2004

TR-2004016: Semantics and Tableaus for LPS4

Melvin Fitting

Semantics and Tableaus for **LPS4**

Melvin Fitting
Dept. Mathematics and Computer Science
Lehman College (CUNY), 250 Bedford Park Boulevard West
Bronx, NY 10468-1589
e-mail: melvin.fitting@lehman.cuny.edu
web page: comet.lehman.cuny.edu/fitting

October 11, 2004

Abstract

An axiomatic formulation of a logic called **LPS4** appears in [3]. Here that logic is proved sound and complete with respect to the weak semantics of [7]. Also a tableau system for **LPS4** is introduced, and also proved sound and complete with respect to the semantics, thus establishing both cut elimination and equivalence with the axiomatic formulation.

1 Introduction

A provability logic, **LP**, with explicit proof terms, was presented in [1, 2]. A Kripke-style semantics for it appears in [5, 7, 6]. Extensions of **LP** to include a standard modal operator $\Box$ with **S4**-like properties appear in [3]—they are called **LPS4** and **LPS4**$^-$. This paper is concerned with **LPS4** only. As with the original **LP**, the extensions were given arithmetic semantics. It is the purpose of this report to do the following. First, it will be shown that the axiom system for **LPS4** from [3] is actually complete with respect to the weak semantics of [7], thus answering Problem 1 from [3]. Second, a tableau system for **LPS4** will be presented. It will be shown to be sound and complete with respect to the weak semantics of [7], establishing its equivalence with the axiomatic version, and also demonstrating cut-elimination (non-constructively). For convenience I will begin with brief summaries of the logic **LPS4** axiomatically, and of the weak semantics being used.

2 The logic LPS4 axiomatically

I begin with the language. This is largely taken from [3].

Proof polynomials for **LPS4** are terms built from proof variables x, y, z,… and proof constants a, b, c,… by means of three operations, application "$\cdot$" (binary), union "$+$" (binary), and proof checker "!" (unary).

Using t to stand for any proof polynomial and S for any sentence variable, formulas are defined by the grammar

$$A = S \mid \bot \mid A_1 \supset A_2 \mid \Box A \mid t{:}A$$

A *constant specification* is a mapping $\mathcal{C}$ from proof constants to sets of formulas (possibly empty). A formula X *has a proof constant* with respect to $\mathcal{C}$ if $X \in \mathcal{C}(c)$ for some proof constant c. A proof constant c is *for* a formula X if $X \in \mathcal{C}(c)$.

The logic of explicit proofs and implicit provability, **LPS4**, has axioms of both **LP** and **S4**, a specific principle connecting explicit proofs with $\Box$, and rules Modus Ponens and Constant Specification as shown below. Let $\mathcal{C}$ be a constant specification.

Classical propositional logic A standard set of axioms, and rule *R1. Modus Ponens*

Basic Epistemic Logic S4 The following axiom schemes and rule.

$E1$	$\Box(X \supset Y) \supset (\Box X \supset \Box Y)$	(implicit application)
$E2$	$\Box X \supset \Box\Box X$	(implicit proof checker)
$E3$	$\Box X \supset X$	(reflexivity)
$R2$	$\vdash X \Rightarrow \vdash \Box X$	(necessitation rule)

LP The following axiom schemes and rule.

LP1	$s{:}(X \supset Y) \supset (t{:}X \supset (s \cdot t){:}Y)$	(application)
LP2	$t{:}X \supset !t{:}(t{:}X)$	(proof checker)
LP3	$s{:}X \supset (s+t){:}X, \quad t{:}X \supset (s+t){:}X$	(union)
LP4	$t{:}X \supset X$	(explicit reflexivity)
$R4$	$\vdash c{:}A$ where A is an axiom and $A \in \mathcal{C}(c)$	(constant specification rule)

Connecting principle

$$C1 \quad t{:}X \supset \Box X \quad \text{(explicit-implicit connection)}$$

A formula X has an **LPS4** axiomatic proof using constant specification $\mathcal{C}$ if X is provable using the axioms and rules above, where $\mathcal{C}$ is the constant specification used in *R4*.

Note the omission of a rule numbered *R3*. This is to maintain correspondence with the numbering in [3]. Axiom **LP**4 can be proved from the others, but is kept to maintain the relationship with the usual formulation of **LP**.

