov and Zakharyashev[1997].

Dov Gabbay, in [1999] and other works, has developed a method of combining logics called "fibring logics". The scope of this method is extremely general, and it is applicable to almost any logics which are complete for some sort of possible world semantics. Its basic idea is the following. Suppose we want to combine two logics with different logical connectives. When $t \Vdash A$ holds, where t is from a semantics of one logic and A is a formula of the combined logics whose main connective is from the other logic, we associate a model of the other logic at the state t by using a function F, regard A as 'atomic' and use the rule $t \Vdash A$ iff $F(t) \Vdash A$. By using this idea, we can understand how the two different semantics are related when we evaluate a formula in a mixed language. In principle, this idea could be applied to CPC if we consider the semantics of CPC as a degenerate case of possible world semantics.

However, there is one twist here. In their [1996], Farias del Cerro and Herzig considered a more specific case that makes use of two types of implication: classical and intuitionistic. They take the union of the axioms of the both logics based on the two different implications, and the different impliciations are connected by a certain bridging axiom. Interestingly enough, they showed that careless combination of these two implications results in a collapse of the system to classical logic, so some special care must be exercised. In fact, in the same paper, Farias del Cerro and Herzig give a restricted system that is free from the collapse.[2] Though the work of Farias del Cerro and Herzig was done independently of Gabbay's idea of fibring, Gabbay's approach also needs careful treatment of this case. In his [1999], Gabbay himself puts a comment on it and refers to their paper about how to avoid the collapse. [3]

Now the question is whether Gabbay's general approach subsumes our ap-

[2]They point out that if they accept an unrestricted version of weakening axiom schema, the system will collapse into a classical system. The formulas substitutable into the schema have to be restricted.

[3]In their [2001], Sernadas, Rasga and Carnielli give another method called "modulated fibring" that can avoid this collapse, following the line of Gabbay's "fibring" and refining it by using the language of category theory.

proach of combining logic by using two kinds of atoms. Our answer to the question is that it does not. Gabbay seems to put focus on the case of a combination of two logics with different 'logical connectives'. Although Gabbay mentions the case where he has disjoint sets of atoms, he does not address the issue of directly restricting the behavior of a set of "atoms" by postulating new axioms regulating these atoms. So, as far as the current situation of research goes, our approach is different from Gabbay's.

In this paper, we emulate classical logic in IPC by adding classical propositional variables. A new axiom schema $X_i \vee \neg X_i$ restricts the behavior of these new propositional variables so that they act classically. We show that our system contains both IPC and CPC and does not collapse to classical logic.

Now we present a Hilbert-style system for the combined logic and also prove completeness of this system with respect to a certain class of Kripke models for intuitionistic logic; namely, we impose additional conditions on the forcing of the new (classical) propositional variables.

The language of IPC_{CA} consists of the usual intuitionistic propositional connectives and two sets of propositional variables: intuitionistic propositional variables

$$Var_I := \{p_1, \ldots, p_n, q_1, \ldots, q_n\}$$

and classical propositional variables

$$Var_C := \{X_1, \ldots, X_n\}\ .$$

The latter variables will satisfy an additional constraint that will provide for their classical behavior. In the following, we denote the language of propositional with variables in Var_I, Var_C and $Var_I \cup Var_C$ as L_{Var_I}, L_{Var_C} and $L_{VarI \cup Var_C}$ respectively.

Definition 1 *IPC_{CA} consists of the following axioms and rules.*[4]

- *Axioms of intuitionistic propositional logic*
  1. $A \to (B \to A)$
  2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
  3. $A \wedge B \to A$, $A \wedge B \to B$
  4. $A \to A \vee B$, $B \to A \vee B$

[4] IPC_{CA} is based on the Hilbert-style system of IPC, see [Trolestra and Schwichtenberg, 1996].

