

City University of New York (CUNY)

CUNY Academic Works

Publications and Research

York College

2016

Development and Separation of Forced Convective Flow

Adefemi Sunmonu
CUNY York College

[How does access to this work benefit you? Let us know!](#)

More information about this work at: https://academicworks.cuny.edu/yc_pubs/213

Discover additional works at: <https://academicworks.cuny.edu>

This work is made publicly available by the City University of New York (CUNY).
Contact: AcademicWorks@cuny.edu

Development and Separation of Forced Convective Flow

Adefemi Sunmonu

Department of Mathematics & Computer Science
York College
The City University of New York
Jamaica, NY 11451, USA

Copyright © 2016 Adefemi Sunmonu. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

Singularities are considered in the solution of the laminar boundary - layer equation at a position of separation. The works of Howarth (1938). Goldstein (1948), Stewartson (1958), Terrill (1960) and Akinrelere [(1981), (1982)] are reviewed to fully establish the existence of singularity in the incompressible boundary layer at separation for both the velocity and thermal fields. A flow at a large Reynold's number along an immersed solid surface around which boundary layer is formed through which the velocity rises rapidly from zero at the surface to its value in the main stream is considered. It is found that whenever separation does occur, the boundary layer equations cease to be valid on the upstream side and also down-stream of separation.

In this paper, the works of Akinrelere [(1981),(1982)] on the thermal field had been extended to include suction through a porous surface. Following Stewartson (1958) the stream function ψ_1 is expanded in a series of the type

$$\psi_1 = 2^{\frac{3}{2}} \xi^3 \sum_{r=0}^6 \xi^r f_r(\eta) + 2^{\frac{3}{2}} \xi^8 \ln \xi [F_5(\eta) + \xi F_6(\eta)] + O(\xi^{10} \ln \xi),$$

where $\xi = x_1^{\frac{1}{4}}$, $\eta = y_1/2^{\frac{1}{2}} x_1^{\frac{1}{4}}$ and (x_1, y_1) are non-dimensional distances measured from the separation point. Analytical solutions for f_1, f_2, f_3, g_1, g_2 and g_3 are presented. Results are obtained both for arbitrary Prandtl number σ and $\sigma = 1$, with and without suction.

Mathematics Subject Classification: 35K55, 35J47, 93C10, 74S05

Keywords: development of flow, separation of flow, forced convective flow, laminar flow, boundary layer flow, boundary layer separation, convective heat transfer, laminar boundary layer, flow velocity, incompressible boundary layer, prandtl number, two dimensional boundary layer

1 Introduction

One of the most important advances in fluid mechanics was contributed by Prandtl in 1904 when he showed that fluid motion around objects could be divided into two regions a thin region close to the object where frictional effects are important (boundary layer) and an outer region where friction is negligible. For a real fluid at high Reynold's numbers, the boundary layer which grows along the body surface is thin and laminar at the up-stream end, but as the layer moves on along the body the continual action of shear stress tends to slow down additional fluid particles causing the thickness of the boundary layer to increase with distance from the up stream point.

The fluid in the layer is also subjected to an adverse pressure gradient that increases its momentum at the expense of the kinetic energy. If this adverse pressure gradient is large enough, then separation can occur followed by a region of reversed flow. Down stream of the point of separation there exists a region of eddying, irregular flow called the wake. This region forms essentially a new boundary in the flow different from that of the bounding surface. Since the boundary between the separated region and the wake is unknown it is impossible to use potential flow theory to predict the flow properties down stream of the point of separation. In fact, large cross velocities are to be expected at separation making the basic assumptions of boundary layer theory invalid at and near separation [4].

Several authors have discussed the problem of singularities in the solution of the laminar boundary layer equations at a position of separation using an external velocity distribution of the form

$$u = U_o \left(1 - \frac{x}{L}\right) \quad (1)$$

where U_o, L are characteristic speed and length respectively.

Until 1938, no work has been reported on possible singularities at separation and as a matter of fact, no analytical solution is known yet for a boundary layer flow involving separation, the methods used are approximate and numerical [4]. Howarth (1938) used a series solution to predict the position of separation and to calculate a velocity profile at separation. Howarth's result has been checked by Hatree [4] using a direct numerical process. In his work

Hatree showed that there was a strong suggestion of the existence of a singularity at the point of separation, both because of the nature of the skin-friction in its neighborhood, and because of a breakdown of the numerical process if any attempt was made to obtain real accuracy at and near separation.

Goldstein (1948) undertook to find some formula that would hold near the singularity (revealed by the work of Hatree) by constructing an asymptotic solution for the immediate neighborhood of the separation. Hatree was quite certain that in his solution, the skin friction $\left. \frac{\partial U_1}{\partial y_1} \right|_{y_1=0}$ behaved near $x_1 = 0$ like a multiple of x_1^r where r is certainly less than 1 and greater than 1/4.

Thus taking

$$U_1 = \frac{1}{2}y_1^2 + a_3y^3 + a_4y^4 + \dots \quad \text{at } x_1 = 0 \quad (2)$$

we find that

$$\left. \frac{\partial U_1}{\partial y_1} \right|_{y_1=0} = 2^{3/2} \left\{ \alpha_1 x_1^{1/2} + \alpha_2 x_1^{3/4} + \alpha_3 x_1 + \alpha_4 x_1^{5/4} + \dots \right\} \quad (3)$$

where α 's are constants [4].

Ten years later, Stewartson (1958) showed that the expansion (3) was incomplete and that logarithmic terms must be added at some stage of the expansion. But these new terms necessitate the need to satisfy certain integral conditions inherent in the Goldstein solution. Also at $x = x_s, y \neq 0$ (x_s is the position of separation), some of the added terms become infinite and so the velocity profile (2) can no longer be considered valid. Unfortunately, it is not clear from the work of Stewartson how the infinite number of arbitrary constants $\alpha_1, \alpha_5, \alpha_9, \dots$ in his solution can be determined. Terrill (1960) extended this work of Stewartson to include suction through porous surface, and also confirmed the need for logarithmic terms to be added to the series. This present work takes after the Stewartson-Terrill approach.

Even though, with all the above notable works in mind, one can conclude that "the existence of the singularity at separation for the velocity field is fully established and understood", it was not until 1981 that Akinrelere succeeded in establishing singularity at separation for the temperature field. This work extends the latter of the two works of Akinrelere in this area to include continuous suction through a porous surface. Suction will be accounted for by prescribing a non-zero normal velocity component $V_s(x) < 0$ at the wall, where the fluid absorbed is assumed to be so small that only fluid particles in the immediate neighborhood of the wall are sucked away.

2 Basic Equations

The momentum equation for the steady incompressible laminar boundary layer flow is

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \quad (4)$$

The continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (5)$$

is satisfied by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (6)$$

where ψ is a stream function.

The boundary conditions are

$$\left. \begin{aligned} u &= 0, & \text{at } y &= 0 \\ v &= -\left(U_o \frac{\nu}{L}\right)^{1/2} V_s(x), & \text{at } y &= 0 \\ u &\rightarrow U \text{ as } y \rightarrow \infty \end{aligned} \right\} \quad (7)$$

where x is the distance measured along the surface of the fluid, y is the distance normal to the surface, u and v are velocity components in the directions of increasing x and increasing y respectively- L is a characteristic displacement- $V_s(x)$ is the given non-dimensional suction velocity. We shall non-dimensionalize by setting

$$x_1 = \frac{x_s - x}{L}, \quad y_1 = \frac{\sqrt{R}y}{L} \quad (8)$$

$$L = \frac{U_s}{\frac{dU_s}{dx}}, \quad U_s = U(x_s), \quad u_1 = u(x_1, y_1), \quad v_1 = v(x_1, y_1) \quad (9)$$

where the subscript 's' refers to the position of separation.

Setting

$$u = u_1 U_s, \quad v = \frac{v_1 U_s}{\sqrt{R}}, \quad \psi = \frac{\psi_1 U_s}{\sqrt{R}} \quad (10)$$

in equations (4) and (6) we obtain

$$-u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} = U_1 \frac{dU_1}{dx_1} + \frac{\partial^2 u_1}{\partial y_1^2} \quad (11)$$

and

$$u_1 = \frac{\partial \psi_1}{\partial y_1}, \quad v_1 = \frac{\partial \psi_1}{\partial x_1} \quad (12)$$

respectively.