A principle called *positive introspection* in [3] has an easy proof in this system: $t{:}X \supset \Box t{:}X$. It is mentioned because it will be needed when we come to tableaus. There is a corresponding principle of *negative introspection*: $\neg(t{:}X) \supset \Box\neg(t{:}X)$, which is not provable. Adding it to **LPS4** yields the system called **LPS4**$^-$ in [3]. It is actually **LPS4**$^-$ that has the natural arithmetic semantics, but I do not pursue this issue here.

3 Weak Semantics

A semantics for **LP** was given in [5] and, in more detail in [7]. There are actually two versions, a weak and a strong. It is the weak version that is needed now, and a sketch of it follows.

$\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$ is a *frame* in the usual sense, where $\mathcal{G}$ is a set of states and $\mathcal{R}$ is a binary relation on $\mathcal{G}$. Given the frame $\mathcal{F}$, a *possible evidence* function $\mathcal{E}$ is a mapping from states and proof polynomials to sets of formulas. Here are the conditions that must be met by $\mathcal{E}$ to be an evidence function with respect to constant specification $\mathcal{C}$:

$\mathcal{E}$ is an evidence function on $\langle \mathcal{G}, \mathcal{R} \rangle$ if, for all proof polynomials s and t, for all formulas X and Y, and for all $\Gamma, \Delta \in \mathcal{G}$:

1. **Application** $(X \supset Y) \in \mathcal{E}(\Gamma, s)$ and $X \in \mathcal{E}(\Gamma, t)$ implies $Y \in \mathcal{E}(\Gamma, s \cdot t)$.
2. **Monotonicity** $\Gamma\mathcal{R}\Delta$ implies $\mathcal{E}(\Gamma, t) \subseteq \mathcal{E}(\Delta, t)$.

3. **Proof Checker** $X \in \mathcal{E}(\Gamma, t)$ implies $t{:}X \in \mathcal{E}(\Gamma, !t)$.

4. **Sum** $\mathcal{E}(\Gamma, s) \cup \mathcal{E}(\Gamma, t) \subseteq \mathcal{E}(\Gamma, s + t)$.

A structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ is a *weak* **LP** model provided $\langle \mathcal{G}, \mathcal{R} \rangle$ is a frame with $\mathcal{R}$ reflexive and transitive, $\mathcal{E}$ is an evidence function on $\langle \mathcal{G}, \mathcal{R} \rangle$, and $\mathcal{V}$ is a mapping from propositional variables to subsets of $\mathcal{G}$.

Given a weak model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$, a forcing relation is defined by the following rules. For each $\Gamma \in \mathcal{G}$:

1. $\mathcal{M}, \Gamma \Vdash P$ for a propositional variable P provided $\Gamma \in \mathcal{V}(P)$.

2. $\mathcal{M}, \Gamma \Vdash \bot$ never holds—written $\Gamma \nVdash \bot$.

3. $\mathcal{M}, \Gamma \Vdash (X \supset Y)$ if and only if $\mathcal{M}, \Gamma \nVdash X$ or $\mathcal{M}, \Gamma \Vdash Y$.

4. $\mathcal{M}, \Gamma \Vdash (t{:}X)$ if and only if $X \in \mathcal{E}(\Gamma, t)$ and, for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, $\mathcal{M}, \Delta \Vdash X$.

We say X is *true at world* Γ if $\mathcal{M}, \Gamma \Vdash X$, and otherwise X is *false at* Γ.

Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$ be a weak **LP** model. And let $\mathcal{C}$ be a constant specification meeting the condition that any formula having a proof constant with respect to $\mathcal{C}$ must be true at every possible world of every weak **LP** model. Then $\mathcal{M}$ is a weak **LP** model that meets constant specification $\mathcal{C}$ provided $\mathcal{C}(c) \subseteq \mathcal{E}(\Gamma, c)$, for each $\Gamma \in \mathcal{G}$.

All of the definitions above were for **LP**, and were taken from [5]. Now this is extended to **LPS4**. For this we simply need to do two things. First, the language is extended to include $\Box$, with all conditions stated above applying to the enlarged language. Second, the forcing conditions are extended in the familiar way:

5. $\mathcal{M}, \Gamma \Vdash \Box X$ if and only if $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma\mathcal{R}\Delta$.

