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ABSTRACT

Hydraulic simulation tools have become trusted tools in water distribution network analysis. However, the simulation tools only provide raw information about the hydraulic state. In this paper, we develop a hydraulic small signal model (HSSM) which is able to derive internal states of the hydraulic system. The model can be directly derived from the output of the hydraulic simulation. We present how the model performs in computing statistics for uncertain demands in a direct way and give a comparison to results obtained by means of Monte Carlo analysis.

INTRODUCTION

To increase the performance, quality and reliability of water distribution systems, implementing efficient computational and algorithmic techniques has become a major task in hydraulic modelling. Examples can be found in online condition monitoring, real time control applications or model based leakage detection and location approaches. All these techniques require extensive hydraulic simulations. Well-known and trusted hydraulic simulation tools like EPANET are deployed within the individual task specific code using provided interface routines. The flow dependent friction models of hydraulic systems require an iterative solution strategy to solve the problem. Although this is done efficiently using Newton-Raphson method, the simulation output provided by those tools is limited to raw information (i.e. flow and head). Yet the superior algorithms often require more information than the raw output. I.e. gradient based optimization methods rely on derivative information.

In this paper, we report on a HSSM which can be directly derived from the output of the hydraulic simulation tool itself. The model provides cheap computational access to internal information like gradients or sensitivity of the hydraulic simulation. The $\boldsymbol{A}^T\boldsymbol{C}\boldsymbol{A}$ equilibrium structure of the model is numerically suitable and provides properties like a positive definite stiffness matrix enabling the efficient use of direct solvers like Cholesky decomposition. Further, the symmetry provides the property of self adjointness which enables the efficient use of Greens functions.

We will present how the model can be assembled from the raw simulator output and present how to use it for the computation of covariance propagation due to uncertain demands.

This paper is structured as follows. In the first section, we recapitulate hydraulic modelling using a compact matrix notation for the nonlinear hydraulic model. Given the compact notation, we derive the HSSM and demonstrate how the model can be obtained from the output of a nonlinear hydraulic simulation. Finally, we present an example for the use of the HSSM where we introduce the efficient computation of statistics for uncertain demands. We illustrate that the HSSM can be used to provide the same results as Monte Carlo methods.

HYDRAULIC MODELING

In this section, we derive the HSSM starting the process from the nonlinear hydraulic model. Hereby we use a linear algebra notation in order to keep all notations in a consistent manner.

With respect to the terminology of applied mathematics, a water distribution network is a directed graph. A graph is an abstract object consisting of nodes and lines. Lines provide the connection between the nodes. Each node is connected to at least one line. Graphs are extensively used to describe/solve physical problems with a certain spatial expansion, e.g. in finite element theory they are used to approximate the solution of partial differential equations.

For problems like water distribution- or electrical networks, graphs present an exact description. The governing physical laws are associated with both elements of the graph, the nodes and the lines, respectively. For the physical description of the network a potential and a flow are used. The potential is associated with the nodes, while the flow belongs to the lines. In water distribution networks the nodal potential is given by the pressure. The flow is the real water flow along the line. Given this, the flow rates at each node are determined by a balance equation. For the lines incompressibility has to hold. Further, a friction law connects the flow rate and the head loss along a line. Given this interpretation of network quantities and network laws it is considerable simple to form a mathematical framework.

We consider $\boldsymbol{p}$ to be a vector of length N, which holds all nodal potentials (pressure). The vector of the differential pressures $\Delta\boldsymbol{p}$ along M lines can be computed by

$$\Delta\boldsymbol{p} = \boldsymbol{A}\boldsymbol{p}, \tag{1}$$

where the $M \times N$ matrix $\boldsymbol{A}$ is referred to as incidence matrix. Due to its purpose for computing differences, the incidence matrix $\boldsymbol{A}$ contains the values $+1$, -1, and 0 only. By choice of the signs, a direction along each line is introduced which defines the positive direction of the flow. The incidence matrix provides connectivity information about the network. It can be constructed from structural information about the network, e.g. EPANETs *.inp- or input file.