We assume that the velocity profile at $x_1 = 0$ (at separation) is given by

$$u_1(0, y_1) = a_1 y_1 + a_2 y_1^2 + a_3 y_1^3 + \dots \tag{13}$$

the pressure gradient by

$$-\frac{dP}{dx_1} = U_1 \frac{dU_1}{dx_1} = - [P_0 + P_1 x_1 + P_2 x_1^2 + P_3 x_1^3 + \dots] \tag{14}$$

and the suction velocity by

$$v_1(x_1, 0) \equiv -V_s(x_1) = - [V_0 + V_1 x_1 + V_2 x_1^2 + V_3 x_1^3 + \dots] \tag{15}$$

It is known that if there is to be a solution without singularities certain equations must be satisfied, that is

$$\begin{aligned} a_1 &= 0 \\ 2a_2 + P_0 &= 0 \\ 3!a_3 &= P_0 V_0 \\ 5!a_5 + 2a_1 P_1 &= P_0 V_0^3 \\ \dots &\vdots \dots \end{aligned} \tag{16}$$

It is important to note here that the condition $a_1 = 0$ implies that we are looking for the solution of a flow over an impermeable surface near a position of separation. The numerical solution of Hartree as cited by Goldstein (1948) shows that the skin-friction $\left. \frac{\partial u_1}{\partial y_1} \right|_{y_1=0}$ behaved near $x_1 = 0$ like a multiple of x_1^r where $1/4 \leq r < 1$. Thus, we take

$$u_1 = \frac{1}{2} y_1^2 + a_3 y_1^3 + a_4 y_1^4 + \dots \text{ at } x_1 = 0 \tag{17}$$

and so,

$$\left. \frac{\partial u_1}{\partial y_1} \right|_{y_1=0} = 2^{3/2} \left(\alpha_1 x_1^{1/4} + \alpha_2 x_1^{3/4} + \alpha_3 x_1 + \alpha_4 x_1^{5/4} + \dots \right) \tag{18}$$

where the α 's are constants and the constant $2^{3/2}$ is inserted to conform with the notation to be discussed latter.

Now the energy equation is given by:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\nu}{\sigma} \frac{\partial^2 T}{\partial y^2} \tag{19}$$

with boundary conditions

$$\left. \begin{aligned} T &= T_w \\ T &\rightarrow T_0 \text{ as } y \rightarrow \infty \end{aligned} \right\} \tag{20}$$

where T_w is the wall temperature.

Setting $\theta = \frac{T-T_w}{T_0-T_w}$ and non-dimensionalizing as in the case of the velocity field using equations (8), (9) and (10) we obtain

$$-u_1 \frac{\partial \theta}{\partial x_1} + v_1 \frac{\partial \theta}{\partial y_1} = \frac{1}{\sigma} \frac{\partial^2 \theta}{\partial y_1^2} \quad (21)$$

Following the method used by Terrill (1960) and Akinrelele (1981), we take

$$\xi = x_1^{1/4}, \quad \eta = \frac{y_1}{\sqrt{2}x_1^{1/4}}, \quad (22)$$

and

$$\psi_1 = 2^{3/2} \xi^3 f(\xi, \eta), \quad \theta_1 = g(\xi, \eta) \quad (23)$$

where

$$f(\xi, \eta) = \sum_{r=0}^6 f_r(\eta) \xi^r + \xi^5 \ln \xi [F_5(\eta) + \xi F_6(\eta)] + \dots \quad (24)$$

and

$$g(\xi, \eta) = \sum_{r=0}^3 g_r(\eta) \xi^r + \xi^4 \ln \xi [G_5(\eta) + \dots] + \dots \quad (25)$$

Then

$$u_1 = 2\xi^2 \frac{\partial f}{\partial \eta}, \quad \frac{\partial u_1}{\partial y_1} = \sqrt{2}\xi \frac{\partial^2 f}{\partial \eta^2}, \quad \frac{\partial^2 u_1}{\partial y_1^2} = \frac{\partial^3 f}{\partial \eta^3} \quad (26)$$

$$\begin{aligned} \frac{\partial u_1}{\partial x_1} &= \frac{\partial}{\partial \eta} \left(2\xi^2 \frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial x_1} + \frac{\partial}{\partial \xi} \left(2\xi^2 \frac{\partial f}{\partial \eta} \right) \frac{\partial \xi}{\partial x_1} \\ &= -\frac{\xi^{-2}}{2} \eta \frac{\partial^2 f}{\partial \eta^2} + \frac{\xi^{-2}}{2} \left(2 \frac{\partial f}{\partial \eta} + \xi \frac{\partial^2 f}{\partial \xi \partial \eta} \right) \end{aligned}$$

$$\begin{aligned} v_1 &= 2^{3/2} \xi^2 \left(3f + \xi \frac{\partial f}{\partial \xi} \right) \left(\frac{1}{4} x_1^{-3/4} \right) + 2^{3/2} \xi^3 \frac{\partial f}{\partial \eta} \left(\frac{-\eta}{4\xi^4} \right) \\ &= 2^{-1/2} \xi^{-1} \left[3f + \xi \frac{\partial f}{\partial \xi} - \eta \frac{\partial f}{\partial \eta} \right]. \end{aligned} \quad (27)$$

substituting these results into (11) we obtain

$$\frac{\partial^3 f}{\partial \eta^3} - 3f \frac{\partial^2 f}{\partial \eta^2} + 2 \left(\frac{\partial f}{\partial \eta} \right)^2 + \xi \left(\frac{\partial f}{\partial \eta} * \frac{\partial^2 f}{\partial \xi \partial \eta} - \frac{\partial^2 f}{\partial \eta^2} * \frac{\partial f}{\partial \xi} \right) - \sum_{r=0}^{\infty} P_r \xi^{4r} = 0 \quad (28)$$

Applying similar procedure to the temperature field gives

$$\frac{1}{\sigma} \frac{\partial^2 g}{\partial \eta^2} - 3f \frac{\partial g}{\partial \eta} - \xi \left(\frac{\partial f}{\partial \eta} * \frac{\partial g}{\partial \xi} - \frac{\partial f}{\partial \xi} * \frac{\partial g}{\partial \eta} \right) = 0. \tag{29}$$

The boundary conditions are

$$f(\xi, 0) = 0, \quad \frac{\partial f}{\partial \eta}(\xi, 0) = 0, \quad g(\xi, 0) = 0$$

and that neither $f(\xi, \eta)$ nor $g(\xi, \eta)$ can be exponentially large at infinity. Also,

$$\frac{1}{\sqrt{2\xi}} \left(3f + \xi \frac{\partial f}{\partial \xi} \right) = -V_s(x_1) \text{ at } \eta = 0 \tag{30}$$

Using the expression (15) in (30) we get

$$\begin{aligned} f(\xi, 0) &= -2^{-3/2} \sum_{r=0}^{\infty} \frac{V_r \xi^{4r+1}}{r+1} \\ &= -2^{-3/2} \left(V_0 \xi + \frac{1}{2} V_1 \xi^5 + \dots \right) \end{aligned} \tag{31}$$

3 Velocity Field Without Suction

The formal solution for the motion upstream is found by writing

$$\xi = x_1^{1/4}, \eta = \frac{y_1}{2^{1/2}} x_1^{1/4} \tag{32}$$

$$\psi_1 = 2^{3/2} \xi^3 f(\xi, \eta) \tag{33}$$

where

$$f(\xi, \eta) = \sum_{r=0}^6 f_r(\eta) \xi^r + \xi^5 \ln [F_5(\eta) + \eta F_6(\eta)] + \dots \tag{34}$$

Then

$$u_1 = 2\xi^2 \frac{\partial f}{\partial \eta}, v_1 = \frac{1}{2^{1/2}\xi} \left(3f + \xi \frac{\partial f}{\partial \eta} \right) \tag{35}$$

and substituting these into (11), the equation of motion is

$$\begin{aligned} &\frac{\partial^3 f}{\partial \eta^3} - 3f \frac{\partial^2 f}{\partial \eta^2} + 2 \left(\frac{\partial f}{\partial \eta} \right)^2 \\ &+ \eta \left(\frac{\partial f}{\partial \eta} + \frac{\partial^2 f}{\partial \xi \partial \eta} - \frac{\partial^2 f}{\partial \eta^2} \frac{\partial f}{\partial \xi} \right) - \sum_{r=0}^{\infty} P_r \xi^{4r} = 0 \end{aligned} \tag{36}$$

with boundary condition

$$\frac{\partial f}{\partial \eta} = 0 \text{ at } \eta = 0. \quad (37)$$

From (36) the leading term of $f(\xi, \eta)$ in (34) is

$$f_0''' - 3f_0''f_0 + 2(f_0')^2 = 1 \quad (38)$$

subject to the boundary conditions

$$f_0'(0) = f_0(0) = 0.$$

From the definition of u_1 (given by (17))

$$u_1 = \xi^2\eta^2 + 2^{3/2}a_3\xi^3\eta^3 + \dots \quad (39)$$

using (22). Recall that using $u_1 = \frac{\partial \psi_1}{\partial y_1}$ which by (24) becomes

$$u_1 = 2\xi^2 [f_0'(\eta) + \xi f_1'(\eta) + \xi^2 f_2'(\eta) + \dots] \quad (40)$$

Equating coefficients of powers of ξ in (39) and (40) we have

$$2f_0' = \eta^2 \implies f_0 = \frac{\eta^3}{6} \quad (41)$$

is the solution of (38) satisfying the given boundary conditions. Alternatively, (38) can be solved by assuming a polynomial of the form

$$f_0(\eta) = a + b\eta + c\eta^2 + d\eta^3.$$

Furthermore,

$$\begin{aligned} 2f_1' &= 2^{3/2}a_3\eta^3, & \text{i.e. } \frac{f_1'}{\eta^3} &= \sqrt{2}a_3 \\ 2f_2' &= 2^2a_4\eta^4, & \text{i.e. } \frac{f_2'}{\eta^4} &= 2a_4. \end{aligned}$$

Continuing in this way we get the r th term as

$$2\frac{f_r'}{\eta^{r+2}} = a_{r+2}2^{(r+1)/2}.$$

In the limit as $\eta \rightarrow \infty$ we obtain in general

$$\lim_{\eta \rightarrow \infty} \frac{f_r'}{\eta^{r+2}} = 2^{r+2}a_{r+2}, \quad r = 0, 1, 2, \dots \quad (42)$$

which f_r satisfies provided $F(\xi, |\eta\xi)$ is negligible.