5. $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
6. $A \to (B \to (A \wedge B))$
7. $\bot \to A$

- *Axioms for classical propositional variables*[5]

  $X_i \vee \neg X_i$

- Modus ponens

  $$\frac{A \quad A \to B}{B}$$

Our Kripke-style semantics for IPC_{CA} is obtained by extending the forcing relation of IPC Kripke models to include our new classical propositional variables X_i. In particular, a model $\mathcal{K}$ is a triple $(K, \leq, \Vdash)$, where K is a set of states, $\leq$ a partial order on K, and $\Vdash$ a relation on $K \times (Var_I \cup Var_C)$. Without loss of generality, we assume that K is a forest with respect to the ordering $\leq$. We will denote roots of this forest by r_k. Additionally, $\Vdash$ satisfies:

- Monotonicity of intuitionistic propositional variables

  $$\text{if } \alpha \Vdash p_i \text{ and } \alpha \leq \beta \text{ then } \beta \Vdash p_i.$$

- **The new condition for classical atoms** For each X_i and for each root $r_k \in K$, exactly one of the following holds:
  - $\beta \Vdash X_i$ for all $\beta \geq r_k$
  - $\beta \nVdash X_i$ for all $\beta \geq r_k$

We extend the forcing relation $\Vdash$ in the usual way to include all formulas of the language IPC_{CA}. Namely, for $\alpha \in K$,

- $\alpha \nVdash \bot$
- $\alpha \Vdash A \wedge B$ iff $\alpha \Vdash A$ and $\alpha \Vdash B$
- $\alpha \Vdash A \vee B$ iff $\alpha \Vdash A$ or $\alpha \Vdash B$
- $\alpha \Vdash A \to B$ iff $\beta \Vdash A$ implies $\beta \Vdash B$ for all $\beta \geq \alpha$

We now prove the soundness of IPC_{CA}.

[5] $\neg X$ is an abbreviation for $X \to \bot$.

Theorem 2 (Soundness) *If $IPC_{CA} \vdash A$, then $\mathcal{K}, \alpha \Vdash A$ for all models $\mathcal{K} = (K, \leq, \Vdash)$ and for all $\alpha \in K$.*

Before going into the proof of the theorem, we will state and prove the following small lemma.

Lemma 3 (Extended Monotonicity) *Let C be a formula and $\mathcal{K} = (K, \leq, \Vdash)$ be a model. For all $\alpha, \beta \in K$ with $\beta \geq \alpha$, $\alpha \Vdash C$ implies $\beta \Vdash C$.*

Proof By induction on the complexity of the formula. Case 1) Both A and B consist of propositional variables from Var_I. Subcase 1.1. $C = A \wedge B$. Suppose $\alpha \Vdash A \wedge B$ and $\alpha \leq \beta$. Then, $\alpha \Vdash A$ and $\alpha \Vdash B$. By IH, $\beta \Vdash A$ and $\beta \Vdash B$. So $\beta \Vdash A \wedge B$ Subcase 1.2. $C = A \vee B$ is similar. Subcase 1.3. $C = A \rightarrow B$. Suppose $\alpha \Vdash A \rightarrow B$ and $\alpha \leq \beta$. Also, suppose that $\delta \geq \beta$ and $\delta \Vdash A$ to show $\delta \Vdash B$. By transitivity, $\alpha \leq \delta$. By definition, for all $\gamma \geq \alpha$, $\gamma \Vdash A \Rightarrow \gamma \Vdash B$. Take δ as γ. Then, $\delta \geq \alpha \Rightarrow (\delta \Vdash A \Rightarrow \delta \Vdash B)$. So we have $\delta \Vdash B$. This allows us to conclude $\beta \Vdash A \rightarrow B$.

Case 2) At least one of A, Bin C contains classical variables X_i from Var_C. The only difference is that at least one of the propositional variables is X_i. If X_i is forced at one node in a tree, then it will be forced everywhere in the tree. So X_i's are also trivially monotonic. So we do not have to make any difference about our induction hypothesis, since that automatically holds, too. Each case of induction is completely similar to purely intuitionistic case. ⊠

Now we get into the proof of the soundness theorem.