4 Axiomatic Soundness and Completeness

Theorem 4.1 *A formula X (in the language of* **LPS4***) has an* **LPS4** *axiomatic proof using constant specification $\mathcal{C}$ if and only if X is true at all worlds of* **LPS4** *models meeting constant specification $\mathcal{C}$.*

The soundness half of this Theorem is quite straightforward, and is omitted—see Theorem 8 in [3]. As it happens, the completeness half is quite straightforward, being a direct combination of standard canonical arguments for **S4**, and for **LP** from [5, 7]. I will sketch the details.

Let $\mathcal{C}$ be a constant specification, fixed for the following construction. Axiomatic proofs are assumed to use this constant specification $\mathcal{C}$.

Call a set S of **LPS4** formulas *inconsistent* if there is some finite subset $\{X_1, \ldots, X_n\} \subseteq S$ such that $(X_1 \wedge \ldots \wedge X_n) \supset \bot$ is a theorem of **LPS4** (with $\wedge$ defined from $\supset$ and $\bot$ in the usual way). Call S *consistent* if it is not inconsistent. Consistent sets extend to maximal consistent sets, via a standard Lindenbaum Lemma construction. Let $\mathcal{G}$ be the set of all maximally consistent sets of formulas. If $\Gamma \in \mathcal{G}$, let $\Gamma^\sharp = \{\Box X \mid \Box X \in \Gamma\}$, and set $\Gamma\mathcal{R}\Delta$ if $\Gamma^\sharp \subseteq \Delta$. This gives us a frame, $\langle \mathcal{G}, \mathcal{R} \rangle$. It is obviously reflexive and transitive. Define a mapping $\mathcal{E}$ by setting $\mathcal{E}(\Gamma, t) = \{X \mid t{:}X \in \Gamma\}$. Finally, define a mapping $\mathcal{V}$ by specifying that for an atomic formula P, $\Gamma \in \mathcal{V}(P)$ if and only if $P \in \Gamma$. This gives us a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V} \rangle$. The claim is that $\mathcal{M}$ is an **LPS4** model that meets $\mathcal{C}$.

First I'll verify two of the conditions or the evidence function $\mathcal{E}$—other conditions are similar. I'll begin with the Application Condition. Suppose we have $X \in \mathcal{E}(\Gamma, t)$ and $(X \supset Y) \in \mathcal{E}(\Gamma, s)$. By the definition of $\mathcal{E}$, we must have $t{:}X \in \Gamma$ and $s{:}(X \supset Y) \in \Gamma$. Since $s{:}(X \supset Y) \supset (t{:}X \supset (s \cdot t){:}Y)$ is an **LPS4** axiom, and Γ is maximally consistent, it follows that $(s \cdot t){:}Y \in \Gamma$, and hence $Y \in \mathcal{E}(\Gamma, s \cdot t)$.

Next, I'll verify the Monotonicity Condition. Suppose $\Gamma, \Delta \in \mathcal{G}$ and $\Gamma \mathcal{R} \Delta$. Also assume $X \in \mathcal{E}(\Gamma, t)$. By definition of $\mathcal{E}$, we have $t{:}X \in \Gamma$. Now $t{:}X \supset !t{:}t{:}X$ is an **LPS4** axiom (**LP**2), and so is $!t{:}t{:}X \supset \Box t{:}X$ (*C1*), so we have $\Box t{:}X \in \Gamma$, and hence $\Box t{:}X \in \Gamma^\sharp$. Since $\Gamma \mathcal{R} \Delta$ we have $\Gamma^\sharp \subseteq \Delta$, so $\Box t{:}X \in \Delta$, and so $t{:}X \in \Delta$, using *E3*, and so $X \in \mathcal{E}(\Delta, t)$.

Other conditions are similar. Thus $\mathcal{M}$ is an **LPS4** model that meets $\mathcal{C}$.

Now a Truth Lemma can be shown: for each formula X and each $\Gamma \in \mathcal{G}$

$$X \in \Gamma \Longleftrightarrow \mathcal{M}, \Gamma \Vdash X \tag{1}$$

Most of the cases are familiar from standard **S4** completeness proofs. I'll verify only one case. Suppose (1) is known for X, and we are considering the formula $t{:}X$.