Substituting (23) and (24) into (28) we have (considering $f(\xi, \eta) = \sum_{r=0}^n \xi^r f_r(\eta)$ only for now),

$$\frac{\partial f}{\partial \eta} = \sum_{r=0}^n \xi^r f'_r(\eta), \quad \frac{\partial^2 f}{\partial \eta^2} = \sum_{r=0}^n \xi^r f''_r(\eta), \quad \frac{\partial^3 f}{\partial \eta^3} = \sum_{r=0}^n \xi^r f'''_r(\eta),$$

$$\frac{\partial f}{\partial \xi} = \sum_{r=1}^n r \xi^{r-1} f_r(\eta), \quad \frac{\partial^2 f}{\partial \xi \partial \eta} = \sum_{r=1}^n r \xi^{r-1} f'_r(\eta).$$

Substituting these into (28) yields

$$\sum_{r=0}^n \{ \xi^r f'''_r - (3\xi^r f_r)(\xi^r f''_r) + 2(\xi^r f_r)(\xi^r f'_r) \}$$

$$\xi \left[\sum_{r=0}^n \{ (\xi^r f'_r)(r \xi^{r-1} f'_r) - (\xi^r f''_r)(r \xi^{r-1} f_r) \} \right] + \sum_{r=0}^{\infty} P_r \xi^{4r} = 0$$

Now notice that

$$\sum_{r=0}^n \xi^r f'''_r = f'''_0 + \xi f'''_1 + \xi^2 f'''_2 + \dots \tag{43}$$

and

$$-3 \sum_{r=0}^n (\xi^r f_r)(\xi^r f''_r) = -3(f_0 + \xi f_1 + \xi^2 f_2 + \dots) * (f''_0 + \xi f''_1 + \xi^2 f''_2 + \dots)$$

In this expression, the coefficient of ξ^r is

$$-3(f_0 f''_r + f_1 f''_{r-1} + \dots + f_r f''_0) = -3(f_0 f''_r + f_r f''_0 + \dots + f_1 f''_{r-1} + f_2 f''_{r-2} + \dots + f_{r-1} f''_1)$$

$$= -3(f_0 f''_r + f_r f''_0 + \sum_{s=1}^{r-1} f_s f''_{r-s}) \tag{44}$$

Also

$$2 \sum_{r=0}^n (\xi^r f'_r) * (\xi^r f'_r) = 2 (f'_0 + \xi f'_1 + \xi^2 f'_2 + \dots)^2$$

The coefficient of ξ^r in this case is

$$2(f'_0 f'_r + f'_1 f'_{r-1} + \dots + f'_r f'_0) = 2(2f'_0 f'_r + \sum_{s=1}^{r-1} f'_s f'_{r-s}) \tag{45}$$

and

$$\begin{aligned}
 \xi \sum_{r=0}^n (\xi^r f_r') * (r \xi^{r-1} f_r') &= (f_0' + \xi f_1' + \xi^2 f_2' + \dots) * (0 + \xi^0 f_1' + 2\xi f_2' + \dots \\
 &+ r \xi^r f_r' + \dots) \xi \\
 &= r f_0' f_r' + (r-1) f_1' f_{r-1}' + \dots + f_{r-1}' f_1' \\
 &= r f_0' f_r' + \sum_{s=1}^{r-1} f_s' f_{r-s}' (r-s)
 \end{aligned} \tag{46}$$

Similarly,

$$-\xi \sum_{r=0}^n (\xi^r f_r'') * (r f_r \xi^{r-1}) = r f_0'' f_r + \sum_{s=1}^{r-1} f_s'' f_s (r-s) \tag{47}$$

Gathering results (43)-(47) together, we have as the sum of the coefficients of ξ^r :

$$\begin{aligned}
 &f_r''' - 3(f_0'' f_r + f_0 f_r'') + (r+4) f_0' f_r' - r f_0'' f_r \\
 &= \sum_{s=1}^{r-1} [(r-s+3) f_s'' f_{r-s} - (r-s+2) f_s' f_{r-s}'] + \sum_{r=0}^{\infty} P_{r/4}
 \end{aligned} \tag{48}$$

Using $f_0 = \frac{1}{6} \eta^3$, earlier obtained, (48) becomes

$$f_r''' - \frac{1}{2} \eta^3 f_r'' + \frac{1}{2} (r+4) \eta^2 f_r' - (r+3) \eta f_r \equiv G_r, \quad (r = 0, 1, 2, \dots, n) \tag{49}$$

$G_1 = 0$, and for $r \geq 2$

$$= \sum_{s=1}^{r-1} [(r-s+3) f_s'' f_{r-s} - (r-s+2) f_s' f_{r-s}'] + \sum_{r=0}^{\infty} P_{r/4} \tag{50}$$

where $P_{r/4} = 0$ if $r/4$ is not an integer.

The boundary conditions are

$$f_r'(0) = f_r(0) = 0, \quad r = 0, 1, \dots \tag{51}$$

The equations for f_r ($r=1,2,3,4$) are

$$f_1''' - \frac{1}{2} \eta^3 f_1'' + \frac{5}{2} \eta^2 f_1' - 4 \eta f_1 = 0 \tag{52}$$

$$f_2''' - \frac{1}{2} \eta^3 f_2'' + 3 \eta^2 f_2' - 5 \eta f_2 = 4 f_1'' f_1 - 3 (f_1')^2 \tag{53}$$

$$f_3''' - \frac{1}{2} \eta^3 f_3'' + \frac{7}{2} \eta^2 f_3' - 6 \eta f_3 = 5 f_1'' f_2 - 4 f_1' f_2' + 4 f_2'' f_1' - 3 f_2' f_1' \tag{54}$$

$$f_4''' - \frac{1}{2} \eta^3 f_4'' + 4 \eta^2 f_4' - 7 \eta f_4 = 6 f_1'' f_3 - 5 f_1 f_2'' + 5 f_2'' f_2 - 4 (f_2')^2 + 4 f_3'' f_1' - 3 f_3' f_1' + P_1 \tag{55}$$

We now proceed to solve the first three of the above equations first without suction. The equation for $f_1(\eta)$ is

$$f_1''' - \frac{1}{2}\eta^3 f_1'' + \frac{5}{2}\eta^2 f_1' - 4\eta f_1 = 0$$

with boundary conditions

$$f_1(0) = f_1'(0) = 0.$$

Assuming $f_1(\eta) = A + B\eta + C\eta^2 + D\eta^3 + E\eta^4$, we get $B = D = 0$ and

$$f_1(\eta) = A + C\eta^2 + \frac{A}{6}\eta^4 \tag{56}$$

Using the boundary conditions, we obtain

$$f_1(\eta) = \alpha_1 \eta^2 \tag{57}$$

where α_1 is a constant. Equation (42) then gives

$$a_3 = \frac{1}{\sqrt{2}} \lim_{\eta \rightarrow \infty} \frac{f_1'}{\eta^3} \equiv 0. \tag{58}$$

The equation for $f_2(\eta)$ is

$$f_2''' - \frac{1}{2}\eta^3 f_2'' + 3\eta^2 f_2' - 5\eta f_2 = 4f_1'' f_1 - 3(f_1')^2 \equiv -4\alpha_1^2 \eta^2 \tag{59}$$

with boundary conditions

$$f_2(0) = f_2'(0) = 0.$$

To find the solution of (59) we assume a trial solution of the form

$$f_2(\eta) = a\eta + b\eta^2 + c\eta^3 + d\eta^4 + e\eta^5 \tag{60}$$

Following the same procedure as in the case of $f_1(\eta)$ above we obtain

$$f_2(\eta) = \alpha_2 \eta^2 - \frac{\alpha_1^2}{15} \eta^5 \tag{61}$$

where α_2 is a constant. From (42) we have

$$4! a_4 = -4\alpha_1^2. \tag{62}$$

The equation for f_3 becomes

$$f_3''' - \frac{1}{2}\eta^3 f_3'' + \frac{7}{2}\eta^2 f_3' - 6\eta f_3 = -10\alpha_1 \alpha_2 \eta^2 - \frac{4}{3} \alpha_1^3 \eta^5 \tag{63}$$

after substituting for f_1 and f_2 .