Proof of Soundness We need to show that all the axioms are valid for the above class of Kripke models and to show that the inference rule preserves validity. Since the class of our Kripke models is the class of usual Kripke models for intuitionistic propositional logic with some additional language and conditions and all classical propositional variables satisfy the axioms for intuitionistic propositional logic, we are essentially done for the IPC part of the axioms. Also, the rule of modus ponens preserves validity w.r.t. the class of our Kripke models. Suppose a formula A of IPC$_{CA}$ is valid w.r.t. our class of models and $A \rightarrow B$ in IPC$_{CA}$ is also valid in the same class of models. So, for all $\mathcal{K}$ and $\alpha \in K$, $\mathcal{K}, \alpha \Vdash A$ and for all $\mathcal{K}$ and all $\alpha \in K$, $\mathcal{K}, \alpha \Vdash A \rightarrow B$. So for all $\mathcal{K}$, the following holds. For all $\beta \geq \alpha, \beta \Vdash A \Rightarrow \beta \Vdash B$. Also, by the first assumption, for all $\beta \geq \alpha, \alpha \Vdash A$. So for all $\beta \geq \alpha, \alpha \Vdash B$. However, since α is arbitrary, for all $\alpha \in K$, $\alpha \Vdash B$. So for all $\mathcal{K}, \alpha \Vdash B$. so B is valid.(The proof is essentially the same as that for IPC)

The rest of the task is to show that one axiom schema for Var_C is valid w.r.t. Kripke models above.

Next, We want to prove the axiom $X_i \vee \neg X_i$ is valid w.r.t. our Kripke models. Suppose there exist a model $\mathcal{K}$ and a state α such that $\alpha \nVdash X_i \vee \neg X_i$. By extended monotonicity, at a root node $r_k \nVdash X_i \vee \neg X_i$. Then, we have $r_k \nVdash X_i$ and $r_k \nVdash \neg X_i$. By $r_k \nVdash \neg X_i$ and the recursive clause for negation, we get $\exists \gamma \geq r_k, \gamma \Vdash X_i$. On the other hand, we have the condition for classical atoms, i.e. either $\forall \beta \geq r_k, \beta \Vdash X_i$ or $\forall \beta \geq r_k, \beta \nVdash X_i$. The second disjunct of the new condition of the model contradicts $\exists \gamma \geq r_k, \gamma \Vdash X_i$. So we have $\forall \beta \geq r_k, \beta \Vdash X_i$. Now we take r_k as an instance of β. Then $r_k \nVdash X_i$, since $\geq$ is reflexive. Thus, we have both $r_k \nVdash X_i$ and $r_k \Vdash X_i$. This is a contradiction. This suffices to show the validity of the axiom for classical propositional variables. $\boxtimes$

We now proceed to the completeness theorem.

Theorem 4 (Completeness) *If a formula A is valid in all models $\mathcal{K}$, then $IPC_{CA} \vdash A$.*

Proof We prove the contrapositive of the statement of completeness. By construction of the canonical model by way of Kripke model for IPC and modifying that. Suppose $\text{IPC}_{CA} \nvdash A$. Take the set of all the subformulas of A, and call it $\text{Sb}(A)$. Let $X_0, X_1, ..., X_n$ be the list of all classical propositional variables in $\text{Sb}(A)$. And let q_0, $q_1, ..., q_n$ be sentence variables not occurring in X. To every $B \in Sb(A)$, we associate a formula of the language of propositional logic B^t such that B is the result of substituting all those occurrences of X_i for q_i throughout B^t for all i with $0 \leq i \leq n$. In other words, B^t is a formula such that if we substitute q_i for all occurrences of X_i for all $0 \leq i \leq n$ in B, then we get B^t. So we have the following situation here.

- B^t is a formula in the language of usual intuitionistic propositional logic.
- B is the result of substituting X_i for q_i for all i with $0 \leq i \leq n$.

Then consider the following sentences.

1. $q_i \vee \neg q_i$ for all i s.t. $0 \leq i \leq n$.