Suppose first that $t{:}X \in \Gamma$. Then, using axiom *C1*, $t{:}X \supset \Box X$, $\Box X \in \Gamma$, and so $\Box X \in \Gamma^\sharp$. Then if Δ is an arbitrary member of $\mathcal{G}$ with $\Gamma \mathcal{R} \Delta$ we have $\Gamma^\sharp \subseteq \Delta$ and hence $\Box X \in \Delta$, and so $X \in \Delta$, using *E3*. By the induction hypothesis, $\mathcal{M}, \Delta \Vdash X$. Also since $t{:}X \in \Gamma$, we have $X \in \mathcal{E}(\Gamma, t)$. It follows that $\mathcal{M}, \Gamma \Vdash t{:}X$.

Next, suppose $\mathcal{M}, \Gamma \Vdash t{:}X$. This case is trivial. By the general definition of $\Vdash$ we must have $X \in \mathcal{E}(\Gamma, t)$, and by definition of $\mathcal{E}$ for $\mathcal{M}$, we must also have $t{:}X \in \Gamma$.

Thus we have the Truth Lemma. Now, as usual, if X does not have an **LPS4** axiomatic proof, $\{X \supset \bot\}$ is consistent. Extend it to a maximal consistent set Γ. Then $\Gamma \in \mathcal{G}$ and by (1) $\mathcal{M}, \Gamma \not\Vdash X$.

Thus axiomatic completeness has been established.

5 Tableaus

I give a tableau system for **LPS4**, divided into subsystems. The classical part is from [10]. The **S4** part appears in several places, ranging from [4] to [8]. The **LP** rules are from [9, 7]. It is assumed that the language is that for **LPS4**, as earlier.

A *signed* formula is $T\,X$ or $F\,X$, where X is a formula of **LPS4**. A *tableau proof* of X is a closed tableau for $F\,X$. A *tableau* for $F\,X$ is a tree, with $F\,X$ at the root, constructed using certain branch extension rules to be given in a moment. A tableau is *closed* if each branch is closed, and a branch is closed if it contains an explicit contradiction, $T\,Z$ and $F\,Z$ for some formula Z, or $T\,\bot$ (there will be one additional closure condition below). What remains is to say what the branch extension rules are. These fall into two categories: those that extend branches (known as α rules) and those that cause branches to split (known as β rules). Since we have only one classical connective to consider, there is only one classical rule of each type, given in Table 1. These are non-deterministic rules—one may freely choose which signed formula on a branch to apply a rule to.

$$\frac{F\,X \supset Y}{\begin{array}{c} T\,X \\ F\,Y \end{array}} \qquad \frac{T\,X \supset Y}{F\,X \mid T\,Y}$$

Table 1: Classical Tableau Rules

Next are the **LP** rules, in Table 2. Note that the branching rule violates the usual tableau subformula principle.

$$\frac{T\, t{:}X}{T\, X} \qquad \frac{F\, !t{:}(t{:}X)}{F\, t{:}X} \qquad \frac{F\, (s+t){:}X}{F\, s{:}X} \qquad \frac{F\, (s+t){:}X}{F\, t{:}X} \qquad \frac{F\, (t \cdot s){:}Y}{F\, t{:}X \supset Y \mid F\, s{:}X}$$

Table 2: **LP** Tableau Rules

I said there would be one more closure condition—now it is time for it. Let $\mathcal{C}$ be a constant specification. I'll say X *has an* **LPS4** *tableau proof using* $\mathcal{C}$ if it has a proof using the machinery above together with the additional rule: a branch closes if it contains $F\, c{:}Z$ where $Z \in \mathcal{C}(c)$. As far as we have gone, we have a sound and complete system for **LP**.

For **S4** machinery, I will use what are called *destructive* rules. These modify branches by removing formulas. For this, it is convenient to define the following operation. If S is a set of signed formulas, let $S^\sharp = \{T\, \Box X \mid T\, \Box X \in S\}$. Now, the **S4** rules are in Table 3.

$$\frac{T\, \Box X}{T\, X} \qquad \frac{S, F\, \Box X}{S^\sharp, F\, X}$$

Table 3: **S4** Tableau Rules

For Table 3, the first rule is straightforward. The second is the destructive one. If $F\, \Box X$ occurs on a tableau branch, with S being the set consisting of the other formulas on the branch, the branch may be *replaced* with a new branch consisting of $S^\sharp$ and $F\, X$. Non-necessary formulas disappear.