Before attempting to solve equation (63) we note that equations for all succeeding f_r 's are non-homogeneous linear equations with the right hand side rapidly becoming more and more complicated. The complementary functions involve integrals of confluent hypergeometric functions (See Appendix for more details).

We now discuss some of the complementary functions involved in the solution of f_3 .

$${}_1F_1(a, b, x) = 1 + \frac{a}{1 \cdot b}x + \frac{a(a+1)}{2!b(b+1)}x^2 + \dots \quad (64)$$

$$\equiv \sum_{m=0}^{\infty} \frac{\Gamma(a+m)}{\Gamma(b+m)} * \frac{\Gamma(b)}{\Gamma(a)} * \frac{x^m}{\Gamma(m+1)} \quad (65)$$

then writing $x!$ for $\Gamma(x+1)$,

$$\bar{g}_r = - \sum_{m=0}^{\infty} \frac{(m - \frac{3}{2} - \frac{r}{4})! \frac{1}{4}! \eta^{4m+1}}{m!(-\frac{3}{2} - \frac{r}{4})!(m + \frac{1}{4})! 8^m (4m-1)} \quad (66)$$

$$= \eta - \eta^2 \int_0^{\eta} \eta^{-2} \left[{}_1F_1(-\frac{1}{2} - \frac{1}{4}r, \frac{5}{4}, \frac{\eta^4}{8}) - 1 \right] d\eta \quad (67)$$

and

$$\begin{aligned} \bar{h}_r &= - \sum_{m=0}^{\infty} \frac{(m - \frac{7}{4} - \frac{r}{4})! (-\frac{1}{4})! \eta^{4m}}{m!(-\frac{7}{4} - \frac{r}{4})!(m - \frac{1}{4})! 8^m (2m-1)} \\ &= 1 - 2\eta^2 \int_0^{\eta} \eta^{-3} \left[{}_1F_1(-\frac{3}{4} - \frac{1}{4}r, \frac{3}{4}, \frac{\eta^4}{8}) - 1 \right] d\eta \end{aligned} \quad (68)$$

We observe that the term $(m - \frac{3}{2} - \frac{r}{4})!$ in \bar{g}_r gives a problem at $r = 4m + 2$ (since \bar{g}_r is exponentially large at this value). Hence \bar{g}_r must terminate at this value if it is to contain no infinite term. Similarly, \bar{h}_r terminates at $r = 4m + 1$.

When x is large and positive [5],

$$\begin{aligned} {}_1F_1(a, b, x) &\approx \frac{(b-1)!}{(a-1)!} e^x x^{a-b} \left\{ 1 + \frac{(b-a)(1-a)}{x} \right. \\ &\quad \left. + \frac{(b-a)(b-a+1)(1-a)(2-a)}{2!x^2} + \dots \right\} \end{aligned} \quad (69)$$

So that if $x = \eta^4/8, b = 5/4, a = -\frac{1}{2} - \frac{1}{4}r$, then

$$\begin{aligned}
 {}_1F_1\left(-\frac{1}{2} - \frac{1}{4}r, \frac{5}{4}, \frac{\eta^4}{8}\right) &\approx \frac{\left(\frac{1}{4}\right)!}{\left(-\frac{3}{2} - \frac{1}{4}r\right)!} \left(\frac{\eta^4}{8}\right)^{\frac{-7-r}{4}} \exp\left(\frac{\eta^4}{8}\right) (1 + \dots) \\
 &= \frac{\left(\frac{1}{4}\right)!}{\left(-\frac{3}{2} - \frac{1}{4}r\right)!} \eta^{-(7+r)} 8^{(7+r)/4} \exp\left(\frac{\eta^4}{8}\right) (1 + \dots) \\
 &= \frac{\left(\frac{1}{4}\right)!}{\left(-\frac{3}{2} - \frac{1}{4}r\right)!} \eta^{-(7+r)} 2^{(21+3r)/4} \exp\left(\frac{\eta^4}{8}\right) (1 + \dots) \\
 &\approx \frac{\frac{1}{4}\left(-\frac{3}{4}\right)!}{\left(-\frac{3}{2} - \frac{1}{4}r\right)!} \eta^{-(7+r)} 2^{(13+3r)/4} 2^2 \exp\left(\frac{\eta^4}{8}\right) (1 + \dots) \\
 &\approx \frac{\left(-\frac{3}{4}\right)!}{\left(-\frac{3}{2} - \frac{1}{4}r\right)!} \eta^{-(7+r)} 2^{(13+3r)/4} \exp\left(\frac{\eta^4}{8}\right) (1 + \dots) \tag{70}
 \end{aligned}$$

$r \neq 4m + 2$. In a similar way,

$${}_1F_1\left(-\frac{3}{4} - \frac{1}{4}r, \frac{3}{4}, \frac{\eta^4}{8}\right) \approx 2^{(3r+10)/4} \left(-\frac{5}{4}\right)! \eta^{-(r+6)} \exp\left(\frac{\eta^4}{8}\right) (1 + \dots), \tag{71}$$

$r \neq 4m + 2$. Hence

$$\bar{g}_r \approx 2^{(3r+17)/4} \left(-\frac{3}{4}\right)! \eta^{-(r+10)} \exp\left(\frac{\eta^4}{8}\right) (1 + \dots), \quad r \neq 4m + 1, \tag{72}$$

and

$$\bar{h}_r \approx \frac{\left(-\frac{5}{4}\right)!}{\left(-\frac{7}{4} - \frac{1}{4}r\right)!} \eta^{-(10+r)} 2^{(18+3r)/4} \exp\left(\frac{\eta^4}{8}\right) (1 + \dots), \quad r \neq 4m + 1. \tag{73}$$

Exponentially large terms must be avoided in the solution for f_r , Goldstein(1948) found that \bar{g}_r and \bar{h}_r must occur in the combination

$$\left(-\frac{5}{4}\right)! \left(-\frac{3}{2} - \frac{r}{4}\right)! \bar{g}_r + 2^{1/4} \left(-\frac{3}{4}\right)! \left(-\frac{7}{4} - \frac{r}{4}\right)! \bar{h}_r \tag{74}$$

which is also a complementary function for equation (49) [5].

We now turn to solving the equation for $f_3(\eta)$. The equation for f_3 is equation (63) with boundary conditions

$$f_3(0) = f_3'(0) = 0.$$

The general integral with a double zero at the origin is found by Goldstein

$$f_3 = \alpha_3 \eta^2 + 4\alpha_1 \alpha_2 (\eta - \bar{g}_3) - \frac{8}{3} \alpha_1^3 \left(1 + \frac{\eta^4}{4} - \bar{h}_3\right) \tag{75}$$

where α_3 is a constant. To avoid exponentially large terms in f_3 , \bar{g}_3 and \bar{h}_3 must appear in (75) in the same combination as in

$$\begin{aligned} k_3 &= \bar{h}_3 - \frac{3 * 2^{1/4} \pi^{3/2}}{10(\frac{1}{4}!)^3} \bar{g}_3 \\ &\approx \frac{\pi}{160(\frac{1}{4}!)^2} \left\{ \eta^6 - \frac{15}{2! \eta^2} + \frac{15 * 1 * 3!}{2 * 3! \eta^6} + \dots \right\} \\ &+ \frac{2\pi \eta^2}{32(\frac{1}{4}!)^2} \left\{ 4 \ln \eta + 2 \ln 2 + \nu - \frac{\pi}{2} - 5 \right\} \end{aligned} \quad (76)$$

as found by Goldstein (1948).

Therefore we must have

$$\alpha_2 = \frac{2^{1/4} * \pi^{3/2}}{5(\frac{1}{4}!)^3} \alpha_1^2 \quad (77)$$

and

$$\begin{aligned} f_3 &= \alpha_3 \eta^2 - \frac{8}{3} \alpha_1^3 \left\{ 1 + \frac{\eta^4}{4} - \bar{h}_3 - \frac{3 * 2^{1/2} * \pi^{3/2}}{10(\frac{1}{4}!)^3} (\eta - \bar{g}_3) \right\} \\ &= \alpha_3 \eta^2 - \frac{8}{3} \alpha_1^3 \left\{ 1 + \frac{\eta^4}{4} - k_3 \right\} \end{aligned} \quad (78)$$

using (77).

It follows then from (42) that

$$a_5 = -2^{1/2} \pi \alpha_1^3$$

The other f's are very involved- for example f_4 involve f_1, f_2, f_3, P_1 and so on. We shall stop our treatments here and introduce suction into these equations.