Obviously, substituting X_i for q_i in the above formulas would give us the axioms for classical part of IPC_{CA}. Make the conjunction of these.

$\bigwedge_{0 \leq i \leq n} (X_i \vee \neg X_i)$.

Then the following meta-statement holds between formulas of IPC_{CA} and those of IPC.

$IPC \vdash \bigwedge_{0\leq i\leq n}(q_i \vee \neg q_i) \to A^t \Rightarrow \mathrm{IPC}_{CA} \vdash A.$

Here we are assuming $\mathrm{IPC}_{CA} \nvdash A$. So we have

$\mathrm{IPC} \nvdash \bigwedge_{0\leq i\leq n}(q_i \vee \neg q_i) \to A^t.$

Then there exists a finite tree model of IPC (by means of Kripke completeness theorem and the finite tree theorem for IPC) [Smorynski,1973].

So, for some finite tree Kripke model based on a canonical model construction $\mathcal{K} = (K, \leq, \Vdash)$ with the root node r, $r \nVdash \bigwedge_{0\leq i\leq n}(q_i \vee \neg q_i) \to A^t$.

So there exists a $\beta \geq r, \beta \Vdash \bigwedge_{0\leq i\leq n}(q_i \vee \neg q_i)$ and $\beta \nVdash A^t$.

Here we want to define a new model that makes the above two statements true and has β as the root node.

First, take the cone above β and define a submodel of our original model. Call it $\mathcal{K}_\beta := (K_\beta, \leq_\beta, \Vdash_\beta)$, where $K_\beta = \{\alpha \in K : \alpha \geq \beta\}$, $\geq_\beta = \geq \cap (K_\beta \times K_\beta)$, $\Vdash_\beta = \Vdash \restriction (K_\beta \times L_{Var_I})$. Obviously, this model satisfies our desired condition.

Next, based on this submodel $\mathcal{K}_\beta$, we define a new model. We are still based on a modified canonical model construction for IPC. We now change the model $\mathcal{K}_\beta$ for IPC into IPC_{CA} by changing q_i into X_i everywhere it is necessary. Since our states consist of saturated set of formulas, we need to change all the occurrence of q_i in the saturated sets in $\mathcal{K}_\beta$ into X_i. We call a new set of states as K^*_β. Then, our partial order gets automatically changed, since it is just an inclusion relation between saturated sets. We call our new partial order as $\leq^*_\beta$. Finally, we define a new forcing relation. The new forcing relation is defined by the following. For all $\gamma \in K^*_\beta$,

- $\gamma \Vdash^*_\beta X_i \rightleftharpoons \gamma \Vdash_\beta p_i$ if $p_i = q_i$. [6]
- $\gamma \Vdash^*_\beta p_i \rightleftharpoons \gamma \Vdash_\beta p_i$ if $\mathrm{p}_i \neq q_i$.

[6]When we move from $\mathcal{K}^*_\beta$ to $\mathcal{K}_\beta$, we no longer have the same states in the two models, since each state of the model $\mathcal{K}^*_\beta$ has to be changed systematically, i.e. X_i should be changed to q_i at each occurrence. So, officially, we should use different symbols for states, but for the sake of simplicity, we keep using the same lower case Greek letters for states.

- $\Vdash^*_\beta$ satisfies the conditions of the recursive clauses for the intuitionistic connectives.

Based on this definition, we claim

Claim 5 *In $\mathcal{K}^*_\beta$, for any $\gamma \in K^*_\beta$ and for all formulas F, $\gamma \Vdash^*_\beta F \Leftrightarrow \gamma \Vdash_\beta F^t$.*