For the further development of the hydraulic model we introduce the line specific friction model, which connects the flow and the potential difference. E.g. the Darcy–Weisbach friction equation is of form

$$\Delta p = kq^2, \tag{2}$$

where Δp is the pressure loss along the line, q is the flow and k is a line specific friction parameter. Recasting the friction law for q results in

$$q = \left(\frac{1}{k}\Delta p\right)^{\frac{1}{2}}, \tag{3}$$

which is denoted by

$$q = \tilde{c}(\Delta p). \tag{4}$$

The nonlinear nature of the friction law is in contrast to the linear algebra scheme we want to develop. However, we can use the ratio

$$\frac{q}{\Delta p} = \hat{c}(\Delta p), \tag{5}$$

for each line and define the vector $\hat{\boldsymbol{c}}(\Delta\boldsymbol{p})$ which holds all elements of equation (5) for the network. Subsequently we introduce the matrix $\boldsymbol{C}$ as

$$\widehat{\boldsymbol{C}}(\Delta\boldsymbol{p}) = \operatorname{diag}(\hat{\boldsymbol{c}}(\Delta\boldsymbol{p})), \tag{6}$$

which is a diagonal matrix. Then we use this result to extend our hydraulic model to

$$\hat{\boldsymbol{q}} = \widehat{\boldsymbol{C}}(\Delta\boldsymbol{p})\boldsymbol{A}\boldsymbol{p}, \tag{7}$$

where the vector $\boldsymbol{q}$ holds the individual flows. Thus, the vector $\boldsymbol{q}$ is of length M. The last equation provides the relation between the nodal pressures and the flow in each line. To complete the model, the flow balance condition for each node has to be incorporated. This can be done by a multiplication of the last equation by $\boldsymbol{A}^T$resulting in

$$\boldsymbol{A}^T\widehat{\boldsymbol{C}}(\Delta\boldsymbol{p})\boldsymbol{A}\boldsymbol{p} = \boldsymbol{q}_{\text{Demand}}, \tag{8}$$

where the vector $\boldsymbol{q}_{\text{Demand}}$ holds the nodal demands.

Equation (8) provides a hydraulic model. The model is nonlinear due to the dependence of the diagonal matrix $\widehat{\boldsymbol{C}}$ by $\Delta\boldsymbol{q}$. The solution can be found using nonlinear methods like Newton-Raphson iteration. The presented derivation is marked by some simplifications, e.g. we did not consider any hydrostatic pressure losses. Nor we did consider more complex hydraulic components like emitter losses. However, we will not stick to this incompleteness. We consider the solution of the nonlinear model to be performed by established software tools like EPANET. But, the derived model equations provide all necessary components for our further derivations.

DERIVATION OF THE HSSM

In this section we will derive a small signal variant of the hydraulic model given by equation (8). The term "small signal" is referred to as a linear approximation around the stationary point, i.e. the stationary solution of the nonlinear hydraulic model. The idea of an HSSM is to investigate the model behavior for small changes $d\boldsymbol{p}$ and $d\boldsymbol{p}_{\text{Demand}}$ for the stationary solution.

Therefore, we can take the derivative of equation (8). The outcome of this is

$$\boldsymbol{A}^T\left[\frac{d\boldsymbol{q}}{d\Delta\boldsymbol{p}}\right]\boldsymbol{A}d\boldsymbol{p} = d\boldsymbol{q}_{\text{Demand}}. \tag{9}$$

For a short notation we define the matrix

$$\left[\frac{\boldsymbol{dq}}{\boldsymbol{d\Delta p}}\right] = \boldsymbol{C}, \tag{10}$$

which is a Jacobian matrix. Equation (9) is the linearized form of equation (8). The structure of equation (8) is referred to as $\boldsymbol{A^T C A}$ equilibrium system [1]. The resulting matrix

$$\boldsymbol{K} = \boldsymbol{A}^T\boldsymbol{CA}, \tag{11}$$

is referred to as stiffness matrix. $\boldsymbol{K}$ is symmetric and positive definite, which enables the efficient use of direct solvers like Cholesky decomposition. Further, the symmetry provides the property of self adjointness which enables the use of efficient Greens functions.

The matrix $\boldsymbol{C}$ is defined by equation (10). However, for a computational use of the HSSM, the numerical realization of $\boldsymbol{C}$ has to be found. We take use of the form of the friction model and compute the derivative of equation (3)

$$\frac{dq}{d\Delta p} = \frac{1}{2}\left(\frac{1}{k}\right)^{\frac{1}{2}}(\Delta p)^{-\frac{1}{2}}. \tag{12}$$

The numerical values of (12) form the matrix $\boldsymbol{C}$. They can be found by analyzing the term $\frac{1}{2}\frac{p}{\Delta q}$ and substituting equation (3). This results in

$$\frac{1}{2}\frac{q}{\Delta p} = \frac{1}{2}\frac{\left(\frac{1}{k}\Delta p\right)^{\frac{1}{2}}}{\Delta p} = \frac{1}{2}\left(\frac{1}{k}\right)^{\frac{1}{2}}(\Delta p)^{-\frac{1}{2}}, \tag{13}$$

which is exactly the result of equation (13). With respect to the HSSM this result is remarkable. The nonlinear solution provides q and Δp for each pipe of the network. Thus, the solution of nonlinear problem provides also the HSSM without any further computational steps.