4 Velocity Field With Suction

We start this section by listing the relevant equations we want to use:

$$f_r''' - \frac{1}{2} \eta^3 f_r'' + \frac{1}{2} (r+4) \eta^2 f_r' - (r+3) \eta f_r = G_r \quad (79)$$

where

$$G_r = \sum_{s=1}^{r-1} [(r-s+3) f_s'' f_{r-s} - (r-s+2) f_s' f_{r-s}'] + P_{r/4}. \quad (80)$$

The boundary conditions (51) become

$$\begin{aligned} f'_r(0) &= 0 & (r = 0, 1, 2, 3, 4, 5) \\ f_r(0) &= 0 & (r = 0, 1, 2, 3, 4) \\ f_1(0) &= -2^{-3/2}V_o \\ f_5(0) &= -2^{-5/2}V_1. \end{aligned}$$

The equation for $f_0(\eta)$ is

$$f_0''' - 3f_0f_0'' + 2(f_0')^2 = 1 \quad (81)$$

with boundary conditions

$$f_0(0) = f_0'(0) = 0.$$

The solution of (81) satisfying these conditions is

$$f_0(\eta) = \frac{1}{6}\eta^3 \quad (82)$$

as before.

We shall endeavor to present the solutions of the following equations here.

$$f_1''' - \frac{1}{2}\eta^3 f_1'' + \frac{5}{2}\eta^2 f_1' - 4\eta f_1 = 0 \quad (83)$$

with boundary conditions

$$f_1'(0) = 0, \quad f_1(0) = -2^{-3/2}V_o \quad (84)$$

$$f_2''' - \frac{1}{2}\eta^3 f_2'' + 3\eta^2 f_2' - 5\eta f_2 = 4f_1'' f_1 - 3(f_1')^2 \quad (85)$$

with boundary conditions

$$f_2'(0) = f_2(0) = 0. \quad (86)$$

and

$$f_3''' - \frac{1}{2}\eta^3 f_3'' + \frac{7}{2}\eta^2 f_3' - 6\eta f_3 = 5f_1'' f_2 - 4f_1' f_2' + 4f_2'' f_1 - 3f_2' f_1' \quad (87)$$

with boundary conditions

$$f_3'(0) = f_3(0) = 0. \quad (88)$$

We now proceed to solve the equations one after the other. The equation for f_1 is given by (83) subject to the boundary conditions (84). By assuming a trial solution of the form

$$f_1(\eta) = a + b\eta + c\eta^2 + d\eta^3 + e\eta^4 + h\eta^5$$

the solution is found to be

$$f_1(\eta) = \alpha_1^2 - 2^{-3/2}V_0 \left(1 + \frac{1}{6}\eta^4\right) \quad (89)$$

where α_1 is a constant.

Then

$$a_3 = -\frac{V_0}{3!} \quad (90)$$

from (42).

Using f_1 in the equation for $f_2(\eta)$ we now have

$$f_2''' - \frac{1}{2}\eta^3 f_2'' + 3\eta^2 f_2' - 5\eta f_2 = -4\alpha_1^2\eta^2 - 2^{3/2}V_0\alpha_1 \left(1 + \frac{\eta^4}{6}\right) + V_0^2\eta^2 \quad (91)$$

with boundary conditions

$$f_2'(0) = f_2(0) = 0.$$

The solution of equation (91) satisfying these boundary conditions is found as before to be

$$f_2(\eta) = -\frac{1}{15}\alpha_1\eta^5 + \alpha_2\eta^2 - \frac{2^{1/2}}{3}\alpha_1V_0\eta^3 + \frac{1}{60}V_0^2\eta^5 \quad (92)$$

where α_2 is a constant and (42) gives

$$a_4 = \frac{V_0^2 - 4\alpha_1^2}{4!}. \quad (93)$$

Proceeding to equation for f_3 , we have

$$\begin{aligned} f_3''' - \frac{1}{2}\eta^3 f_3'' + \frac{7}{2}\eta^2 f_3' - 6\eta f_3 &= -10\alpha_1\alpha_2\eta^2 - \frac{4}{3}\alpha_1^3\eta^5 + 2^{1/2}V_0 \left[-2\alpha_2 + 4\alpha_1^2\eta^3 - \frac{1}{2}\alpha_2\eta^4\right] \\ &+ \alpha_1V_0^2 \left[4\eta + \frac{\eta^5}{3}\right] - \frac{2^{1/2}}{3}V_0^3\eta^3 \end{aligned} \quad (94)$$

subject to the boundary conditions

$$f_3'(0) = f_3(0) = 0.$$

Following the same treatments as in the latter part of section 2.1, the general solution of (94) satisfying these boundary conditions is

$$\begin{aligned} f_3(\eta) &= \alpha_3\eta^2 + 4\alpha_1\alpha_2\eta - \frac{8}{3}\alpha_1^3 \left(1 + \frac{\eta^4}{4}\right) + \frac{8}{3}\alpha_1^3k_3 \\ &+ \frac{2^{1/2}}{3}V_0 \left(\frac{1}{10}\alpha_1^2\eta^6 - \alpha_2\eta^3\right) + \frac{1}{6}\alpha_1V_0^2\eta^4 \\ &- \frac{2^{3/2}}{6!}V_0^3\eta^6 \end{aligned} \quad (95)$$

From equation (42)

$$a_5 = -\frac{2^{1/2}\pi\alpha_1^3}{40\left(\frac{1}{4}!\right)^2} - \frac{V_0}{5!} (V_0^2 - 12\alpha_1^2)$$

Note that from the expression for k_3 (equation (76)), α_2 is found to be

$$\alpha_2 = \frac{2^{1/4}\pi^{3/2}\alpha_1^2}{5\left(\frac{1}{4}!\right)^3} \tag{96}$$

as before.

The remaining f 's are complicated to obtain and are not used in our computation of the temperature field.

5 The Temperature Field

Recall that the equation of motion for the temperature field is

$$\frac{1}{\sigma} \frac{\partial^2 g}{\partial \eta^2} - 3f \frac{\partial g}{\partial \eta} + \xi \left(\frac{\partial f}{\partial \eta} \frac{\partial g}{\partial \xi} - \frac{\partial f}{\partial \xi} \frac{\partial g}{\partial \eta} \right) = 0 \tag{97}$$

with boundary condition

$$g(\xi, 0) = 0.$$

The leading term of $g(\xi, \eta)$ in (97) is

$$\frac{1}{\sigma} g_0'' - 3f_0 g_0' + \xi \left(f_0' \frac{\partial g_0}{\partial \xi} - g_0' \frac{\partial f_0}{\partial \xi} \right) = 0. \tag{98}$$

Using $f_0 = \frac{1}{6}\eta^3$, (98) becomes

$$\frac{1}{\sigma} g_0'' - \frac{1}{2}\eta^3 g_0' = 0 \tag{99}$$

The general solution of this equation is

$$g_0(\eta) = A + B \int_0^\eta e^{\left(\frac{\sigma}{8}\eta^4\right)} d\eta \tag{100}$$

with boundary conditions $g_0(0) = 0$ and $g_0(\eta)$ is not exponentially large as $\eta \rightarrow \infty$. Hence

$$g_0(\eta) = 0. \tag{101}$$

Substituting $g(\xi, \eta)$ as given by (25) into (97) and equating the coefficients of powers of ξ as we did for the velocity field we obtain

$$\frac{1}{\sigma} g_r'' - \frac{1}{2}\eta^3 g_r' + \frac{1}{2}r\eta^2 g_r = Q_r \tag{102}$$

where $Q_1 = 0$ and for $r > 1$

$$Q_r = \sum_{s=1}^{r-1} [(r-s+3)f_{r-s}g'_s - sf'_{r-s}g_s]. \quad (103)$$

The boundary conditions are that $g_r(0) = 0$ and $g_r(\eta)$ cannot be exponentially large at infinity.