We need something to connect the **LP** part with the **S4** part. This is in Table 4. It does make one of the **LP** rules redundant, but never mind that.

$$\frac{T\, t{:}X}{T\, \Box X}$$

Table 4: Connecting Rule

It would be nice if the system just presented were complete for **LPS4**, but I do not believe it is (though I have no proof of this). I need one more item for a completeness proof. One can introduce *assumptions* or *premises* into tableau proofs. These come in two varieties, *local* and *global*. Never mind the details of the distinction just now—I need to take the positive introspection formulas as global assumptions. This means they can be used (with a sign of T) at any point of a tableau construction. This is summarized in Table 5.

6 Soundness and Completeness

Soundness says: if X has a tableau proof using constant specification $\mathcal{C}$ then X is true at every world of every **LPS4** model meeting $\mathcal{C}$. It is proved by a standard argument. Call a set of signed formulas S *satisfiable* if there is an **LPS4** model $\mathcal{M}$ meeting $\mathcal{C}$, and a possible world Γ of it, such that $T\, X \in S \Longrightarrow \mathcal{M}, \Gamma \Vdash X$ and $F\, X \in S \Longrightarrow \mathcal{M}, \Gamma \nVdash X$. Call a tableau *satisfiable* if the set of signed formulas on some branch is satisfiable. The key thing to show is: any tableau rule applied to a satisfiable branch yields another satisfiable branch. I omit the argument. Now, suppose X has a tableau proof using constant specification $\mathcal{C}$, but there is a model $\mathcal{M}$ meeting constant specification $\mathcal{C}$, and a world Γ of it such that $\mathcal{M}, \Gamma \nVdash X$. Then the set $\{F\, X\}$ is satisfiable, so the tableau construction begins with a satisfiable tableau. Every subsequent tableau must be satisfiable. So there must be a closed, satisfiable tableau, and this is not possible.

$$\overline{T\, t{:}X \supset \Box(t{:}X)}$$

Table 5: Positive Introspection Rule

The completeness argument is more work. From now on, we allow a tableau to start with a finite set of signed formulas, instead of with a single one. The order in which these formulas are added to the initial branch does not matter. Let $\mathcal{C}$ be a constant specification, fixed for the rest of the section. Call a set S (not necessarily finite) *tableau consistent* if there is no closed tableau beginning with any finite subset of S, using $\mathcal{C}$. By the usual Lindenbaum argument, tableau consistent sets extend to maximal ones.

Suppose S is maximally tableau consistent, and $Z \in S$ for some *signed* formula Z. If Z is one of the signed formulas above the line in a non-branching, non-destructive rule in Table 1, 2, 3, 4, or 5, each signed formula below the line in that rule is also in S. As a typical example, if $F\,(s+t){:}X \in S$, then we should also have $F\,s{:}X \in S$. Well if not, since S is maximally tableau consistent, $S \cup \{F\,s{:}X\}$ must not be tableau consistent, and hence there is a closed tableau for a finite subset, say $\{Z_1, \ldots, Z_n, F\,s{:}X\}$. But $\{Z_1, \ldots, Z_n, F\,(s+t){:}X\}$ is a finite subset of S itself, and there is a closed tableau for it, since we can make the first move in the tableau construction the addition of $F\,s{:}X$ using a tableau rule from Table 2, and then proceed as we did in the closed tableau for $\{Z_1, \ldots, Z_n, F\,s{:}X\}$. In a similar way, if Z is one of the signed formulas above the line in a branching rule, and $Z \in S$, then one of the two formulas below the line in that rule is in S. Note that since the rule in Table 5 has no premise, all instances of positive introspection will be in S.

Construct a weak **LPS4** model as follows. Let $\mathcal{G}$ be the collection of all maximal tableau consistent sets. For $\Gamma, \Delta \in \mathcal{G}$, let $\Gamma\mathcal{R}\Delta$ provided $\Gamma^\sharp \subseteq \Delta$. Let $X \in \mathcal{E}(\Gamma, t)$ provided $F\,t{:}X \notin \Gamma$. And let $\Gamma \in \mathcal{V}(P)$ provided $T\,P \in \Gamma$, for a propositional letter P. This gives us a model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{E}, \mathcal{V}\rangle$.