In the following section, we will present one powerful example for the efficient use of the HSSM.

EXAMPLE: EFFICIENT COMPUTATION OF RANDOM DEMANDS

In this section we want to present a use case for the derived small signal model. For our example we use the network presented in Poulakis *et al.* [2]. The network is depicted in the left part of figure 1. Nominal demands are defined for each node leading to the nominal solution which can be found by solving the nonlinear hydraulic model for $\boldsymbol{q}_{\text{Dem. Nom.}}$.

We are interested in the impact of randomness in the nodal demands. This means that the demand $\boldsymbol{q}_{\text{Demand}}$ becomes a random variable with the expectation

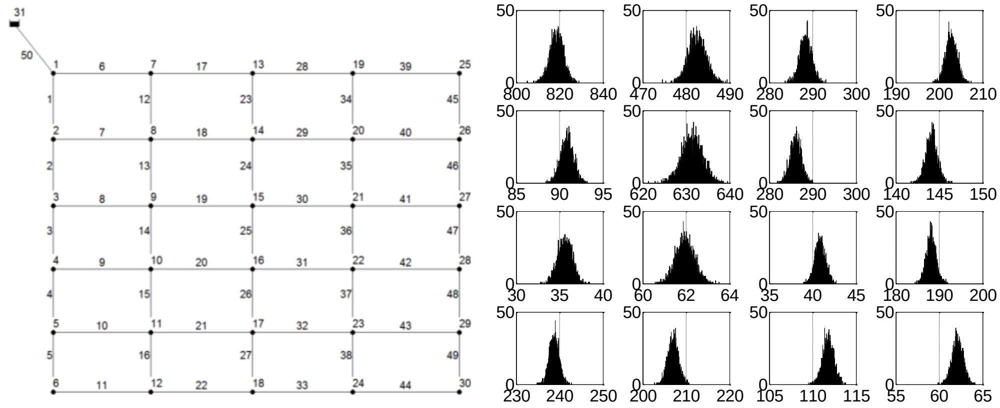

Figure 1.: Left: Network presented in [2]. We refer to the nodal demands as $\boldsymbol{q}_{\text{Dem. Nom.}}$. Right: Exemplary distribution of the flow rates due to random demand fluctuations as output of a Monte Carlo simulation.

$$\mathcal{E}\{\boldsymbol{q}_{\text{Demand}}\} = \boldsymbol{q}_{\text{Dem. Nom.}} \tag{14}$$

The impact of random fluctuations can be computed by means of Monte Carlo simulations. The right part of figure 1 depicts the flow variation of some pipes of the network as the result of a Monte Carlo simulation. In the simulation setup, we generated uniform distributed random demands with a deviation of $\pm$ 10 % of the nominal flow values. As can be seen, the random flows rates follow a Gaussian distribution. This is a result of the central limit theorem. We therefore take use of the mean and the standard deviation to describe randomness for the demand and the flow.

The incorporation of randomness is of large importance for many applications. E.g. in model based leakage detection and localization it is the aim to find a leakage flow from spatial measurements given the network model and knowledge about the nominal demands. However, demand uncertainties can cover/hide/overlay the leakage. The incorporation of statistical knowledge can help to improve the algorithms [3]. However, a Monte Carlo simulation can become time-consuming. We can use the HSSM to provide the same result within a single computation step.

As the flow rates and demands have become random variables, we have to take use of the expectation operator to analyze the problem. In a first step we analyze

$$\mathcal{E}\{d\boldsymbol{p}\} = \mathcal{E}\{\boldsymbol{K}^{-1} d\boldsymbol{q}\}, \tag{15}$$

which is the expectation value of the solution. Due to the fact that the stiffness matrix $\boldsymbol{K}$ is constant, equation (15) can be rewritten to

$$\mathcal{E}\{d\boldsymbol{p}\} = \boldsymbol{K}^{-1} \mathcal{E}\{d\boldsymbol{q}\}\ . \tag{16}$$

Due to equation (14) the expectation of $d\boldsymbol{q}$ becomes zero, which means, that the nominal solution is also the expectation of the flow rates and nodal pressures in the network due to random demands. This result could be verified by computing the mean of the results of the Monte Carlo simulation, which equals the solution for the nominal demand.