Proof By induction on the complexity of the formulas. Case 1. Atoms. 1.1. $F = X_i$. By definition, $\gamma \Vdash^*_\beta X_i \Leftrightarrow \gamma \Vdash_\beta q_i \Leftrightarrow \gamma \Vdash_\beta F^t$. 1.2. $F = p_i$. By definition, $\gamma \Vdash^*_\beta p_i \Leftrightarrow \gamma \Vdash_\beta F^t$. (Here t is vacuous.) Case 2. $F = A \wedge B$. Then $\gamma \Vdash^*_\beta A \wedge B \Leftrightarrow \gamma \Vdash^*_\beta A$ and $\gamma \Vdash^*_\beta B \Leftrightarrow (IH) \gamma \Vdash_\beta A^t$ and $\gamma \Vdash_\beta B^t \Leftrightarrow \gamma \Vdash_\beta (A \wedge B)^t$. Case 3. $F = A \vee B$. This case is similar to conjunction. Case 4. $F = A \rightarrow B$. $\gamma \Vdash^*_\beta A \rightarrow B \Leftrightarrow \forall \delta \geq \gamma, \delta \Vdash^*_\beta A \Rightarrow \delta \Vdash^*_\beta B \Leftrightarrow (IH) \forall \delta \geq \gamma, \gamma \Vdash_\beta A^t \Rightarrow \gamma \Vdash_\beta B^t \Leftrightarrow \gamma \Vdash_\beta (A \rightarrow B)^t$. ⊠

By this claim, we now have $\beta \nVdash^*_\beta A$. We have constructed a particular model. So we can claim that there is such a model $\mathcal{K}$. By taking the contrapositive, we have shown (for all $\mathcal{K}$ and for all α, $\mathcal{K}, \alpha \Vdash A$) $\Rightarrow$ IPC$_{CA} \vdash A$.

The only remaining thing is to check whether this model $\mathcal{K}^*_\beta = (K^*_\beta, \leq^*_\beta, \Vdash^*_\beta)$ satisfies the condition of a model for IPC$_{CA}$. The only condition we need to check is that for any $X_i \in Var_C$, $[(\forall \gamma \in K^*_\beta, \gamma \Vdash^*_\beta X_i)$ or $(\forall \gamma \in K^*_\beta, \gamma \nVdash^*_\beta X_i)]$.

Claim 6 *For any $X_i \in Sb(A)$, exactly one of the following holds:*

- *$\gamma \Vdash^*_\beta X_i$ for all $\gamma \geq^*_\beta \beta$*
- *$\gamma \nVdash^*_\beta X_i$ for all $\gamma \geq^*_\beta \beta$*

Proof Suppose not for contradiction. Then for some $X_i, \exists \gamma \geq^*_\beta \beta, \gamma \nVdash^*_\beta X_i$ and $\exists \gamma \geq^*_\beta \beta, \gamma \Vdash^*_\beta X_i$. Pick a particular γ_1 s.t. $\gamma_1 \geq^*_\beta \beta$ and $\gamma_1 \nVdash^*_\beta X_i$. Also, pick a particular γ_2 s.t. $\gamma_2 \geq^*_\beta \beta$ and $\gamma_2 \Vdash^*_\beta X_i$. By the above definition of our partial order $\geq^*_\beta$ and our forcing relation $\Vdash^*_\beta, \gamma_1 \geq_\beta \beta$ and $\gamma_1 \nVdash_\beta q_i$. In the same way, we get $\gamma_2 \geq_\beta \beta$ and $\gamma_2 \Vdash_\beta q_i$. However, in the above argument to establish the completeness of IPC$_{CA}$, our $\Vdash_\beta$ satisfies the condition that $\beta \Vdash_\beta \bigwedge_{0 \leq i \leq n} (q_i \vee \neg q_i)$. In particular, $\beta \Vdash_\beta q_i \vee \neg q_i$. So $\beta \Vdash_\beta q_i$ or $\beta \Vdash_\beta \neg q_i$. Since $\mathcal{K}_\beta$ is a submodel of $\mathcal{K}$, monotonicity still holds in $\mathcal{K}_\beta$. So by monotonicity of the case where $\beta \Vdash_\beta q_i$, $\gamma_1 \Vdash_\beta q_i$, which is contradictory to $\gamma_1 \nVdash_\beta q_i$. Also, $\beta \Vdash_\beta \neg q_i$ implies that for all $\delta \geq_\beta \beta$, $\delta \nVdash_\beta q_i$. In particular, $\gamma_2 \nVdash_\beta q_i$, which is contradictory to $\gamma_2 \Vdash_\beta q_i$. So, anyways we get a contradiction. ⊠

Now, we have completed the proof of Kripke completeness for IPC$_{CA}$. $\boxtimes$

Next, we state some important properties of IPC$_{CA}$.