I'll check two cases of the verification that $\mathcal{E}$ is an evidence function, beginning with the Sum condition. Suppose $X \notin \mathcal{E}(\Gamma, s+t)$; I'll show $X \notin \mathcal{E}(\Gamma, s)$. But this is easy. It amounts to saying $F\,(s+t){:}X \in \Gamma$ implies $F\,s{:}X \in \Gamma$ and, as we saw above, this is the case since Γ is maximal, and we have the Sum rules. Next I'll consider the monotonicity condition. Suppose $\Gamma\mathcal{R}\Delta$ and $X \in \mathcal{E}(\Gamma, t)$; I'll show $X \in \mathcal{E}(\Delta, t)$. By the definition of $\mathcal{E}$ we have $F\,t{:}X \notin \Gamma$. But using the Positive Introspection Rule, $T\,t{:}X \supset \Box t{:}X \in \Gamma$, so either $F\,t{:}X \in \Gamma$ or $T\,\Box t{:}X \in \Gamma$. Then we must have $T\,\Box t{:}X \in \Gamma$. Since $\Gamma\mathcal{R}\Delta$, then $\Gamma^\sharp \subseteq \Delta$, so $T\,\Box t{:}X \in \Delta$. It follows from one of the **S4** rules that $T\,t{:}X \in \Delta$. Since Δ is tableau consistent, we must have $F\,t{:}X \notin \Delta$, and so $X \in \mathcal{E}(\Delta, t)$.

Clearly $\mathcal{R}$ is reflexive and transitive. And obviously $\mathcal{M}$ meets constant specification $\mathcal{C}$ since no set containing $F\,c{:}X$ is tableau consistent, where $X \in \mathcal{C}(c)$. We thus have that $\mathcal{M}$ is an **LPS4** model that meets $\mathcal{C}$.

The main thing now is a version of the Truth Lemma appropriate for tableaus. Unlike the earlier axiomatic version, it takes the form of two implications instead of an equivalence. For each $\Gamma \in \mathcal{G}$ and formula X:

$$T\,X \in \Gamma \Longrightarrow \mathcal{M}, \Gamma \Vdash X \tag{2}$$

$$F\,X \in \Gamma \Longrightarrow \mathcal{M}, \Gamma \not\Vdash X \tag{3}$$

The proof is by induction on the degree of X. Most of the cases are straightforward—I'll consider the modal cases in detail. Assume (2) and (3) are known for Z.

As one case, consider $t{:}Z$. Suppose first that $T\,t{:}Z \in \Gamma$. Since Γ is tableau consistent, $F\,t{:}Z \notin \Gamma$ and so $Z \in \mathcal{E}(\Gamma, t)$. Using the Connecting Rule from Table 4, $T\,\Box Z \in \Gamma$. Let Δ be an arbitrary member of $\mathcal{G}$ with $\Gamma\mathcal{R}\Delta$. By definition of $\mathcal{R}$, $T\,\Box Z \in \Delta$, and using one of the **S4** rules from Table 3, $T\,Z \in \Delta$. By the induction hypothesis, item (2) gives us $\mathcal{M}, \Delta \Vdash Z$. Since Δ was arbitrary, $\mathcal{M}, \Gamma \Vdash t{:}Z$.

Next, suppose that $F\,t{:}Z \in \Gamma$. Then $Z \notin \mathcal{E}(\Gamma, t)$, so trivially $\mathcal{M}, \Gamma \not\Vdash t{:}Z$.

As another case, consider $\Box Z$. Suppose that $T\,\Box Z \in \Gamma$. If $\Gamma\mathcal{R}\Delta$, then $\Gamma^\sharp \subseteq \Delta$, so $T\,\Box Z \in \Delta$. Then by one of the **S4** rules, $T\,Z \in \Delta$. By the induction hypothesis, $\mathcal{M}, \Delta \Vdash Z$. Since Δ was arbitrary, $\mathcal{M}, \Gamma \Vdash \Box Z$. Next, suppose that $F\,\Box Z \in \Gamma$. The set $\Gamma^\sharp \cup \{F\,Z\}$ is easily seen to be tableau consistent, using one of the **S4** rules. Extend it to a maximal tableau consistent set, Δ. Then $\Delta \in \mathcal{G}$, $\Gamma\mathcal{R}\Delta$, and by the induction hypothesis, $\mathcal{M}, \Delta \not\Vdash Z$. Hence $\mathcal{M}, \Gamma \not\Vdash \Box Z$.

With the tableau version of the Truth Lemma established, completeness is as usual. If X does not have a tableau proof, $\{F\,X\}$ is tableau consistent. A maximal extension of this set will be a member of $\mathcal{G}$ at which X is false.