6 Solutions to the thermal field without suction

The equations to be solved are

$$\frac{1}{\sigma}g''_1 - \frac{1}{2}\eta^3g'_1 + \frac{1}{2}\eta^2g_1 = 0 \quad (104)$$

$$\frac{1}{\sigma}g''_2 - \frac{1}{2}\eta^3g'_2 + \eta^2g_2 = 4g'_1f_1 - g_1f'_1 \quad (105)$$

and

$$\frac{1}{\sigma}g''_3 - \frac{1}{2}\eta^3g'_3 + \frac{3}{2}\eta^2g_3 = 5g'_1f_2 + 4g'_2f_1 - 2g_2f'_1 - g_1f'_2 \quad (106)$$

The equation for $g_1(\eta)$ is

$$\frac{1}{\sigma}g''_1 - \frac{1}{2}\eta^3g'_1 + \frac{1}{2}\eta^2g_1 = 0 \quad (107)$$

On putting $x = \frac{\sigma}{8}\eta^4$ in (107) (and changing the variable) we have

$$\frac{1}{\sigma} \left[2^{5/2}\sigma^{1/2}x^{3/2}g''_1 + 3\sigma \left(\frac{x}{8\sigma}\right)^{1/2} g'_1 \right] - \frac{\sigma}{4} \left(\frac{8x}{\sigma}\right)^{3/2} g'_1 + \frac{1}{2} \left(\frac{8x}{\sigma}\right)^{1/2} g_1 = 0$$

which reduces to

$$xg''_1 + \left(\frac{3}{4} - x\right)g'_1 + \frac{1}{4}g_1 = 0. \quad (108)$$

This is Kummer's equation with a regular singularity at $x = 0$ and an irregular singularity at infinity. The solutions to (108) is worked out in the appendix and are given by ${}_1F_1(-1/4, 3/4, x)$ and ${}_1F_1(0, 5/4, x)$ where

$${}_1F_1(a, b, x) = 1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}\frac{x^2}{2!} + \dots \quad (109)$$

The general solution of (108) is therefore

$$\begin{aligned} g_1(\eta) &= \beta\sigma^{1/4}\eta_1F_1(0, 5/4, \frac{\sigma}{8}\eta^4) + \gamma_{11}F_1(-1/4, 3/4, \frac{\sigma}{8}\eta^4) \\ &= \beta\sigma^{1/4}\eta + \gamma_{11}F_1(-1/4, 3/4, \frac{\sigma}{8}\eta^4) \end{aligned} \quad (110)$$

Since g_1 is not exponentially large at infinity, $\gamma_1 = 0$ and so

$$g_1(\eta) = \beta\sigma^{1/4}\eta, \tag{111}$$

$$g'_1(0) = \beta\sigma^{1/4}. \tag{112}$$

In general, the homogeneous equation satisfied by g_r is

$$\frac{1}{\sigma}g''_r - \frac{1}{2}\eta^3g'_r + \frac{r}{2}\eta^2g_r = 0. \tag{113}$$

Making the same transformation $x = \frac{\sigma}{8}\eta^4$ as before, (113) becomes

$$xg''_r + \left(\frac{3}{4} - x\right)g'_r + \frac{r}{4}g_r = 0. \tag{114}$$

with solutions

$${}_1F_1\left(-\frac{r}{4}, \frac{3}{4}, \frac{\sigma}{8}\eta^4\right) \text{ and } {}_1F_1\left(\frac{1-r}{4}, \frac{5}{4}, \frac{\sigma}{8}\eta^4\right).$$

From (105) the equation for g_2 is

$$\frac{1}{\sigma}g''_2 - \frac{1}{2}\eta^3g'_2 + \eta^2g_2 = 2\alpha_1\beta_1\eta^2\sigma^{1/4} \tag{115}$$

The complementary function which is regular at infinity is

$$q_2 = {}_1F_1\left(-\frac{1}{2}, \frac{3}{4}, \frac{\sigma}{8}\eta^4\right) - b_2\eta_1F_1\left(-\frac{1}{4}, \frac{5}{4}, \frac{\sigma}{8}\eta^4\right) \tag{116}$$

where

$$b_2 = \frac{\sigma^{1/4}\pi^{3/2}}{2^{11/4}(\frac{1}{4}!)^2} \approx 1.11155\sigma^{1/4} \tag{117}$$

which is obtained from the asymptotic expansions of the two confluent hypergeometric functions involved.

The particular integral of the polynomial part is sought by writing $g_2(\eta) = A + B\eta + C\eta^2 + D\eta^3$ and the part involving q_2 is obtained by writing

$$g_2(\eta) = aq_2 + bq'_2.$$

The general solution is

$$g_2(\eta) = 2\alpha_1\beta_1(1 - q_2)\sigma^{1/4} \tag{118}$$

$$g'_2(0) = 2\alpha_1\beta_1b_2\sigma^{1/4} \approx 2.2231\alpha_1\beta_1\sigma^{1/2}. \tag{119}$$

The equation for $g_3(\eta)$ is now

$$\frac{1}{\sigma}g_3'' - \frac{1}{2}\eta^3g_3' + \frac{3}{2}\eta^2g_3 = 3\alpha_2\beta_1\eta^2 - 8\alpha_1^2\beta_1\eta + (8\alpha_1^2\beta_1\eta)(q_2 - \eta q_2'). \quad (120)$$

The particular integral for the polynomial part is

$$(2\alpha_2\beta_1 - 4\alpha_1^2\beta_1\sigma\eta^3)\sigma^{1/4} \quad (121)$$

The part involving q_2 has $-4\alpha_1^2\beta_1q_2'$ as its particular integral. If the complementary solution that is regular at infinity is

$$g_3(\eta) = {}_1F_1\left(-\frac{3}{4}, \frac{3}{4}, \frac{\sigma}{8}\eta^4\right) - b_3\eta {}_1F_1\left(-\frac{1}{2}, \frac{5}{4}, \frac{\sigma}{8}\eta^4\right), \quad (122)$$

where

$$b_3 = \frac{3\pi^{3/2}\sigma^{1/4}}{(\frac{1}{4}!)^3 2^{21/4}} \approx 0.5849\sigma^{1/4}.$$

Hence,

$$g_3(\eta) = 2\alpha_2\beta_1\left(1 - \frac{9}{4}q_3\right) - \alpha_1^2\beta_1\left(\frac{4}{3}\sigma\eta^3 + 4q_2\right) \quad (123)$$

$$g_3'(0) = 2.65271\alpha_2\beta_1\sigma^{1/4}. \quad (124)$$

7 Solutions To The Thermal Field With Suction

The equations to be solved are given by (104), (105) and (106). The equation for $g_1(\eta)$ is

$$\frac{1}{\sigma}g_1'' - \frac{1}{2}\eta^3g_1' + \frac{1}{2}\eta^2g_1 = 0 \quad (125)$$

Transform this by setting $x = \frac{\sigma}{8}\eta^4$ to get

$$xg_1'' + \left(\frac{3}{4} - x\right)g_1' + \frac{1}{4}g_1 = 0. \quad (126)$$

with solutions

$${}_1F_1\left(-\frac{r}{4}, \frac{3}{4}, \frac{\sigma}{8}\eta^4\right) \text{ and } {}_1F_1\left(\frac{1-r}{4}, \frac{5}{4}, \frac{\sigma}{8}\eta^4\right).$$

The solution to (126) was found in section 6 and it is given by

$$g_1(\eta) = \beta_1\sigma^{1/4}\eta \quad (127)$$

$$g_1'(0) = \beta_1\sigma^{1/4} \quad (128)$$

In general, the homogeneous equation satisfied by $g_r, r \geq 1$ is

$$\frac{1}{\sigma}g_r'' - \frac{1}{2}\eta^3g_r' + \frac{r}{2}\eta^2g_r = 0 \tag{129}$$

Making the transformation $x = \frac{\sigma}{8}\eta^4$, we find that as before, the general homogeneous equation satisfied by g_r is

$$xg_r'' + \left(\frac{3}{4} - x\right)g_r' + \frac{r}{4}g_r = 0. \tag{130}$$

with solutions

$${}_1F_1\left(-\frac{r}{4}, \frac{3}{4}, \frac{\sigma}{8}\eta^4\right) \text{ and } x^{1/4}{}_1F_1\left(\frac{1-r}{4}, \frac{5}{4}, \frac{\sigma}{8}\eta^4\right). \tag{131}$$

The equation for g_2 is now

$$\frac{1}{\sigma}g_2'' - \frac{1}{2}\eta^3g_2' + \eta^2g_2 = 2\alpha_1\beta_1\eta^2\sigma^{1/4} - 2^{1/2}\beta_1V_0\sigma^{1/4} \tag{132}$$

The complementary function that is not exponentially large at infinity is deduced from (131) as

$$q_2(\eta) = {}_1F_1\left(-\frac{1}{2}, \frac{3}{4}, \frac{\sigma}{8}\eta^4\right) - b_2\eta {}_1F_1\left(-\frac{1}{4}, \frac{5}{4}, \frac{\sigma}{8}\eta^4\right), \tag{133}$$

where

$$b_2 = \frac{\sigma^{1/4}\pi^{3/2}}{2^{11/4}(\frac{1}{4}!)^2} \approx 1.11155\sigma^{1/4} \tag{134}$$

The particular integral of the polynomial part for g_2 which is sought as in section 6 is found to be

$$g_2(\eta) = 2\alpha_1\beta_1\sigma^{1/4} - 2^{-\frac{1}{2}}\beta_1V_0\eta^2\sigma^{5/4} \tag{135}$$

while the integral part involving q_2 is

$$g_2(\eta) = -2\alpha_1\beta_1q_2\sigma^{1/4} \tag{136}$$

The general solution which satisfies $g_2(0) = 0$ is then found to be

$$g_2(\eta) = 2\alpha_1\beta_1(1 - q_2)\sigma^{1/4} - 2^{-\frac{1}{2}}\beta_1V_0\eta^2\sigma^{5/4} \tag{137}$$

$$g_2'(0) = 2\alpha_1\beta_1b_2\sigma^{1/4} = 2.2231\alpha_1\beta_1\sigma^{\frac{1}{2}} \tag{138}$$