Definition 7 *Let S be a logic in the language L. Let S' be a logic in a language $L' \supseteq L$. S' is called a* conservative extension *of S if whenever A is a formula in language L and $S' \vdash A$, then $S \vdash A$.*

Now we prove the following two statements.

Theorem 8 *IPC$_{CA}$ is a conservative extension of IPC.*

Theorem 9 *IPC$_{CA}$ is a conservative extension of CPC.*

Proof Proof of the former theorem. We show for any $A \in L_{Var_I}$, IPC $\nvdash$ A $\Rightarrow$ IPC$_{CA}$ $\nvdash$ A. By means of Kripke completeness and soundness of both logics, it suffices to prove the following.

Proposition 10 *If for an IPC model $\mathcal{K} = (K, \leq, \Vdash)$ and some $\alpha \in K$ we have $\alpha \nVdash A$, then there exists a model $\mathcal{K}^*_\beta$ with a state $\gamma \geq^*_\beta \beta$ such that $\gamma \nVdash^*_\beta A$, where β is a node of a finite tree model $\mathcal{K}$ such that $\beta \Vdash \bigwedge_{0 \leq i \leq n}(q_i \vee \neg q_i)$ as above.*

Proof Suppose for some $\mathcal{K}$ and for some $\alpha \in K, \alpha \nVdash A$. By the finite tree theorem, we can construct a finite tree model from $\mathcal{K}$. For the sake of simplicity, we also call the finite tree model as $\mathcal{K}$. So, for some finite tree Kripke model $\mathcal{K}$ for IPC with the root node r_i, $\mathcal{K}, r_i \nVdash^* A$. Since $A \in L_{Var_I}$, $\bigwedge_{0 \leq i \leq n}(q_i \vee \neg q_i)$ in the construction of a model $\mathcal{K}^*_\beta$ is an empty conjunction. So we do not have to worry about β that is not identical with r_i in K itself. We can simply take r_i of $\mathcal{K}$ as β, so our model $\mathcal{K}^*_\beta$ is $\mathcal{K}$ itself. So there exists $\mathcal{K}^*_\beta$ and γ s.t. $\gamma \geq^*_\beta \beta, \gamma \nVdash^*_\beta A$. $\boxtimes$

Now if $IPC \nvdash A$, then by the completeness theorem, for some $\mathcal{K}$ and $\alpha \in K$, $\alpha \nVdash A$. By the proposition, for some $\mathcal{K}^*_\beta$ and some $\gamma \geq^*_\beta \beta, \gamma \nVdash^*_\beta A$. By the soundness theorem of IPC$_{CA}$, IPC$_{CA}$ $\nvdash A$. $\boxtimes$ (Theorem 8)

Proof Proof of the latter theorem. We want to show for $B \in L_{Var_C}, CPC \nvdash B \Rightarrow$ IPC$_{CA}$ $\nvdash B$. By Kripke soundness of IPC$_{CA}$ and completeness of CPC, it suffices to show the following.

Proposition 11 *If for a model $\mathcal{M}$ of CPC we have $\mathcal{M} \nvDash B$, then there exists a model $\mathcal{K}^*_\beta$ with a state $\gamma \geq^*_\beta \beta$ such that $\gamma \nVdash^*_\beta B$.*

Proof Suppose for some model of classical propositional logic $\mathcal{M}, \mathcal{M} \not\models B$. Since $B \in L_{Var_C}$, if we consider the behavior of B as a formula of IPC$_{CA}$, it is entirely determined by whether it is forced or not at the root node of a model of IPC$_{CA}$. This means that a degenerate Kripke model $\mathcal{K}^*_\beta$ for IPC$_{CA}$ with only one state β, partial order $\{<\beta, \beta>\}$ and valuation $\beta \Vdash p \Leftrightarrow \mathcal{M} \models p$ makes the same formula true as $\mathcal{M}$ does. By supposition for the proposition and by using this degenerate Kripke model, we have $\mathcal{K}^*_\beta, \beta \nVdash^*_\beta B$. ⊠