7 Comments and Conclusions

It has now been established, via equivalence with a semantics, that the axiomatic and the tableau formulations of **LPS4** are equivalent. As usual, soundness and completeness of the tableau system (together with soundness of a cut rule) establish cut elimination non-constructively. Treating Positive Introspection as assumptions is contrary to the usual nature of tableaus. On the other hand, the tableau rule for multiplication already violates the usual tableau subformula principle.

It should be noted that Positive Introspection instances can be proved using the tableau rules, without the need for taking them as assumptions. The problem is, to make use of this in other tableau arguments, we need a cut elimination principle, which we seem not to have unless we strengthen the system by treating Positive Introspection as we did. I conjecture that, without the rule of Table 5 the tableau system is not complete, and cut elimination does not hold. It should be noted, however, that the Positive Introspection Rule is, itself, a form of cut elimination. As a referee pointed out, instead of a simple cut on X, it amounts to a cut between $t{:}X$ and $\Box t{:}X$.

Soundness and completeness have been established for what amounts to **LP** + **S4**, both axiomatically and using tableaus. But weaker logics can be considered as well (see Section 10 in [7]). As one example, if we drop axiom schemes *E2* and **LP***2*, completeness can be shown with respect to models in which $\mathcal{R}$ need not be transitive, and in which $\mathcal{E}$ is not required to be monotonic, or to satisfy the proof checker condition (conditions 2 and 3). For tableaus, we drop the rule for "!", and most importantly we drop the rule from Table 5. It is transitivity that seems to force Positive Introspection on us. This needs to be better understood.

Finally, many people are happier with sequent-style rules instead of tableau rules. For the convenience of these, I give a sequent-style formulation that is equivalent to the tableau system of Section 5. In it, σ, with or without subscripts, denotes a finite sequence of formulas of **LPS4**, possibly empty. Also $\sigma^\sharp$ is the sequence that results when all formulas are deleted from σ except those of the form $\Box Z$. and $\sigma^\flat$ is the sequence that results when all formulas are deleted except those of the form $\Box Z \supset \bot$ (abbreviated by $\neg\Box Z$).

Let $\mathcal{C}$ be a constant specification, fixed for the following. In addition to structural rules, which I omit, we have the following.

Axioms $Z \longrightarrow Z$ $\quad \bot \longrightarrow$ $\quad \longrightarrow c{:}Z$, provided $Z \in \mathcal{C}(c)$

Classical Rules

$$\frac{\sigma_1, X \longrightarrow \sigma_2, Y}{\sigma_1 \longrightarrow \sigma_2, X \supset Y} \qquad \frac{\sigma_1 \longrightarrow \sigma_2, X \qquad \sigma_1, Y \longrightarrow \sigma_2}{\sigma_1, X \supset Y \longrightarrow \sigma_2}$$

LP Rules

$$\frac{\sigma_1 \longrightarrow \sigma_2, t{:}(X \supset Y) \qquad \sigma_1 \longrightarrow \sigma_2, s{:}X}{\sigma_1 \longrightarrow \sigma_2, (t \cdot s){:}Y}$$

$$\frac{\sigma_1 \longrightarrow \sigma_2, s{:}X}{\sigma_1 \longrightarrow \sigma_2, (s + t){:}X} \qquad \frac{\sigma_1 \longrightarrow \sigma_2, t{:}X}{\sigma_1 \longrightarrow \sigma_2, (s + t){:}X}$$

$$\frac{\sigma_1, X \longrightarrow \sigma_2}{\sigma_1, t{:}X \longrightarrow \sigma_2}$$

$$\frac{\sigma_1 \longrightarrow \sigma_2, t{:}X}{\sigma_1 \longrightarrow \sigma_2, !t{:}(t{:}X)}$$

S4 Rules

$$\frac{\sigma_1, X \longrightarrow \sigma_2}{\sigma_1, \Box X \longrightarrow \sigma_2} \qquad \frac{\sigma_1^\sharp \longrightarrow \sigma_2^\flat, X}{\sigma_1 \longrightarrow \sigma_2, \Box X}$$

Connecting Rule

$$\frac{\sigma_1, \Box X \longrightarrow \sigma_2}{\sigma_1, t{:}X \longrightarrow \sigma_2}$$

Positive Introspection Rule

$$\frac{\sigma_1, t{:}X \supset \Box X \longrightarrow \sigma_2}{\sigma_1 \longrightarrow \sigma_2}$$

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