The equation for g_3 is

$$\begin{aligned}
\frac{1}{\sigma}g_3'' - \frac{1}{2}\eta^3g_3' + \frac{3}{2}\eta^2g_3 &= -2^{3/2}\alpha_1\beta_1V_o\sigma^{5/4}\eta^3 \\
&+ 3\alpha_2\beta_1\sigma^{1/4}\eta^2 + (2\beta_1V_o^2\sigma^{5/4} - 8\alpha_1^2\beta_1\sigma^{1/4})\eta \\
&+ \left[\frac{2^{1/2}}{3}\alpha_1\beta_1V_o\sigma^{1/4}\eta^4 - 8\alpha_1^2\beta_1\sigma^{1/4}\eta^2 + 2^{3/2}\alpha_1\beta_1V_o\sigma^{1/4} \right] q_2' \\
&- \left[\frac{2^{3/2}}{3}\alpha_1\beta_1V_o\sigma^{1/4}\eta^3 - 8\alpha_1^2\beta_1\sigma^{1/4}\eta \right] q_2
\end{aligned} \tag{139}$$

The particular integral for the polynomial part is sought (as was done for g_2) as

$$\begin{aligned}
g_3(\eta) &= 2\alpha_2\beta_1\sigma^{1/4} - 2^{3/2}\alpha_1\beta_1V_o\sigma^{5/4}\eta \\
&+ \frac{1}{3}(\beta_1V_o^2\sigma^{9/4} - 4\alpha_1^2\beta_1\sigma^{5/4})\eta^3
\end{aligned} \tag{140}$$

The particular integral of the part involving q_2 is found to be

$$\begin{aligned}
g_3(\eta) &= \left(\frac{2^{1/2}}{9}\alpha_1\beta_1V_o\sigma^{1/4}\eta^2 - 4\sigma_1^2\beta_1\sigma^{1/4} \right) q_2' \\
&- 2^{1/2}\alpha_1\beta_1V_o\sigma^{1/4} \left(\sigma - \frac{1}{9} \right) \eta q_2
\end{aligned} \tag{141}$$

We take the complementary solution which is not large at infinity as

$$q_3 = {}_1F_1 \left(-\frac{3}{4}, \frac{3}{4}, \frac{\sigma}{8}\eta^4 \right) - b_3\eta_1 F_1 \left(-\frac{1}{2}, \frac{5}{4}, \frac{\sigma}{8}\eta^4 \right) \tag{142}$$

where

$$b_3 = \frac{3\pi^{3/2}\sigma^{1/4}}{2^{21/4}(\frac{1}{4!})^3} = 0.58949\sigma^{1/4}. \tag{143}$$

The solution of (139) which satisfy $g_3(0) = 0$ and which is not exponentially large at infinity is

$$\begin{aligned}
g_3(\eta) &= 2\alpha_2\beta_1 \left(1 - \frac{9q_3}{4} \right) \sigma^{1/4} - 2^{3/2}\alpha_1\beta_1V_o\sigma^{5/4}\eta \\
&+ \frac{1}{3}(\beta_1V_o^2\sigma^{9/4} - 4\alpha_1^2\beta_1\sigma^{5/4})\eta^3 \\
&+ \alpha_1\beta_1 \left(\frac{2^{1/2}}{9}V_o\eta^2 - 4\alpha_1 \right) \sigma^{1/4} q_2 \\
&+ 2^{1/2}\alpha_1\beta_1V_o\sigma^{1/4} \left(\sigma - \frac{1}{9} \right) \eta q_2
\end{aligned} \tag{144}$$

Using the fact that $\alpha_1^2 b_2 = \frac{5}{8} \alpha_2$; $q_2''(0) = 0$, and $q_2(0) = 1$:

$$g_3'(0) = -\frac{9}{2} \alpha_2 \beta_1 q_3'(0) \sigma^{1/4} - 2^{1/2} \alpha_1 \beta_1 V_o \sigma^{5/4} - \frac{2^{1/2}}{9} \alpha_1 \beta_1 V_o \sigma^{1/4}$$

But $q_3'(0) = -b_3 = -0.58949 \sigma^{1/4}$.

Hence

$$g_3'(0) = 2.65271 \alpha_2 \beta_1 \sigma^{1/2} - 2^{1/2} \alpha_1 \beta_1 V_o \sigma^{5/4} - \frac{2^{1/2}}{9} \alpha_1 \beta_1 V_o \sigma^{1/4} \tag{145}$$

7.1 Heat Transfer

The heat transfer on the surface is given by $Q_w = K \left(\frac{\partial T}{\partial y} \right)_{y=0}$. Transforming this to (ξ, η) variables defined by equation (32), Q_w becomes

$$Q_w = \frac{KR^{1/2}}{2^{1/2}L} \left(\frac{T_o - T_w}{\xi} \right) \left(\frac{\partial \theta}{\partial \eta} \right)_w \tag{146}$$

Substituting for $\left(\frac{\partial \theta}{\partial \eta} \right)_w$ we have

$$Q_w = \frac{KR^{1/2}(T_o - T_w)}{2^{1/2}\xi L} [g_o'(0) + g_1'(0)\xi + g_2'(0)\xi^2 + g_3'(0)\xi^3 + \dots] \tag{147}$$

Substituting for $g_o'(0), g_1'(0), \dots$, the Nusselt number $Nu = \frac{Q_w L}{K(T_o - T_w)}$ is then given by

$$\begin{aligned} \frac{Nu}{R^{1/2}} = \frac{\beta_1 \sigma^{1/4}}{2^{1/2}} \{ & 1 + 2.2231 \alpha_1 \sigma^{1/4} \xi \\ & + [4.7177916 \alpha_1^2 \sigma^{1/4} - \alpha_1 V_o (0.157135 + 1.4142 \sigma)] \xi^2 + \dots \} \end{aligned} \tag{148}$$

8 Conclusion

The equations of all f 's except f_0, f_1 and f_2 are non-homogeneous with the right hand side rapidly becoming more and more complicated. For instance, f_5 involves f_1, f_2, f_3, f_4 , and P_1 . Higher f 's get even more involved. The complementary functions are also found to involve integrals of confluent hypergeometric functions which yield complicated particular integrals. Since the f 's are made up of infinite series that must be combined in such a way that large terms cancel out, we find that the α 's and the a 's have special relations depending on the f 's involved. For instance, the condition for the absence of exponentially large terms in f_3 is

$$\alpha_2 = \frac{2^{1/4} \pi^{3/2}}{5 \left(\frac{1}{4!} \right)^3} \alpha_1^2$$

and

$$a_5 = -\frac{2^{1/2}\pi}{40\left(\frac{1}{4!}\right)^2}\alpha_1^3$$

while that for f_4 it is choosing α_3 such that

$$\alpha_3 = \frac{\pi^3}{40\left(400\frac{1}{4!}\right)^6}\left(35 - \frac{8 \times 2^{1/2}}{\alpha_1^3}\right)$$

so that

$$a_6 = \alpha_1^4 \left[\frac{1}{9} - \frac{7\pi^2}{600\left(\frac{1}{4!}\right)^4} \right] - \frac{P_1}{360}$$

We find that the a 's are expressed in terms of α_1 and P_1 .

In general we find that apart from the complementary function $\alpha_r\eta^2$, the equation for f_r has a complementary function whose asymptotic expansion starts with a multiple of $\eta^{-(r+10)}\exp(\eta^4/8)$ and another complementary function whose asymptotic expansion commences with a multiple of η^{r+3} followed by a multiple of η^{r-1} . This is true for all the f_r 's because if we assume it is true for $r \leq n-1$, then G_n (from (80)) has a leading term a multiple of η^{n+4} which cancels out. Hence the expansion of G_n commences with a multiple of η^{n+2} . This implies that the equation for f_n has a particular integral, I , which begins with a multiple of η^{n+1} . Since any particular integral with a double zero at the origin can be expressed as the sum of I and the complementary functions which must not be exponentially large at infinity, it follows that the required asymptotic expansion of the solution will begin with a multiple of η^{n+3} (from the complementary function) followed by a multiple of η^{n+1} from I . Hence by the principle of induction, we conclude that f_r begins generally with a multiple of η^{r+3} followed by a multiple of η^{r+1} .