Now if we have $CPC \nvdash B$, then for some $\mathcal{M}, \mathcal{M} \not\models B$. So by the proposition, we have some $\mathcal{K}^*_\beta$ with one node β, s.t. $\beta \nVdash^*_\beta B$. So we have IPC$_{CA} \nvdash B$. ⊠(Theorem 9)

Let us think about the consequence of these conservativeness results. Theorem 8 above essentially suffices to show that IPC$_{CA}$ does not collapse to classical logic, i.e. it is not the case that classical laws hold for all formulas of $L_{Var_I \cup Var_C}$. We can easily see this by using a simple example of formulas with L_{Var_I} that *is not* provable in IPC$_{CA}$ but would be provable if we took the logical connectives as classical.

Corollary 12 *$IPC_{CA} \nvdash p \vee \neg p$*

Proof $p \vee \neg p \in L_{Var_I}$, and by conservativeness we have $IPC \nvdash p \vee \neg p \Rightarrow$ IPC$_{CA} \nvdash p \vee \neg p$. But $\nvdash p \vee \neg p$. So IPC$_{CA} \nvdash p \vee \neg p$. ⊠

Next, we show that a certain version of the disjunction property holds for this logic. First, we need to introduce the following notions.

Definition 13 *Let F_I be a formula that has no classical atoms. Let F_C be a formula that has classical atoms and no intuitionistic atoms. Let F_M be a formula containing both intuitionistic and classical atoms.*

Theorem 14 (The Disjunction Property, DP)

1. *$IPC_{CA} \vdash F_I \vee G_I \Rightarrow IPC_{CA} \vdash F_I$ or $IPC_{CA} \vdash G_I$*
2. *$IPC_{CA} \vdash F_I \vee G_M \Rightarrow IPC_{CA} \vdash F_I$ or $IPC_{CA} \vdash F_M$*
3. *$IPC_{CA} \vdash F_M \vee G_M \not\Rightarrow IPC_{CA} \vdash F_M$ or $IPC_{CA} \vdash G_M$*

Proof The key idea of the proof is to apply "the gluing method" of two (tree-style) Kripke models into one Kripke model.

First of all, we consider the contrapositive of each statement. By completeness theorem, there must be a model for each formula F and G s.t. $\mathcal{K}_1, r_1 \nVdash^*_{r_1} F$ and $\mathcal{K}_2, r_2 \nVdash^*_{r_2} G$, (where r_i (i=1 or 2) is the root node for each model). We will try to combine these two models into one model that is a model of $F \vee G$. Here is how we do it.

1. Add one node r_k with no intuitionistic variables forced below the two root nodes of the two Kripke models of F and G.

2. Extend the p.o.'s of the models to $K \cup \{r_k\}$ by putting all the necessary ordered pairs into the p.o. and combine them into a new p.o. with the root node r_k.

3. Then, we may have one glued Kripke model if we can successfully define a new forcing relation, though we may fail. Whether the theorem holds or not depends on whether both of F and G contain classical variables or not.

   (1) If we do not have any classical variables in either F or G, then we can safely do the gluing procedure of the two models into one in a consistent manner, since at r_k we force neither F nor G.

   (2) If we have classical variables in only one of F and G, then we first parse the formula by using a forcing condition of the formula and check whether a variable is forced at the root node or not. If we find it forced at the root by the forcing condition of the entire formula containing the variable, then we have to force it variables at r_k. Otherwise, we just leave the model unchanged. This change does not affect the other formula, since the other formula has nothing to do with the newly forced classical variable. So we can glue two models in a consistent manner in this case, too.