We are now interested in the variation of flow rates and nodal pressures due to random demands. For this we evaluate the covariance of $\boldsymbol{dp}$ which results in

$$\begin{aligned}
\mathcal{E}\{d\boldsymbol{p}d\boldsymbol{p}^T\} &= \mathcal{E}\{\boldsymbol{K}^{-1}d\boldsymbol{q}(\boldsymbol{K}^{-\mathbf{1}}d\boldsymbol{q})^T\},\\
\mathcal{E}\{d\boldsymbol{p}d\boldsymbol{p}^T\} &= \mathcal{E}\{\boldsymbol{K}^{-1}d\boldsymbol{q}d\boldsymbol{q}^T\boldsymbol{K}^{-T}\},\\
\mathcal{E}\{d\boldsymbol{p}d\boldsymbol{p}^T\} &= \boldsymbol{K}^{-1}\mathcal{E}\{d\boldsymbol{q}d\boldsymbol{q}^T\}\boldsymbol{K}^{-T}.
\end{aligned} \tag{17}$$

As can be seen, the covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{dq}}$ of the demand and the covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{dp}}$ of the pressure variation are related by

$$\boldsymbol{\Sigma}_{\boldsymbol{dp}} = \boldsymbol{K}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{dq}}\boldsymbol{K}^{-T}. \tag{18}$$

Given the HSSM we can directly compute the statistics of the pressure measurements due to random demands without the use of a computational expensive Monte Carlo simulation. For flow measurements we can derive following equation

$$\boldsymbol{\Sigma}_{\boldsymbol{dq}} = \boldsymbol{CA}\boldsymbol{\Sigma}_{\boldsymbol{dp}}\boldsymbol{A}^T\boldsymbol{C}^T. \tag{19}$$

Equation (18) appears to be computational extensive due to the inverse stiffness matrices. However, with respect to the typical dimensions of the stiffness matrix, the computational costs for the evaluation of (18) are acceptable. Further, some properties of the stiffness matrix can be used to reduce the computational costs.

Figure (2) depicts a comparison of computed standard deviations for head and flow using the results of a Monte Carlo Analysis and equation (18) or (19), respectively.

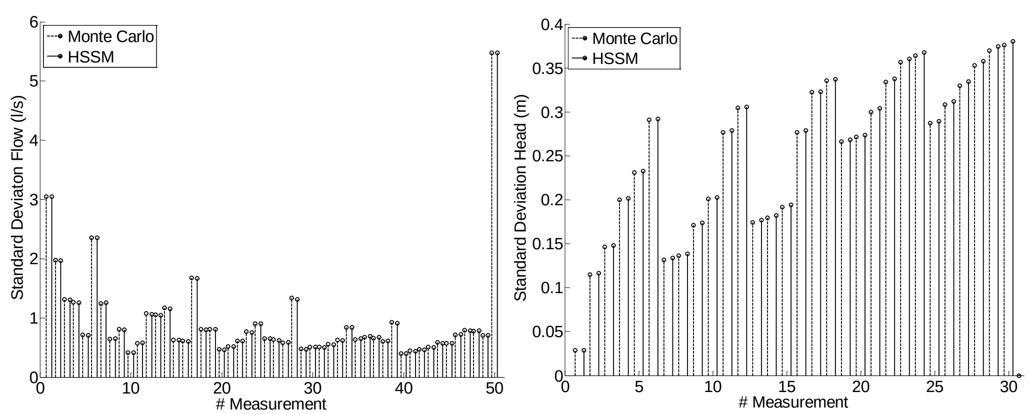

Figure 2.: Comparison between the results of a Monte Carlo analysis and computation of the standard deviation for head and flow using equations (18) and (19).

CONCLUSION

In this paper we presented a HSSM which can be directly derived from the output of a nonlinear hydraulic simulation. The model provides a linearization around the current state of the hydraulic system and can be used for the efficient computation of different characteristics within superior algorithms. We introduced the application of the HSSM for efficient computation of statistics due to uncertain demands and presented a fast approach to calculate covariance matrices. We demonstrated that the model provides equal results as provided by time-consuming Monte Carlo methods.

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