In the expression for the velocity u_1 at the position of separation (2) the coefficient y_1^2 is $\frac{1}{2}$ as a result of choice of unit. To get the other a 's, equations (4) and (6) are solved subject to the boundary condition $u = v = 0$ at $y = 0$. This leads to the determination of all the a 's in terms of a_1 and P_1 . Since η^2 is a complementary solution of (75) α_r are not determinable in (3), but according to Goldstein, they can be expressed in terms of P_1 and α_1 , where α_1 could possibly be found from the condition that $u \rightarrow U$ as $y \rightarrow \infty$.

Analytical solutions for the functions f_0, f_1, f_2, f_3 have been presented and these solutions reduce to those obtained by Goldstein when the velocity of suction is zero.

For the thermal field, let us first consider the case of zero suction and Prandtl number $\sigma = 1$. The transformation $x = \frac{\eta^4}{8}$ makes all the homogeneous equations for the g 's tractable. The result comes out perfectly well in form of a confluent hypergeometric series and a general expression is therefore easily obtainable for the complementary solution of g_r . As in the case of the velocity

field, solutions of the thermal field are expressed as a sum of infinite series which may be large at infinity. However, since g must be regular at infinity, the series are combined in such a way that the large terms cancel out.

When arbitrary Prandtl number is considered, the thermal equation is of the form

$$\frac{1}{\sigma}g_r'' - \frac{1}{2}\eta^3g_r' + \frac{r}{2}\eta^2g_r = Q_r \quad (149)$$

On putting $x = \frac{\sigma}{8}\eta$ this reduces to

$$xg_r'' \left(\frac{3}{4} - x \right) g_r' + \frac{r}{4}\eta^2g_r = Q_r \quad (150)$$

The way σ is absorbed in the transformation is remarkable. It provides a neat procedure for obtaining solutions to equation (149). Notice also how σ is completely absorbed in the confluent hypergeometric functions. Both σ and V_o appear in the particular integrals.

Since we are only interested in the what happens in the neighborhood of separation, only four terms of the expansion are worked out. The heat transfer is found to have the same result as in [2] when $\sigma = 1$ and V_o is zero. It is found that the heat transfer at the position of separation is not zero even when suction is applied which confirms the earlier results obtained by Akinrelere [[2],[3]].

The effect of suction here, as it is always the case consists in the removal of decelerated fluid particles from the boundary layer before they are given a chance to separate. Suction reduces the boundary layer thickness thereby reducing the skin friction and the heat transfer (compare equation (30) in [3] with (148)). New boundary layer which is capable of over coming a certain adverse pressure gradient is allowed to form in the region behind the suction slit. With suitable arrangements of slits and under favorable conditions, separation can totally be avoided.

References

- [1] D. J. Acheson, *Elementary Fluid Dynamics*, Oxford, Clarendon Press, 1990.
- [2] E. A. Akinrelere, Forced convection near Laminar Separation, *Aeronautical Quarterly*, **32** (1981), 212-227.
<https://doi.org/10.1017/s000192590000915x>
- [3] E. A. Akinrelere, Effects of arbitrary Prandtl number on a forced convection flow near laminar separation, *Boundary and interior layers - Computational and asymptotic methods, Proceedings of the First B.A.I.L. Conference*, Dublin, Ireland, Boole Press, (1980), 193-198.

- [4] N. Curle, The steady compressible laminar boundary layer with arbitrary pressure gradient and uniform wall temperature, *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, **249** (1959), no. 1257, 206-224. <https://doi.org/10.1098/rspa.1959.0017>
- [5] S. Goldstein, On laminar boundary-layer flow near a position of separation, *Quarterly Journal of Mechanics And Applied Mathematics*, **1** (1949), no. 1, 43-69. <https://doi.org/10.1093/qjmam/1.1.43>
- [6] L. Howarth, On the solution of the laminar boundary layer equations, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, **164** (1938), 547-579. <https://doi.org/10.1098/rspa.1938.0037>
- [7] H. Dumitrescu, V. Cardo, N. Alexandrescu, Computation of Separating Laminar Boundary-layer Flows *Proceedings of the Romainan Academy, Series A*, **3** (2003), no. 3, 1-6.
- [8] C. W. Jones, On a solution of the laminar boundary-layer equation near a position of separation, *Quarterly Journal of Mechanics And Applied Mathematics*, **1** (1948), no. 1, 385-407. <https://doi.org/10.1093/qjmam/1.1.385>
- [9] H. Schlichting, *Boundary Layer Theory*, McGraw-Hill, New York, 1968.
- [10] F. W. J. Olver, *Introduction to Asymptotics and Special Functions*, Academic Press 1974.
- [11] K. Stewartson, On Goldstein's theory of laminar boundary layer separation, *Quarterly Journal of Mechanics and Applied Mathematics*, **11** (1958), no. 4, 399-410. <https://doi.org/10.1093/qjmam/11.4.399>
- [12] K. Stewartson, The behaviour of a laminar compressible boundary layer near a point of zero skin-friction, *Journal of Fluid Mechanics*, **12** (1960), 117-128. <https://doi.org/10.1017/s0022112062000075>
- [13] R. M. Terrill, Laminar boundary-layer flow near separation with and without suction, *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, **253** (1960), 55-100. <https://doi.org/10.1098/rsta.1960.0018>
- [14] V. A. Wehrle, Determination of the separation point in laminar boundary-layer flows, *AIAA Journal*, **24** (1986), no. 10, 1636-1641. <https://doi.org/10.2514/3.9494>

Appendix

Consider the equation

$$x \frac{d^2y}{dx^2} + (b - x) \frac{dy}{dx} - ay = 0. \tag{151}$$

This equation is the confluent hypergeometric equation, otherwise known as Kummer's equation and any solution of (151) is a confluent hypergeometric function. Since the equation has a removable (non-essential) singularity at $x = 0$, its solution may be developed directly by series method about $x=0$. Choosing the series as

$$y = \sum_{r=0}^{\infty} A_r x^{k+r}. \tag{152}$$

Substituting for $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (151) and equating the coefficients of the first and the general term (that is x^{k+r}) to zero. It is easy to find that indicial equation

$$k(k + b - 1) = 0 \tag{153}$$

gives $k = 0, 1 - b$ when $A_0 \neq 0$, and the recurrence relation is

$$A_{r+1} = \frac{k + r + a}{(k + r + 1)(k + r + b)} A_r \tag{154}$$

which for $k = 0$ gives

$$y = A_0 \left[1 + \frac{a}{b}x + \frac{a(a + 1)}{1 \times 2 \times b \times (b + 1)}x^2 + \dots \right] = A_0 {}_1F_1(a, b, x) \tag{155}$$

and for $k = 1 - b$ gives

$$\begin{aligned} y &= A_0 x^{1-b} \left[1 + \frac{a^c}{b^c}x + \frac{a^c(a^c + 1)}{1 \times 2 \times b^c \times (b^c + 1)}x^2 + \dots \right] \\ &= A_0 x^{1-b} {}_1F_1(a^c, b^c, x) \\ &= A_0 x^{1-b} {}_1F_1(a - b + 1, 2 - b, x) \end{aligned} \tag{156}$$

where $a^c = a - b + 1, b^c = 2 - b$. The function ${}_1F_1(a - b + 1, 2 - b, x)$ is called the confluent hypergeometric function of the second kind.

The general solution is therefore,

$$y = A_1 {}_1F_1(a, b, x) + B x^{1-b} {}_1F_1(a - b + 1, 2 - b, x) \tag{157}$$

The function ${}_1F_1(a, b, x)$ is an analytic function of a and x for all values of x real or complex, but it is not an analytic function of b , when $b \leq 0$. The function has simple poles at $b = 0, -1, -2, \dots$ However, since

$$\lim_{b \rightarrow 1-m} \frac{1}{\Gamma(b + n)} = \begin{cases} 0 & \text{for } n = 0, 1, 2, \dots, m - 1 \\ \frac{1}{(n-m)!} & \text{for } n = m, m + 1, m + 2, \dots \end{cases} \tag{158}$$

we have

$$\lim_{b \rightarrow 1-n} \frac{{}_1F_1(a, b, x)}{\Gamma(b)} = (a)_n x^n {}_1F_1(a+n, 1+n, x) \quad (159)$$

and so the modified solution

$$\frac{{}_1F_1(a, b, x)}{\Gamma(b)}$$

is an analytical function of b as well as a and x and is defined for all values of a, b with x being real or complex.

Notationally, we denote (155) by

$${}_1F_1(a, b, x) \equiv \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} x^n = \frac{a(a+1)(a+2)\cdots}{n!b(b+1)(b+2)\cdots} x^n.$$

We define an alternative form of solution to Kummer's equation as

$$y = U(a, b, x) \equiv \frac{\Gamma(1-b)}{\Gamma(1+a-b)} {}_1F_1(a, b, x) + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} {}_1F_1(1+a-b, 2-b, x) \quad (160)$$

This solution is the most important because it remains finite as $x \rightarrow \infty$.

Received: September 6, 2016; Published: December 31, 2016