   However, if we have any classical variables in both F and G, then it may be the case that we have some conflict about the forcing relation for X_i, because we need to have either X_i or $\neg X_i$ in all connected nodes by the definition of our models of IPC$_{CA}$. So gluing does not necessarily work in this case and in general we may not be able to get a new glued model.

Now we go through all the cases. Note that though we define F_C, F_M separately, all the cases of F_C in this theorem can be treated as a special case of F_M

where F_M has no intuitionistic propositional variables. By regarding F_C as a special case of F_M, we can reduce the number of cases in the theorem and the proof.

Case 1) This is just the same as DP in IPC.

Case 2) If X_i in G_M (or G_C) is forced at r_M, then do gluing with forcing X_i at r_k. If X_i is not forced at r_M, then do gluing without X_i at r_k. The resulting model will be a model $\mathcal{K}$, s.t. $\mathcal{K}, r_k \nVdash F_I \vee G_M$.

For the case 3), there may be a conflict between forcing X_i at r_1 and forcing X_i at r_2, because we may have classical variables forced at all the nodes in one model and in the other model we may have negation of these forced at all the nodes. We may not be able to construct a new model with the desired property. So DP does not necessarily hold in the case 3). Here we will give a particular counter example for this case.

We will show the case where both F_M, G_M have classical variables and intutionisitic variables in detail. In other cases where one of them has (or both have) only classical variables, we have obvious counter examples such as $\mathrm{IPC}_{CA} \vdash X \vee \neg X$ and $\mathrm{IPC}_{CA} \vdash X \vee (X \to p)$. Now we have $(X \to p) \vee (p \to X)$ as a counter example. We claim that $\mathrm{IPC}_{CA} \vdash (X \to p) \vee (p \to X)$ and $\mathrm{IPC}_{CA} \nvdash X \to p$ and $\mathrm{IPC}_{CA} \nvdash p \to X$. We can prove the formula purely syntactically.[7] We give an example of a deduction using IPC_{CA}, since we have not done any in this paper.

1. $\mathrm{IPC}_{CA} \vdash X \to (p \to X)$ Ax.1
2. $\mathrm{IPC}_{CA} \vdash X \to ((p \to X) \to (X \to p) \vee (p \to X))$ Ax.4, Ax.1, MP
3. $\mathrm{IPC}_{CA} \vdash X \to ((X \to p) \vee (p \to X))$ L1, L2, MP
4. $\mathrm{IPC}_{CA} \vdash X \to (\bot \to p)$ Ax.1, Ax.7, MP
5. $\mathrm{IPC}_{CA} \vdash (X \to \bot) \to (X \to p)$ L4, Ax.2, MP
6. $\mathrm{IPC}_{CA} \vdash \neg X \to (X \to p)$ by Def. of $\bot$
7. $\mathrm{IPC}_{CA} \vdash \neg X \to ((X \to p) \to ((X \to p) \vee (p \to X)))$ Ax.4, Ax.1, MP

[7]Here is one remark about a semantics of this formula. This formula is an analogue of Dummett formula in the context of intermediate logic. In our semantics, we do not have to take linear ordered models in order to validate this formula.

8. $\mathrm{IPC}_{CA} \vdash \neg X \to ((X \to p) \vee (p \to X))$ L6, L7, Ax.2, MP
9. $\mathrm{IPC}_{CA} \vdash (X \vee \neg X) \to ((X \to p) \vee (p \to X))$ L3, L8, Ax.5, MP
10. $\mathrm{IPC}_{CA} \vdash (X \to p) \vee (p \to X)$ Ax. for Var_C, MP

Now we construct a counter model for each disjunct. For $X \to p$, we can take any model in which $r_1 \nVdash p$ and $\forall \beta \geq r_1, \beta \Vdash X$. For $p \to X$, we can take any model in which $r_2 \Vdash p$ and $\forall \beta \geq r_2, \beta \nVdash X$. Each of them is obviously a counter model of the respective formula, and we cannot glue them into one model because of the conditions for X.

These counter examples are sufficient to show that the case 3) does not hold in general. ⊠

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